In praise of order units

Abstract We show that the ordered rings naturally associated to compact convex polyhedra with interior satisfy a positivity property known as order unit cancellation, and obtain other general positivity results as well.

David Handelman[1]

An element u of a partially ordered abelian group G is called an *order unit* (relative to G) if for all g in G, there exists a positive integer M such that $-Mu \leq g \leq Mu$. A *trace* of G is a positive additive nonzero function $G \to \mathbf{R}$, and a *pure trace* is a trace that is not a nontrivial convex linear combination of other traces. The partially ordered abelian group is *unperforated* if for g in G and positive integer n, $ng \in G^+$ implies $g \in G^+$. When G is unperforated and admits an order unit, then u will be an order unit if and only if for all (pure) traces τ on G, $\tau(g) > 0$ [GH, Lemma 4.1]. An *order ideal* I of a partially ordered abelian group G is a subgroup with two additional properties, *convexity* ($0 \leq g \leq h$, $h \in I$, $g \in G$ entails $g \in I$) and directedness (I is spanned by its positive elements).

In this paper, a *partially ordered ring* R will be a commutative unital ring with a subset R^+ making R into a partially ordered group with respect to addition, such that in addition, $R^+ \cdot R^+ = R^+$, R is unperforated, and 1 is an order unit (relative to R). By [H85, 1.2(a)], any order ideal of a partially ordered ring is itself an ideal (this depends of course on 1 being an order unit). The pure traces on an ordered ring with 1 as order unit, normalized so that $1 \to 1$, are exactly the multiplicative traces. The set of these, equipped with the point-open topology, form a compact set (the extremal boundary of the Bauer simplex consisting of all the normalized traces).

We say the partially ordered ring R satisfies *order unit cancellation* (informally described as *order cancellation* in [H87A, p 22]) if whenever u is an order unit of R and a nonzero divisor, and a is an element of R, $ua \geq 0$ implies $a \geq 0$. (As usual, we use $r \geq s$ to denote $r - s \in R^+$.) This is a relatively weak property for an ordered ring, but it has been useful. For example, it is the key tool in [H87A, II.1], showing that if P, Q, and f are real polynomials in several variables such that P and Q have no nonnegative coefficients and the same monomials appear, then

there exists m such that $P^m f$ has no negative coefficients

if and only if there exists M such that $Q^M f$ has no negative coefficients.

(The actual result is sharper than this.)

If $U \equiv U(R)$ denotes the set of nonzero divisor order units of the partially ordered ring R, then U is multiplicatively closed (and also additively closed if R is a domain, but this seems to play a limited role in what follows). We may thus form the overring $U^{-1}R$, which can be equipped with the direct limit partial ordering, $(U^{-1}R)^+$, obtained by taking the directed limit over all maps of the form $\times u : R \to R$, every element u appearing infinitely often. Then

$$(U^{-1}R)^+ = \left\{ u^{-1}r \mid u \in U, \text{there exists } v \text{ in } U \text{ such that } vr \in R^+ \right\}.$$

(We could just as well take this as the definition of $(U^{-1}R)^+$ rather than using the limit formulation; however, direct limits appear throughout the examples.) Then order unit cancellation is equivalent to $(U^{-1}R)^+ \cap R = R^+$. The effect of this localization is to eliminate the multiplicative homomorphisms that are not traces (i.e., not positive), so that the maximal ideal space of $U^{-1}R$ is identifiable in the obvious way with the pure trace space; this is used in [H87A, App A] to

[1] Supported in part by a Discovery Grant from NSERC.

deduce properties of decompositions of convex polyhedra as sums of others. This should also be the precursor to the use of localization at prime order ideals.

Our principle result will be to show that a class of partially ordered rings studied in [H88] satisfies order unit cancellation. Let K be a compact convex polyhedron (a "*polytope*") with interior, in $\mathbf{R}^n$. For each facet, F_i of K, choose a linear form (unique up to positive scalar multiple) $\beta_i = \sum a_i x_i + a_0$ (where $a_0, a_1, \dots, a_n$ are real) such that $\beta_i | F_i \equiv 0$ and $\beta_i | K \geq 0$. Let $R = \mathbf{R}[K]$ denote the polynomial ring $\mathbf{R}[x_1, \dots, x_n]$ equipped with the positive cone generated additively and multiplicatively (allowing positive scalars) by $\{\beta_i\}$. It was shown in [H88, I.2] that this is a partially ordered ring in the sense used here, including the fact that 1 is an order unit. We will show that $\mathbf{R}[K]$ has order unit cancellation. We will also obtain a number of positivity results in a similar vein to order unit cancellation.

Before doing so, we note a number of classes of examples and non-examples.

Trivial examples Let X be a compact set, and let R be any unital subring of the ring of continuous real-valued functions, $C(X)$, equipped with the pointwise order (that is, for f, g in R, $f - g \in R^+$ if and only if $f(x) \geq g(x)$ for all x in X). Order unit cancellation is obvious.

A partially ordered abelian group is *simple* if it has no proper order ideals, or what amounts to the same thing, all nonzero positive elements are order units. If the partially ordered ring R is simple as a partially ordered abelian group, then obviously order unit cancellation holds.

If F is a formally real subfield of the algebraic closure of the rationals equipped with the sum of squares ordering, then F is a partially ordered ring that is simple as a partially ordered group—but this falls in the previous class of examples, with X being the Galois group of the real algebraic closure of the rationals.

A larger class consists of partially ordered rings R (with 1) equipped with a unital algebra map $\pi : R \to C(X)$ (not necessarily one to one), such that a nonzero element of R is in R^+ if and only its image in $C(X)$ is strictly positive (as a function). This in fact describes completely the class of partially ordered rings which are simple as partially ordered groups. •

Before dealing with more examples, we cite a result that we will prove later.

