# An Explicit Solution to the Chessboard Pebbling Problem

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**Abstract**

We consider the chessboard pebbling problem analyzed by Chung, Graham, Morrison and Odlyzko [3]. We study the number of reachable configurations $G(k)$ and a related double sequence $G(k,m)$. Exact expressions for these are derived, and we then consider various asymptotic limits.

**Keywords:** Chessboard pebbling; reachable configurations; asymptotics.

## 1 Introduction

The following problem has attracted some attention recently. We begin with an infinite chessboard, which consists of the lattice points $\{(i,j) : i,j \geq 0\}$ in the first quadrant. We

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refer to an individual lattice point as a "cell". We start with a single pebble placed at $(0,0)$. The first "step" consists of removing this pebble and placing two pebbles at the cells $(0,1)$ and $(1,0)$. At each subsequent step we remove a pebble at cell $(i,j)$ and place two pebbles at cells $(i+1,j)$ and $(i,j+1)$, provided that the latter two cells are unoccupied. We consider all possible choices of $(i,j)$. After $k$ steps, there will be a total of $k+1$ pebbles on the board, in various arrangements or configurations.

We let $G(k)$ be the total number of reachable configurations with $k$ pebbles. We define the level sets $L(l) = \{(i,j) : i+j = l\}$, so that $\cup_{l\geq 0} L(l)$ is the entire quadrant. The original problem, posed by Kontsevich [1], was to show that $L(1) \cup L(2) \cup L(3)$ is unavoidable, in that such a set must contain at least one pebble for any reachable configuration. A partial analysis of this fact was published thereafter by Khodulev [2]. The first complete proof was given by Chung, Graham, Morrison and Odlyzko [3]. It was also shown in [3] that $L(1) \cup L(2)$ is an unavoidable set, as well as certain properties relating to the number of unavoidable sets with $k$ pebbles, including the geometric growth rate of this quantity as $k \to \infty$. In [4], Knessl obtained some further asymptotic properties relating to the enumeration of unavoidable sets. Various extensions of this problem are studied by Eriksson [5] and Warren [6].

Suppose we allow more than one pebble per cell and start with an initial configuration of one pebble in cells $(0, m+1)$ and $(m+1, 0)$ and two pebbles in each of the cells $(1,m), (2, m-1), ..., (m-1, 2), (m, 1)$. Thus there are a total of $2m+2$ pebbles in the level set $L(m+1)$, and we assume that $L(M)$ are empty for $M > m+1$. Again the pebble at $(i,j)$ can only be moved if the cells $(i+1,j)$ and $(i,j+1)$ are empty. Let the number of reachable configurations corresponding to this starting arrangement be denoted by $G(k,m)$. In [3], a recurrence relation for $G(k,m)$ is derived, and it is shown that, for $k \geq 2$, $G(k,0) = G(k)$. It is also established that as $k \to \infty$, $G(k) \sim c_* a^k$ where $a = 2.321642199494\cdots$ and $c_* = 0.12268707\cdots$. These constants are characterized in terms of a continued fraction representation of the generating function of $G(k)$. In [7], Knessl obtained a more explicit analytic characterization of the growth rate $a$ of the number of reachable configurations, and showed that this constant can be obtained by solving a transcendental equation that involves two series, each resembling Jacobi elliptic functions. Other asymptotic properties of $G(k,m)$ for $k$ and/or $m \to \infty$ are also established in [7].

In this paper, we derive an exact expression for the number of reachable configurations $G(k)$ and $G(k,m)$. Using the exact expression, we recover the asymptotic results given in [7], and obtain more explicit analytic expressions for each asymptotic scale. These asymptotic scales are $k \to \infty$ with $m = O(1)$; $k$ and $m \to \infty$ simultaneously with $2k/m^2 > 1$; and $k, m \to \infty$ with $l = k - m(m+5)/2 = O(1)$.

The paper is organized as follows. In Section 2, we state the basic equations and summarize the main results for $G(k,m)$ in Theorem 2.1 and 2.2, and for $G(k)$ in Corollary 2.1 and 2.2. In Section 3, we provide brief derivations.