THEOREM 1 Let R be a noetherian partially ordered ring (unperforated, and with 1 as order unit). Suppose that for every proper order ideal I of R the following holds:

> for every prime ideal P minimal over I, there exists a proper order ideal J of R and an integer k such that $P^k \subseteq J$.

Then R satisfies order unit cancellation.

Toy examples Let R be the ring $\mathbf{R}[x]$, which we equip with two partial orderings, but these can obviously be modified.

$$R_1^+ = \left\{ x^j (1 + x) f \mid j \geq 1,\ f | [0, 1] > 0 \right\} \cup \{ f \mid f | [0, 1] > 0 \}$$
$$R_2^+ = \left\{ x^j f \mid j = 0, 2, 3, \dots;\ f | [0, 1] > 0 \right\}$$

In both cases, we see the putative positive cones are closed under addition, positive scalar multiplication, and multiplication, and 1 is an order unit. In both cases, the pure traces are point evaluations at the points of $[0, 1]$, so the order units are precisely the polynomials that are strictly positive thereon.

In the case of R_1, we note that x is not in the positive cone, but $x(1 + x)$ is, and $1 + x$ is an order unit. Thus order unit cancellation fails here. On the other hand, in R_2, it is easy to check that the nontrivial order ideals are exactly (x^j) with $j \geq 2$. The only prime ideal containing a nontrivial order ideal is thus (x), which although it is not an order ideal, its square is, and so the criterion of Theorem 1 applies—R_2 does satisfy order unit cancellation.

The difference in the behaviour of these two examples lies in the behaviour of the maximal order ideals. In R_1, it is $(x(1+x))$ which (as an ideal) is a product of two distinct prime ideals one of which is not contained in any order ideal (the split case), while for R_2, the maximal order ideal is (x^2) which is contained in just one prime ideal of which it is the square (the ramified case).

Interesting examples [H85] Let H be a compact group with identity e, and representation ring denoted $\mathcal{R}(H)$, and $\pi : H \to M_d\mathbf{C}$ a d-dimensional representation with character χ. By reducing to a power and possibly factoring out a normal subgroup, we may assume in what follows that π (respectively χ) is *projectively faithful*, that is, $\pi^{-1}(\mathbf{C} \cdot \mathrm{I}) = \{e\}$ ($\{h \in H \mid |\chi(h)| = \chi(e)\} = \{e\}$). Multiplication by χ is a positive $\mathcal{R}(H)$-module endomorphism of $\mathcal{R}(H)$, and we can take the direct limit, forming $\mathcal{R}(H)[\chi^{-1}] = \lim \times\chi : \mathcal{R}(H) \to \mathcal{R}(H)$, with the direct limit ordering. The convex subgroup generated by $(\chi_0, 1)$ the image of the trivial character at the first level, or equivalently, the identity in $\mathcal{R}(H)[\chi^{-1}]$ is a partially ordered ring (in our strong sense), denoted R_χ.

When $H = \boldsymbol{T}^n$, the n-torus, $\chi = P$ is simply a Laurent polynomial in n variables with nonnegative integer coefficients, and we can permit nonnegative real coefficients by replacing $\mathcal{R}(H)$ by $\mathcal{R}(H) \otimes \mathbf{R}$. The resulting ordered rings, R_P, are the main object of study in [H87A], and the proof that they satisfy order unit cancellation [H87A, II.5] is rather intricate, but yields the result referred to earlier about polynomials in several variables.

If H is a finite group, then R_χ is simple (as an abelian group) with unique trace (as follow quickly from the Perron-Frobenius theorem and the assumption of projective faithfulness), so order unit cancellation holds in these examples as well.

However, if $H = \boldsymbol{T} \times \mathbf{Z}_2$, the product of the circle with the two-element group, there is a projectively faithful character χ such that R_χ does not satisfy order unit cancellation [H93, Section 5].

If H is a simple connected Lie group (i.e., finite centre), then generically in χ, R_χ satisfies order unit cancellation, but it is unknown whether this holds for all projectively faithful χ. For example, let P be the Laurent polynomial obtained by restricting χ to a maximal torus. The Weyl group, W, acts on R_P, and there is a natural inclusion $R_\chi \to R_P$ whose image is contained in R_P^W, the fixed point subring under the action of W. If all the extreme points of the convex hull of the weights of all the irreducibles in χ do not lie on the boundary of the Weyl chamber, then the map $R_\chi \to R_P^W$ is an isomorphism as partially ordered rings (where R_P^W inherits the ordering from R_P) [H95, Theorem 1.1], and it follows immediately from R_P satisfying order unit cancellation that R_χ does as well.

In the remaining situations, we do not know whether order unit cancellation holds. There are plenty of cases wherein the map $R_\chi \to R_P^W$ is an isomorphism of rings but not of ordered rings (ring isomorphism occurs but not necessarily order isomorphism, e.g., if H is simply laced and χ is a power of an irreducible; more generally, iff some power of χ is saturated in a certain sense ([H95, Proposition 2.7] using a result of O'Brien [O'B]). There are even cases wherein the map is not onto. But in all cases, the order units are the same, in the sense that for u in R_χ, u is an order unit for R_χ if and only if its image in R_P is an order unit therein.

Power series versions of these constructions have also been studied. Let $P = \sum a_n x^n = \sum (P, x^n) x^n$ be a convergent Maclaurin series with radius of convergence 1, and all coefficients nonnegative. Define R_P to be what we would get from the direct limit construction (modified for functions analytic on the disk),

$$R_P = \left\{ f/P^k \;\middle|\; \text{there exists positive integer } N \text{ such that for all } n,\, |(f, x^n)| \leq N(P^k, x^n) \right\}.$$

This is a partially ordered ring. If P is continuous at 1 and the coefficients behave reasonably smoothly (no large gaps, etc; e.g., $P = \sum x^n/n^2$), then R_P is noetherian, satisfies order unit cancellation for very easy reasons (the only relevant prime ideal consists of the functions vanishing at zero). On the other hand, if $P = (a + bx)/(1 - x^2)$ with a and b positive, and $a \neq b$, then R_P does not satisfy order unit cancellation [H03, Proposition B.2 and Corollary B.6] (and there are many other examples of this kind). However, it would be interest to decide whether R_P satisfies order unit cancellation when $a = b$, i.e., $P = (1 - x)^{-1}$. In all these cases, provided the coefficients behave reasonably, the pure point evaluations at $[0, 1)$ are dense in the pure trace space, but when $P(1)$ does not exist, just which compactification occurs is unknown.

Matrix-related examples Yet another class of interesting partially ordered rings (not all, but generically, commutative non-noetherian domains) appears in [H09]. Let M be a square matrix whose entries are real polynomials in several variables with no negative entries, form the direct limit (repeating M), and let S be the centralizer of the induced automorphism in the direct limit module, and finally, let R be the convex subgroup thereof generated by the identity. This is a partially ordered ring (not always commutative, or a domain, or noetherian, but having 1 as an order unit). The example mentioned above arising from $\boldsymbol{T} \times \mathbf{Z}_2$ also appears in this context, so order unit cancellation need not hold—but it should hold when R has no zero divisors.

Some positivity criteria

Here we give criteria for positivity, mostly recalled from earlier work. For s in R, let $S(s) = \{r \in R \mid rs \geq 0\}$; then $sS(s)$ is the set of products rs such that $rs \geq 0$, and in particular is a subset of R^+. Define $S^+(s) = S(s) \cap R^+$.

If A is a subset of R, then as usual (A) denotes the ideal generated by R. If T is a subset of R^+, the *order ideal generated by* T is

$$< T >:= \left\{ r \in R \,\middle|\, \exists \text{ positive integer } M \text{ and finite } \{t_i\} \subseteq T \text{ such that } -M \sum t_i \leq r \leq M \sum t_i \right\}.$$

This is the smallest order ideal containing T, and when 1 is an order unit, $< T >$ is an order ideal. (In general, there is no satisfactory way of defining $< T >$, the order ideal generated by T, if T has nonpositive elements.)

The approach in the following is mostly based on that of [H87A, Section II].

LEMMA 2 Let R be a commutative ordered ring with 1 as an order unit, and let s be an element of R. Suppose that with $J :=< sS^+(s) >$, the element $s + J$ is in $(R/J)^+$ (this includes the case that $s + J = 0$, i.e., $s \in J$). Then $s \in R^+$.

Proof. We may write $s = p + q$ where $p \in R^+$ and $q \in J$. There exists a finite set r_i of elements of $S^+(s)$ such that both $\pm q \leq \sum sr_i$ (the r_is may be repeated, hence we don't need an integer multiple appearing on the right). Let $J_0 :=< \{sr_i\} >$, and let $J_1 :=< \{sr_i\}, p >$ (there are no problems with the order ideals generated by these sets, since all the members thereof are in R^+). Then $s \in J_1$, and since J_1 is generated as an order ideal by a finite set of elements of the positive cone of R, J_1 admits an order unit. We will show that $\gamma(s) > 0$ for all pure (normalized) traces of J_1, from which it follows that s is an order unit of J_1 and thus is in R^+.

By [H85, Lemma 2.1(c)], γ is of either of two forms. Either (i) there exists a pure trace L of R such that $\gamma = \alpha L\,|_R$ (i.e., up to renormalization by the value of L at the order unit of J_1, γ is the restriction of L to J_1) for some real $\alpha > 0$, or (ii) there exists a pure trace L of R such that $L(J_1) = 0$ and for all j in J_1 and r in R, $\gamma(jr) = \gamma(j)L(r)$.

Suppose $\gamma(s) = 0$. If (i) holds, then $\gamma(sr_i) = \alpha L(sr_i) = \alpha L(s)L(r_i) = 0$, whereas if (ii) holds, then $\gamma(sr_i) = \gamma(s)L(r_i) = 0$. Hence in either case, $\gamma(sr_i) = 0$, and thus $\gamma(q) = 0$, whence $\gamma(p) = 0$. But this means γ vanishes on the order ideal generated by $\{sr_i\} \cup \{p\}$, that is, on J_1, a contradiction. Hence $\gamma(s) \neq 0$.

If $\gamma(s) < 0$, then since $sr_i \in R^+$ and $0 \leq \gamma(sr_i) = \gamma(s)L(r_i)$ (in either case (i) or (ii)) and $r_i \geq 0$, we deduce $L(r_i) = 0$ for all i. Therefore $\gamma(sr_i) = 0$, which again implies $\gamma(q) = 0$, and thus $\gamma(s) = \gamma(p) \geq 0$ a contradiction. •

This can be extended. But for now, we note a consequence: if u is an order unit and $us \geq 0$, then u is in $S^+(s)$, and we may form the order ideal I generated by us. If s belongs to I (or if $s + I \in (R/I)^+$), then $s \in J$ (defined in the lemma), so that $s \in R^+$. It is also true if instead of $s \in I$, we have $-s + I \in (R/I)^+$, by an almost identical proof (but there is no particular use apparent for this weird result).

Now we can give a proof of Theorem 1.