## 2 Summary of results

It is shown in [3] that $G(k,m)$ satisfies the following recurrence equations:

$$G(k,0) = 2G(k-1,0) + G(k,1) + \delta(k,2) \tag{2.1}$$

$$G(k,1) = G(k-3,0) + 2G(k-2,1) + G(k-1,2) + G(k-4,1) \tag{2.2}$$

$$G(k,m) = G(k-m-2,m-1) + 2G(k-m-1,m) + G(k-m,m+1), \quad m \geq 2. \tag{2.3}$$

Here $\delta(i,j)$ is the Kronecker delta symbol. As shown in [7], the boundary condition (2.1) can be replaced by

$$G(k,0) = 2^{k-2} + \sum_{l=1}^{k} 2^{k-l} G(l,1), \quad k \geq 2. \tag{2.4}$$

Using (2.4) in (2.2), $G(k,0)$ can be eliminated, which leads to

$$G(k,1) = 2G(k-2,1) + G(k-1,2) + G(k-4,1) + 2^{k-5} + \sum_{l=1}^{k-3} 2^{k-l-3} G(l,1),\ k \geq 5. \quad (2.5)$$

We introduce a generation function of $G(k,m)$, with

$$V_m(z) = (-1)^m z^{-m} \sum_{k=0}^{\infty} G(k,m) z^k. \quad (2.6)$$

By explicitly solving for $V_m(z)$, we obtain the following integral representation for $G(k,m)$.

**Theorem 2.1** *The number of reachable configurations $G(k,m)$ has the following exact expression*

$$G(k,m) = \frac{1}{2\pi i} \oint_{\mathcal{C}} (-1)^m z^{m-k-1} V_m(z) dz. \quad (2.7)$$

*Here $\mathcal{C}$ is a closed counterclockwise contour around the origin in the $z$-plane, with $|z| < 1/2$ on $\mathcal{C}$,*

$$V_m(z) = \frac{z^{1+m(m+1)/2}}{S(z)} \sum_{n=1}^{\infty} (-1)^{n+m} z^{n(n+1)/2+nm} \prod_{L=0}^{m} \frac{1}{1-z^{L+n}} \prod_{L=1}^{n-1} \frac{1}{(1-z^L)^2} \quad (2.8)$$

*and*

$$S(z) = (2z^2 - 3z + 2)S_1(z) - (4z^2 - 4z + 1)S_2(z) + 2z^2 - z - 1, \quad (2.9)$$

*where*

$$S_k(z) = \sum_{i=1}^{\infty} (-1)^{i+1} z^{i^2/2+(2k-1)i/2} \prod_{j=1}^{i} \frac{1}{(1-z^j)^2}. \quad (2.10)$$

Since the total number of reachable configurations for the original problem is $G(k) = G(k,0)$, using (2.7) in (2.4) with $m=1$ we get the exact expression of $G(k)$:

**Corollary 2.1** *An exact expression for the total number of reachable configurations $G(k)$ is*

$$G(k) = 2^{k-2} + \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{2^k - z^{-k}}{1-2z} V_1(z)dz,$$

*where $V_1(z)$ is given by (2.8) with $m = 1$.*

Using (2.7), we obtain the following asymptotic expressions for $G(k, m)$. These asymptotic results were first obtained in [7], but there some parameters could only be determined numerically.

**Theorem 2.2** *The number of reachable configurations $G(k, m)$ has the following asymptotic expressions:*

1. *When $k \to \infty$ and $m = O(1)$,*

$$G(k,m) \sim \frac{z_*^{m(m+3)/2-k}}{S'(z_*)} \sum_{n=1}^{\infty} (-1)^{n+1} z_*^{n(n+1)/2+nm} \prod_{L=0}^{m} \frac{1}{1-z_*^{L+n}} \prod_{L=1}^{n-1} \frac{1}{(1-z_*^L)^2}, \qquad (2.11)$$

*where $z_* > 0$ is the unique root of $S(z) = 0$ for $|z| < 1/2$ and $S'(z_*)$ is the first derivative of $S(z)$ evaluated at $z = z_*$. The numerical approximation of $z_*$ to 15 decimal places is given below*

$$z_* = 0.43072\,95931\,37930\cdots.$$

2. *When $k$, $m \to \infty$ with $2k/m^2 > 1$,*

$$G(k,m) \sim \frac{z_*^{m(m+5)/2-k+1}}{S'(z_*)} \prod_{L=0}^{\infty} \frac{1}{1-z_*^{L+1}}. \qquad (2.12)$$