Proof of Theorem 1 Suppose u is an order unit of R and a is an element thereof such that $ua \in R^+$. Let I be the order ideal generated by ua; then sufficient for $a \in R^+$ is that $a \in I$.

By the Lasker-Noether theorem, there exist meet-irreducible (hence primary) ideals Q_j with associated minimal (over I) ideals P_j such that $I = \cap Q_j$. In particular, for each j, $ua \in Q_j$. Moreover, no power of u can belong to any proper order ideal, so if u^m belonged to Q_j, then u^{mk} would belong to P_j^k (where $k \equiv k(P_j)$ as hypothesized), whence it would belong to the order ideal J, a contradiction. Since meet-irreducible ideals are primary (zero divisors are nilpotent), $ua \in Q_j$ and $u^n \notin Q_j$ for all n imply that $a \in Q_j$, whence $a \in I$. Now the observation above applies. •

(Thus far at least, all the noetherian domains which fail to satisfy the hypotheses in Theorem 1 also fail to satisfy order unit cancellation.)

We will show that all the rings of the form $\mathbf{R}[K]$ with K a compact convex polyhedral body in Euclidean space, satisfy order unit cancellation, while some of their close relatives do not. Recall that the positive cone is generated multiplicatively and additively by the linear forms $\beta_i = a_{0i} + \sum a_{ji} x_j$ with real coefficients a_{ji}, such that $\beta_i^{-1} 0 \cap K = F_i$ run over all the facets (maximal proper faces) of K with $\beta_i | K \geq 0$.

Let K be a compact convex polyhedron with interior in $\mathbf{R}^n$, and let R be the partially ordered ring $\mathbf{R}[K]$. Let I be an order ideal therein, and let $Z(I)$ be zero set of I; since order ideals are generated by monomials in linear forms, the zero set (in $\mathbf{C}^n$) is an intersection of a union of hyperflats, i.e., a union of flats, and moreover, is just the complexification of the set of real zeros (again a union of flats, this time in $\mathbf{R}^n$). Now $Z(I) \cap K$ is a union of faces in K, and our immediate aim is to show that it is Zariski dense in $Z(I)$, that is, for f in R, $f | Z(I) \cap K \equiv 0$ entails $f | Z(I) \equiv 0$. Without loss of generality, we may assume (since order ideals are generated by monomials in the βs), that the zero set notation $Z(I)$ refers to real zeros (i.e., $Z(I)$ can be reduced to $Z(I) \cap \mathbf{R}^n$, which we relabel as $Z(I)$.

To see why this is not entirely trivial, consider a trapezoid, K, which is not a parallelogram, and let F_1 and F_3 be the two non-contiguous but non-parallel edges. Their intersection is empty, but the intersection of the their affine hulls (i.e., the real flats they generate) is a single point, outside K. If β_1 and β_3 are the corresponding linear forms (inward normals) exposing the edges, then the ideal I generated by $\{\beta_1, \beta_3\}$ is generated by two of the generators of $\mathbf{R}[K]^+$, is a maximal ideal, its zero set consists of a singleton (the point where the flats meet) outside K, and the intersection of the zero set with K is empty.

Of course, in this case $\beta_1 + \beta_3$ is an order unit of R, so any order ideal containing it would have to be R itself—in particular, even though I is generated by monomials in the generating linear forms, $Z(I) \cap K$ is not Zariski dense in $Z(I)$. So the order ideal hypothesis is crucial.

Another similar example arises from a pyramid with square base. If we take two non-contiguous triangular faces abutting the apex v, their intersection is just a singleton consisting of the apex; however, the intersection of their affine hulls is a line, and in this case, the linear form $\beta_1 + \beta_3$ is a (relative) order unit for the maximal ideal (and in this case, order ideal) consisting of functions vanishing at v.

PROPOSITION 3 Let G be a proper face of K. Then

$$I(G) := \{ f \in R \mid f | G \equiv 0 \}$$

is an order ideal, and the quotient $R/I(G)$ is order isomorphic to $\mathbf{R}[G]$ where G is embedded in $\mathbf{R}^{n-d}$. If $S = \{\beta_i\}$ is a linearly independent collection of cardinality $n - \dim G$ such that $\beta_i|G \equiv 0$, then S is a generating set for I as an ideal, and $\sum \beta_i$ is an order unit of I (as a partially ordered vector space).

Proof. First, we note that if $G \subset F_i$ where F_i is a facet exposed by the (positive in R) linear form β_i, then β_i belongs to $I(G)$. First, we show that the set of all these β_i generate $I(G)$ as an ideal. If $\dim G = 0$, then G is a vertex (extreme point), so after a change of variables and relabelling, we may assume that it is the origin and $\beta_i = x_i$ for $1 \leq i \leq n$, and there may be subsequent β_i exposing the origin. Now (x_i) is a maximal ideal contained in $I(G)$, and therefore must equal $I(G)$.

More generally, let $g = \dim G$; we may assume that the origin is in the relative interior of G, and we find a collection of facets $F_1, F_2, \ldots, F_{n-g}$ such that $\cap F_i = G$, and F_i is the zero set of x_i (after a change of variables). Then it is easy to check that $I(G) = (x_i)$.

In particular, $I(G)$ is generated as an ideal by positive elements, hence is directed (equal to the set of differences of its positive elements). It is also convex, that is, if $0 \leq h \leq f$ (meaning with respect to R^+) and $f \in I(G)$ then $0 \leq h|G \leq f|G \equiv 0$, so $h|G \equiv 0$, and thus $h \in I(G)$. Thus $I(G)$ is an order ideal.

With the origin in the relative interior of G, let the copy of Euclidean space for realizing the quotient algebra be the vector space spanned by G. It is easy to check that the ordering on the quotient is exactly that spanned by the inward normals to relative facets of G, which is the ordering that we put on $R[G]$.

The linear independence argument is straightforward.

For any order ideal J, if $\{a_\alpha\}$ is a finite generating set consisting of elements of $J^+ = J \cap R^+$, it is trivial to verify that $\sum a_\alpha$ is a (relative) order unit of J (this requires the fact that 1 is an order unit of R, which is part of the definition of partially ordered ring that we are using here). •

PROPOSITION 4 Let G be a proper face of K, and let β be a linear form that is in R^+ and for which $\beta^{-1}0 \cap K = G$.

(i) If γ is any other linear form in R^+ such that $\gamma|G \equiv 0$, then there exists a positive integer M such that $M\beta - \gamma \in R^+$.

(ii) Suppose $\{F_i\}_{i\in S}$ runs over *all* facets of K that contain G, and let β_i denote the corresponding linear forms. Then there exist real $a_i > 0$ such that $\beta = \sum_S a_i\beta_i$.

(iii) β is an order unit of the order ideal $I(G)$.

Proof. Without loss of generality, we may assume that either G is the singleton consisting of the origin, or the origin is in the relative interior of G. Translate G by an element v so that $v + G$ is in the interior of K, and draw a hyperflat L through $v + G$ that misses G; we can make the norm of v sufficiently small so that $E = L \cap K$ only hits facets that abut G.

Now since the origin is in G, β and γ are linear, and kill the vector space spanned by G. Since E is compact, and E misses G, we have that $\inf \beta|E = t > 0$, and similarly, $\sup \gamma|E = u > 0$ exists, so that on E, we have $\beta - (t/2u)\gamma|E \geq 0$. Since β and γ are linear, so is $\rho := \beta - (t/2u)\gamma$, and thus ρ is positive on $\cup_{\lambda>0}\lambda E$, and this includes $K \setminus G$. Since ρ vanishes on G, we have that $\rho|K \geq 0$. By [H88, Lemma 1.1], $\rho \in R^+$. We can thus set $M = 2u/t$. This yields (i).

Form $\gamma = \sum_S \beta_i$, the β_i the linear forms corresponding to all the facets containing G, and find M such that $\rho := M\beta - \gamma \in R^+$. By [H88, Lemma 1.1], we can write $\rho = \sum b_i\beta_i + \sum B_\alpha\beta_\alpha$ where the β_α run over the remaining linear forms (whose facets don't contain G), and the *a*s are all nonnegative. Evaluating at any point v in the relative interior of G, we deduce $0 = \sum B_\alpha\beta_\alpha(v)$; but all β_α are strictly positive on the relative interior of G. Hence all the $B_\alpha = 0$. Part (ii) follows immediately (since $b_i \geq 0$).

Part (iii) is a consequence of part (i) and the previous result. •

Let I be an ideal of the commutative (unital) ring R. As usual, we define the radical of I, $\operatorname{rad} I := \{r \in R \mid \text{there exists } n \text{ such that } r^n \in I\}$. As is well known, $\operatorname{rad} I$ is an ideal, and the primes minimal over I are the same as those minimal over $\operatorname{rad} I$. If R is now a partially ordered ring (i.e., 1 is an order unit of R), and I is an order ideal, $\operatorname{rad} I$ need not be an order ideal, but at least it is convex, that is, if $0 \leq a \leq b \in \operatorname{rad} I$, then $a \in I$—to see this, there exists n such that $b^n \in I$, so $0 \leq a^n \leq b^n \in I$, and since I is convex, $a^n \in I$, and thus $a \in \operatorname{rad} I$.

This means we can define an order ideal of R, denoted $\operatorname{rad}^+ I$, via $\operatorname{rad}^+ I = (\operatorname{rad} I)^+ - (\operatorname{rad} I)^+$ (where $(\operatorname{rad} I)^+ = (\operatorname{rad} I) \cap R^+$. It is routine to verify, in view of the convexity result of the previous paragraph, that $\operatorname{rad}^+ I$ is an order ideal, and the inclusions $I \subseteq \operatorname{rad}^+ I \subseteq \operatorname{rad} I$ guarantee that the minimal prime ideals over each of these ideals are identical. Thus for the purpose of showing $Z(I) \cap K$ is Zariski dense in $Z(I)$, we may assume that $I = \operatorname{rad}^+ I$, that is, the quotient ordered ring R/I has no positive nilpotents.

We define a *flat* in $\mathbf{R}^n$ to be a translate of a subspace, and a *hyperflat* is a flat of dimension $n-1$. If $\beta^w = \prod \beta_i^{w(i)}$ is a monomial in the β_i, then the zero set of β^w is a union of hyperflats, each of whose intersection with K is a facet, $F_i = \beta_i^{-1} 0 \cap K$. Since order ideals are generated (as ideals) by monomials in the β_i, $Z(I)$ will thus be an intersection of union of hyperflats, in particular, it will be a union of flats, which we may write as $Z(I) = \cup E_j$, where the E_j are pairwise incomparable. As we saw in an earlier example (without the order ideal hypothesis, merely generation by monomials), it could happen that for some j, $E_j \cap K$ has smaller dimension than E_j or could even be empty.

However, each $E_j \cap K$ (if nonempty) is a face, call it G_j, of K, and we have $K \cap (\cup E_j) = \cup G_j$. For a face G of K, denote by $\tilde{G}$ the affine hull of G, that is, the flat it generates, alternatively, $\{\sum a_s g_s \mid a_s \in \mathbf{R},\ \sum a_s = 1, \text{ and } g_s \in G\}$.