3. *When $l = k - m(m+5)/2$ and $2 \le l \le m+3$,*

$$G(k,m) = -\frac{1}{(l-2)!} \lim_{z\to 0} \frac{d^{l-2}}{dz^{l-2}} \left[ \frac{1}{S(z)} \prod_{L=0}^{\infty} \frac{1}{1-z^{L+1}} \right]. \qquad (2.13)$$

*When $k$ and $m \to \infty$ with $l > m+3$, (2.13) holds asymptotically, and then reduces to (2.12).*

Now we consider the asymptotic expression for $G(k)$ when $k \to \infty$. Since $z_* < 1/2$, we can neglect the term $2^{k-2}$ in the right of (2.4) and use (2.11) in (2.4) with $m = 1$. Since $z_* < 1/2$, we have

$$\sum_{l=1}^{k} 2^{k-l} z_*^{-l} \sim \frac{z_*^{-k}}{1-2z_*},\ k \to \infty,$$

and hence the following corollary.

**Corollary 2.2** *The total number of reachable configurations $G(k)$ has the following asymptotic expression, when $k \to \infty$:*

$$G(k) \sim \frac{z_*^2}{(1-2z_*)S'(z_*)} \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z_*^{n(n+3)/2}}{(1-z_*^n)(1-z_*^{n+1})} \prod_{L=1}^{n-1} \frac{1}{(1-z_*^L)^2} \right] \left( \frac{1}{z_*} \right)^k.$$

*Here $S(z)$ is as in (2.9) and (2.10).*

# 3 Brief derivations

Using the generating function (2.6) in (2.3), we obtain the following recurrence equation for $V_m(z)$

$$V_{m+1}(z) + V_{m-1}(z) = (2 - z^{-m-1})V_m(z).$$

We notice that this equation is of the same form as (3.12) given in [7], with $z = 1/a$. Two linearly independent solutions are given by (3.29) and (3.30) in [7]. We reject the growing solution given by (3.30) in [7] since we expect that $V_m(z)$ will be bounded as $m \to \infty$. Hence, we have

$$
\begin{aligned}
V_m(z) &= C(z)\frac{2\pi}{\log z}\exp\left(\frac{\pi^2}{2\log z}\right)\sum_{J=m+1}^{\infty}(-1)^{m+J+1}z^{-(m-J)^2/2-(m-J)/2} \\
&\quad \times\prod_{L=0}^{m}\frac{1}{1-z^{L-J}}\prod_{L=m+1,\,L\neq J}^{\infty}\frac{1}{(1-z^{L-J})^2} \\
&= C(z)\frac{2\pi}{\log z}\exp\left(\frac{\pi^2}{2\log z}\right)\sum_{n=1}^{\infty}(-1)^{n+1}z^{-n^2/2+n/2}\prod_{L=0}^{m}\frac{1}{1-z^{-L-n}} \\
&\quad \times\prod_{L=1,\,L\neq n}^{\infty}\frac{1}{(1-z^{L-n})^2},
\end{aligned}
\tag{3.1}
$$

where $C(z)$ is independent of $m$. We note that we can rewrite the two products in (3.1) as follows:

$$
\prod_{L=0}^{m}\frac{1}{1-z^{-L-n}} = (-1)^{m+1}z^{(n+m/2)(m+1)}\prod_{L=0}^{m}\frac{1}{1-z^{L+n}} \tag{3.2}
$$

and

$$
\prod_{L=1,\,L\neq n}^{\infty}\frac{1}{(1-z^{L-n})^2} = z^{n(n-1)}\prod_{L=1}^{n-1}\frac{1}{(1-z^L)^2}\prod_{L=1}^{\infty}\frac{1}{(1-z^L)^2}. \tag{3.3}
$$

Using (3.2) and (3.3) in (3.1) leads to

$$
\begin{aligned}
V_m(z) &= C(z)\frac{2\pi}{\log z}\exp\left(\frac{\pi^2}{2\log z}\right)\prod_{L=1}^{\infty}\frac{1}{(1-z^L)^2}\left\{\sum_{n=1}^{\infty}(-1)^{n+m}z^{n^2/2-n/2+(n+m/2)(m+1)}\right. \\
&\quad \left.\times\left[\prod_{L=0}^{m}\frac{1}{1-z^{L+n}}\right]\left[\prod_{L=1}^{n-1}\frac{1}{(1-z^L)^2}\right]\right\}.
\end{aligned}
\tag{3.4}
$$