Write, for each j, $E_j = \cap_t \tilde{F}_{j,t}$, where the $F_{j,t}$ vary over the facets that arise from the monomials that yield the E_j—that is, there exist monomials β^w in I such that $\beta_{j,t}$ appear (with nonzero exponent) in β^w.

Form the face $H_j = E_j \cap K = \cap_j F_{j,t}$, expressed as an intersection of facets. Let $\mathcal{F}_j$ denote the set of *all* facets of K that contain H_j. Form $\beta^{(j)} = \sum_t \beta_{j,t}$; this is a linear form and a positive element of R for which $\beta^{-1} 0 \cap K = H_j$. By the earlier lemma, we can decompose $\beta^{(j)} = \sum_{\beta_{i,j}^{-1} 0 \cap K \in \mathcal{F}_j} a_{i,j} \beta_{i,j}$ with $a_{i,j}$ being *positive* real numbers.

Set $\gamma = \prod_j \beta^{(j)}$; since $\cap \beta_{j,t}^{-1} 0 = E_j$, we have from the real (here a trivial consequence of the usual) version of the Nullstellensatz, that some power of γ belongs to I; as $\gamma \in R^+$, our reduction to $I = \operatorname{rad}^+ I$ entails that $\gamma \in I$.

For each j, select a $\beta_{i,j}$ (for which $\beta_{i,j}^{-1} 0 \cap K \in \mathcal{F}_j$), and take the product over j; take all such products. From $\gamma \in I$ and I being an order ideal, it easily follows that all such products belong to I. The zero set of each such product is a union of hyperflats, and it is easy to check that their intersection (over all the products) is $\cap \tilde{H}_j$. Hence $Z(I) = \cap \tilde{H}_j$, and $Z(I) \cap K = H_j$, as desired.

THEOREM 5 Let K be a compact convex polyhedral body in $\mathbf{R}^n$. Then the partially ordered ring $R = \mathbf{R}[K]$ satisfies order unit cancellation.

Proof. Let I be an order ideal, with $Z(I) \cap K$ the union of faces, $\cup G_j$. By above, $Z(I) = \cup \tilde{G}_j$, and by the Nullstellensatz, any minimal prime ideal over I is minimal over the ideal of functions vanishing on $Z(I)$ (which we do not assume is an order ideal). For any face G, $I(G)$ is an order ideal, and from the generating elements, it is immediate that its zero set is $\tilde{G}$. Moreover $I(G)$ is a prime ideal. Hence the minimal prime ideals sitting over I are contained in at least one of the $I(G_j)$, and the criterion of Theorem 1 is now satisfied. •

More exciting examples
Close relatives of the $\mathbf{R}[K]$ were described in [H88, Example I.5]. Take $R = \mathbf{R}[x, y]$ with the partial ordering generated additively and multiplicatively by the set $\{\mathbf{R}^+, x, y, 1 - (x + 3/5)^2 - (y + 3/5)^2 := \alpha\}$; it was shown [op. cit.] that 1 is an order unit, and obviously R is unperforated.

The pure traces correspond to points in the disk $(x + 3/5)^2 + (y + 3/5)^2 \leq 1$ in the positive quadrant (since x and y must also be sent to nonnegative reals). Then $\beta := 1/5 - y$ is nonnegative as a function on the pure trace space, vanishing only at the point $(0, 1/5)$, and $u = y + 7/5$ doesn't vanish at all on the pure trace space, so is an order unit of R. Now

$$\begin{aligned}\left(\frac{1}{5} - y\right)\left(y + \frac{7}{5}\right) &= \frac{7}{25} - \frac{6}{5}y - y^2 \\ &= 1 - \left(x + \frac{3}{5}\right)^2 - \left(y + \frac{3}{5}\right)^2 + x^2 + \frac{6}{5}x = \alpha + x^2 + \frac{6}{5}x.\end{aligned}$$

In particular, $u\beta \in R^+$. However, it is easy to show (now) that β itself is not in the positive cone. Any element of the positive cone is a sum of products of powers of x, y, α. The zero set (in $\mathbf{R}^2$) of these elements are respectively the y-axis, the x-axis, and the circle $\alpha = 0$. Any sum of products of the generators of the positive cone has a zero set whose intersection with the union can only be one of the following types: the empty set, the two-point sets $\{(0, 1/5), (0, -7/5)\}$ and $\{(1/5, 0), (-7/5, 0)\}$, the singleton $(0, 0)$, the zero sets of x, y, and α, and all possible unions of these. In particular, if $(0, 1/5)$ belongs to the zero set of a positive element, then so must $(0, -7/5)$. Hence $\beta = 1/5 - y$ is not positive.

The particular property which helps to explain this is the maximal order ideal $I = (\{x, \alpha\})$ (i.e., generated as an ideal). To see that this *is* an order ideal, form $r := \alpha + 6/5x + x^2$, and observe that it is in the ideal, is positive, and vanishes at the trace evaluation $(0, 1/5)$; the latter implies it is not an order unit, so the order ideal it generates, $J = \langle r \rangle$, is proper. Moreover, x and α belong to J, and since order ideals are ideals (when 1 is an order unit), so $I \subseteq J$.

On the other hand, the zero set of the ideal I has just two real points (and these are the only two complex zeroes as well), so I has codimension 2, and R/I is spanned by $\{1/5 - y + I, 7/5 + y + I\}$. Any proper order ideal must have codimension at least two (by the remark about zeros of positive elements), so that $I = J$ and J is a maximal order ideal. Similarly, the ideal (y, α) is a maximal order ideal of codimension two. The only other maximal order ideal is (x, y) which of course is a maximal ideal.