To determine $C(z)$, we use (2.6) in the boundary condition (2.5), which yields

$$
\left(z^4+2z^2-1+\frac{z^3}{1-2z}\right)V_1(z) - z^2V_2(z) = \frac{z^4}{1-2z}. \tag{3.5}
$$

We note that (3.5) holds for $|z| < 1/2$. We introduce the function $U_m(z)$

$$U_m(z) = \sum_{n=1}^{\infty} (-1)^{n+m} z^{n^2/2-n/2+(n+m/2)(m+1)} \prod_{L=0}^{m} \frac{1}{1-z^{L+n}} \prod_{L=1}^{n-1} \frac{1}{(1-z^L)^2},$$

and then $V_m(z)$ in (3.4) is

$$V_m(z) = C(z) \frac{2\pi}{\log z} \exp\left(\frac{\pi^2}{2\log z}\right) \left[\prod_{L=1}^{\infty} \frac{1}{(1-z^L)^2}\right] U_m(z). \tag{3.6}$$

Using (3.6) with $m = 1$ and $m = 2$ in (3.5), we obtain $C(z)$ as

$$\begin{aligned} C(z) &= \Big[(-2z^5 + z^4 - 3z^3 + 2z^2 + 2z - 1)U_1(z) - z^2(1-2z)U_2(z)\Big]^{-1} \\ &\quad \times \frac{\log z}{2\pi} \exp\left(-\frac{\pi^2}{2\log z}\right) z^4 \prod_{L=1}^{\infty} (1-z^L)^2. \end{aligned} \tag{3.7}$$

Instead of using (3.7) in (3.6), we introduce the functions $S_k(z)$ defined in (2.10) and rewrite $C(z)$ in terms of the $S_k(z)$. (The reason for making this change is that it allows us to verify the equivalence of $S(z) = 0$ with equation (3.39) in [7], which we will discuss later.) Using (2.10), $U_1(z)$ and $U_2(z)$ in (3.7) can be expressed as

$$U_1(z) = -S_1(z) + \left(1 + \frac{1}{z}\right) S_2(z) - \frac{1}{z} S_3(z) \tag{3.8}$$

and

$$U_2(z) = -S_1(z) + \left(1 + \frac{1}{z} + \frac{1}{z^2}\right) S_2(z) - \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}\right) S_3(z) + \frac{1}{z^3} S_4(z). \tag{3.9}$$

We rewrite $S_k(z)$ in (2.10) as

$$S_k(z) = \sum_{i=1}^{\infty} (-1)^{i+1} z^{i^2/2+(2k-1)i/2} \prod_{j=1}^{i+1} \frac{1}{(1-z^j)^2} (1 - 2z^i + z^{2i}),$$

and then, after some calculation, we obtain the following recurrence equation

$$z^{1-k}S_{k-1}(z) + (1-2z^{1-k})S_k(z) + z^{1-k}S_{k+1}(z) = 1. \tag{3.10}$$

Hence, we can use (3.10) with $k=2$ and $k=3$ to eliminate $S_3(z)$ and $S_4(z)$ from (3.8) and (3.9). Thus, after some simplification, we obtain $C(z)$ as

$$C(z) = \frac{z}{S(z)}\frac{\log z}{2\pi}\exp\left(-\frac{\pi^2}{2\log z}\right)\prod_{L=1}^{\infty}(1-z^L)^2, \tag{3.11}$$

where $S(z)$ is defined in (2.9). Using (3.11) in (3.6), we obtain (2.8) in Theorem 2.1. Then we have the integral expression of $G(k,m)$ in (2.7).

In the remainder of this section, we discuss the asymptotic approximations to $G(k,m)$. We first consider $k\to\infty$ and $m=O(1)$. By using Rouché's theorem, or by plotting $S(z)$ given in (2.9) numerically, we notice that there is a single real root for $|z|<1/2$. We denote this single root as $z_*$, which satisfies $S(z_*)=0$. Then in the $z$-plane with $|z|<1/2$, the integrand in (2.7) has a simple pole at $z=z_*$, which is the dominant singularity. We use the residue theorem to evaluate the integral in (2.7) asymptotically, which yields (2.11). We note that it's not hard to verify that $S(z)=0$ is equivalent to (3.39) in [7], after we substitute $1/z$ for $a$ in (3.39). Thus $z_*^{-1}=a$, whose numerical approximation to 100 decimal places is given in [7]. By comparing (2.11) in this paper with formula (3.37) in [7], we obtain an analytic expression for the constant $c_1$ in (3.38) in [7] as follows:

$$c_1 = -\frac{\log z_*}{2\pi}\exp\left(-\frac{\pi^2}{2\log z_*}\right)\frac{1}{S'(z_*)}\prod_{j=1}^{\infty}(1-z_*^j)^2. \tag{3.12}$$

By evaluating (3.12) numerically, we find that the numerical values of $c_1$ and $c_1K_*$ provided in (3.42) and (3.43) in [7] are only correct to about 5 decimal places, though they are given to 15 places. The correct values to 15 decimal places are

$$c_1 = 2.02740\ 20474\ 68498 \cdots$$

and

$$c_1 K_* = 0.28777\ 77049\ 35052 \cdots .$$

In the limit when $k$, $m \to \infty$ simultaneously, the $n = 1$ term in (2.11) dominates, which yields (2.12). This recovers the result given in (4.18) in [7], but now with $c_1$ computed explicitly.

Next, we consider $k$ and $m$ large with $l = k - m(m+5)/2 = O(1)$. Note that $l = 2$ corresponds to the smallest number of possible configurations, as $G(k, m) = 0$ for $l \le 1$. Following [7] we denote $G(k, m)$ as $W(l, m)$ and rewrite (2.7) as

$$\begin{aligned} G(k,m) \equiv W(l,m) &= \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{1}{z^{l-1}} \frac{1}{S(z)} \sum_{n=1}^{\infty} (-1)^n z^{(n-1)(n/2+m+1)} \\ &\quad \times \prod_{L=0}^{m} \frac{1}{1 - z^{L+n}} \prod_{L=1}^{n-1} \frac{1}{(1-z^L)^2} dz. \end{aligned} \tag{3.13}$$

For a sufficiently small closed contour $\mathcal{C}$ around the origin in the $z$-plane and $l \ge 2$, $z = 0$ is the only pole inside of $\mathcal{C}$, and it is of order $l - 1$. Thus, using the residue theorem in (3.13) leads to

$$\begin{aligned} W(l,m) &= \frac{1}{(l-2)!} \lim_{z\to 0} \frac{d^{l-2}}{dz^{l-2}} \Bigg\{ \frac{1}{S(z)} \sum_{n=1}^{\infty} (-1)^n z^{(n-1)(n/2+m+1)} \\ &\quad \times \prod_{L=0}^{m} \frac{1}{1 - z^{L+n}} \prod_{L=1}^{n-1} \frac{1}{(1-z^L)^2} \Bigg\}. \end{aligned} \tag{3.14}$$

The first two terms from the infinite sum in the right-hand side of (3.14) are

$$-\prod_{L=0}^{m} \frac{1}{1-z^{L+1}} + \frac{z^{m+2}}{(1-z)^2} \prod_{L=0}^{m} \frac{1}{1-z^{L+2}}, \tag{3.15}$$

and the remaining terms are of order $O(z^{2m+5})$ as $z \to 0$. We notice that after taking $l-2$ derivatives, the second term in (3.15) is of order $O(z^{m-l+4})$ as $z \to 0$. This implies that as long as $m - l + 4 \geq 1$, i.e., $l \leq m+3$, only the $n = 1$ term in the infinite sum in (3.14) contributes to the derivative at $z = 0$, which yields

$$W(l,m) = -\frac{1}{(l-2)!} \lim_{z\to 0} \frac{d^{l-2}}{dz^{l-2}} \left[ \frac{1}{S(z)} \prod_{L=0}^{m} \frac{1}{1-z^{L+1}} \right], \ 2 \leq l \leq m+3. \tag{3.16}$$

Rewriting the expression in the brackets in (3.16) as

$$\frac{1}{S(z)} \prod_{L=0}^{m} \frac{1}{1-z^{L+1}} = \frac{1}{S(z)} \prod_{L=0}^{\infty} \frac{1}{1-z^{L+1}} \prod_{L=m+1}^{\infty} (1-z^{L+1}),$$

we see that (3.16) is independent of $m$. This leads to (2.13), and gives an analytic expression for $W_0(l)$, which appears in [7] but there no explicit expression is given. This concludes our derivation.

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