Now I is the intersection of the two maximal ideals $(x, 1/5 - y)$ and $(x, 7/5 + y)$, and this is what makes the counter-example work—multiplication by $7/5 + y$ brings the element $1/5 - y$ into the order ideal I—if $1/5 - y$ were positive, then any order ideal containing $u(1/5 - y)$ would also contain $1/5 - y$, yielding an alternative proof of non-positivity.

But we have slightly more in this example. It is easy to check that the principal ideals (x), (y), and (α) are all order ideals (and of course prime)—this just uses the fact that if a polynomial in two variables vanishes on a relatively open subset of a one-dimensional irreducible algebraic curve, then it vanishes on the whole curve. We may thus form the quotient ordered ring $Q = R/(\alpha)$ with the quotient ordering. As a ring, $Q = \mathbf{R}[\cos\theta, \sin\theta]$, and is the integral closure of $\mathbf{R}[\cos\theta]$ (read $\mathbf{R}[X]$) in the quadratic extension field $\mathbf{R}(\cos\theta)[\sin\theta]$ (read $\mathbf{R}(X)[\sqrt{1 - X^2}]$), an easy exercise. The partial ordering is obtained from the substitution $(x, y) \mapsto (\cos\theta - 3/5, \sin\theta - 3/5)$.

Being integrally closed, finitely generated, and of Krull dimension one, Q is a Dedekind domain; the maximal ideals correspond precisely to the points of the circle $\alpha = 0$, and rotation induces an isomorphism among all the maximal ideals, and again it is easy to check that the class group has order two. As a partially ordered ring, 1 is an order unit, and the trace space is just the portion of the circle in the positive quadrant. Even here, we obtain a counter-example to order unit cancellation. Let $B = \beta + (\alpha)$ be the image of $1/5 - y$ in Q; it vanishes at a surviving trace (evaluate at $(0, 1/5)$, so could not be an order unit; therefore if it were positive, it would generated a proper order ideal, call it J. If $\pi : R \to Q$ denotes the quotient map, then $\pi^{-1}J$ would be an order ideal in R, and this would contain the ideal $(\alpha, 1/5 - y)$, which is not contained in any maximal order ideal (we have listed the only three), a contradiction. The equation $(u+I)B \in Q^+$ holds directly, and of course $\pi(u)$ is an order unit.

So we have a counter-example in the ring Q. We can generalize this somewhat. We have two maximal ideals sitting over the order ideal $I + (\alpha)$, one of which, M_1, is evaluation at $y = 1/5$, a trace, and the other, M_2 is evaluation at $y = -7/5$, and we also have $M_1 M_2 \subseteq I + (\alpha)$. Let X be the pure trace space; then there exists r in M_2 such that $r(t) \neq 0$ from X being a compact subset of the maximal ideal space (identify the pure traces with their kernels) missing M_2. Then $\pm r$ is necessarily an order unit (since the pure trace space in this case is connected, the sign of $r(t)$ is constant—can fix this even if pure trace space is not connected, just uses density of the polynomial ring in continuous function space: if $R \subset C(X)$ separating points and p is a point of X and Y is a compact subset of X missing p, then there exists r in R such that $r(p) = 0$ and $r|Y$ never zero—take sum of squares). Then r is an order unit, and maps M_1 into I.

By adjusting centres and radii of the circle, we can make the arc approximate as closely as we like the line segment joining $(0, 1)$ to $(1, 0)$, and in all these cases order unit cancellation fails. In some sense, these are perturbations of $\mathbf{R}[K]$ where K is the standard triangle, which of course does satisfy order unit cancellation.

Other positivity results

Some of these might eventually be useful.

PROPOSITION 6 Suppose that R is a partially ordered ring such that every order ideal which is meet-irreducible as an order ideal is primary as an ideal (in particular, if it is meet-irreducible as an ideal and the ring is noetherian). Then R satisfies order unit cancellation.

Proof. Suppose u is an order unit of R, a is an element of R, and $ua \in R^+$. Form I, the order ideal generated by the positive element ua. If $a \in I$, we are done, by Lemma 2. Otherwise, assume a does not belong to I. An easy Zorn's lemma argument yields a proper order ideal, J, maximal with respect to containing I but not a. Any larger order ideal would contain a, so J is meet-irreducible as an order ideal. By hypothesis, J is primary as an ideal. Since all powers of u are order units and order units cannot belong to a proper order ideal, it follows that $u^n \notin J$ for all n. Since $ua \in J$ and J is primary, it follows that $a \in J$, a contradiction. •

Let $C(R) = \left\{b \in R^+ \;\middle|\; \exists r \in R^+ \setminus \{0\}, \exists M \in \mathbf{N}, \exists \text{ positive invertible } u \text{ such that } r \geq b^M u\right\}$. For example, if R^+ is finitely generated (meaning there is a finite subset of R^+ such that every element of R^+ can be expressed as positive real linear combination of products of elements of the set), then $C(R)$ is nonempty (take the product of the generators). We can weaken the definition of finite generation to replace the positive real combination by positive combination with coefficients order units in place of reals. This includes all the ordered rings of the form R_P studied in [H87a], e.g., if $P = 1 + x + y$, set $b = xy/P^3$. The rings obtained from power series in [H96], [H03] (also denoted R_P) do not have finitely generated positive cones, but $C(R_P)$ are not empty here as well—if $P(0) \neq 0$, take $b = P^{-1}$.

Finally, for s in R, let $S(s) = \{r \in R \,|\, rs \geq 0\}$; then $sS(s)$ is the set of products rs such that $rs \geq 0$, and in particular is a subset of R^+.

The following is a partial improvement on [H03, Lemma 4.2(b)], which in turn was based on [H95, Corollary 1.3]. For R a partially ordered ring (with 1 as order unit), the pure trace space (consisting of the positive ring homomorphisms $R \to \mathbf{R}$ and equipped with the point-open topology) is denoted $X(R)$, and is a compact Hausdorff space.

PROPOSITION 7 Suppose $X(R)$ is connected and s is an element of R with the following properties
(a) $\{x \in X \mid x(s) = 0\}$ is nowhere dense in X
(b) $(< sS(s) >, S(s)) = R$.
Then one of $\pm s$ belongs to R^+.

COROLLARY 8 Suppose $X(R)$ is connected, and s is a nonzero-divisor of R whose zero set (as a function on X) is nowhere dense. Suppose in addition, $\mathbf{Z}[1/p] \subset R \cdot 1$ for some positive integer $p > 1$. If $(< C(R), sS(s) >, S(s)) = R$, then one of $\pm s$ belongs to R^+.

Proof of Corollary 8 (from Proposition 7). Suppose $rs \geq 0$; as $rs \neq 0$, for any b in $C(R)$, there exists a positive integer M and a positive invertible u in R such that $rs \geq b^M u$.

There exists an integer n such that $s \geq -n$. Now consider for any multiple of the identity $\epsilon \cdot 1$,

$$(r + \epsilon b^M u)s = b^M u + \epsilon b^M us \geq b^M u(1 - n\epsilon).$$

Select $\epsilon < 1/n$ (all that is required is that the image of the invertible constants in R be dense in the reals). Thus $r + \epsilon b^M u$ belongs to $S(s)$, so that b^M belongs to $(S(s))$. Hence for each b in $C(R)$, there exists M (depending on b) such that $b^M \in (< sS(s) >, S(s))$.

Since $(< C(R) >, S(s)) = R$, there exist $B_1, \ldots, B_N$ in $< C(R) >$ and r_i in $S(s)$ together with t_i in R such that $1 = \sum B_i + \sum r_i t_i$ (an order ideal in an ordered ring with 1 as order unit is automatically an ideal). Write each $B_i = d_{i1} - d_{i2}$ with $d_{ij} \geq 0$, so there exists K such that $d_{ij} \leq K \sum b_l$ for some finite collection of b_l in $C(R)$. Hence there exist (large) L such that $d_{ij}^L \in (< sS(s) >, S(s))$, and even larger J such that B_i^J also belongs, and it easily follows that $(< sS(s) >, S(s))$ is improper, so the preceding result applies. •

Proof of Proposition 8. By [H03, 4.2(B)], it suffices to show $S(s)$ generates the improper ideal. As in the argument in the proof of the corollary, if $r \in S(s)$, then $(r + \epsilon rs)s \geq rs(1 - M\epsilon)$, so if ϵ is chosen sufficiently small, $r + \epsilon rs$ belongs to $S(s)$, and thus $rs \in (S(s))$. If $0 \leq c_{t1}, c_{t2} \leq K \sum r_i s$ for a finite selection of r_i in $S(s)$ and c_{tj} in the positive cone of $< sS(s) >$ and $1 = \sum(c_{t1} - c_{t2}) + \sum r_i s w_i$, then each $r_i s$ belongs to $(S(s))$. Moreover, since $S(s)$ is closed under sums, the term $\sum r_i s$ belongs to $(S(s))$, and now the preceding argument works for each c_{tj} (with appropriately small ϵ). Hence the c_{tj} belong to $(S(s))$, so the latter is improper. •

Unfortunately, this is not enough to get the order unit cancellation results of [H87A, section 2]. With $P = 1 + x + y$ and $b = xy/P^3$ (or anything similar; it must vanish on the boundary of the Newton polytope). For example, $Y + 3 = y/P + 3$ is an order unit and $(b, Y + 3)$ is proper (both terms vanish when $Y = -3$ and $x = 0$) (in this case, $< C(R) >= bR$).

Sums of order ideals
Consider the condition on a partially ordered ring R that (finite) sums of order ideals are order ideals This is a consequence of (Reisz) interpolation. Rather remarkably, for the ordered rings $\mathbf{R}[K]$, it actually implies interpolation, and therefore that up to affine equivalence, K is a product of simplices.

The intersection between the two class of ordered rings, the collection of $\mathbf{R}[K]$ on the one hand, and of R_P (with P a polynomial in several variables with only nonnegative coefficients) is very tiny. The ordered rings in the latter class satisfy interpolation (in fact, they are dimension groups, and their direct limit structure follows from their definition). It was shown in [H88] that if $\mathbf{R}[K]$ satisfies interpolation, then K is affinely equivalent to a product of simplices, and in particular,

$\mathbf{R}[K]$ is of the form R_P for very special choices of P (and as a result, positivity of a given element can be determined explicitly by the main result of [H86]).

An examination of the argument shows that the hypotheses can be reduced considerably. In fact, it requires hardly any additional effort to obtain the following.

THEOREM 9 Let K be a compact convex polyhedron with interior in $\mathbf{R}^n$. The following are equivalent.

(a) $\mathbf{R}[K]$ satisfies the Riesz interpolation property
(b) sums of order ideals in $\mathbf{R}[K]$ are order ideals
(c) any ideal generated by a subset of $\{\beta_i\}$ (linear forms associated to the facets) is an order ideal
(d) K is $\mathrm{AGL}(n, \mathbf{R})$-equivalent to a product of simplices
(e) there exist $n(i)$-dimensional simplices K_i with $\sum \mathrm{dim} K_i = n$ such that $\mathbf{R}[K]$ is order isomorphic to $\otimes_{\mathbf{R}} \mathbf{R}[K_i]$ (tensor product ordering).

Remark In part (e), if K_i is $n(i)$ dimensional, then $\mathbf{R}[K_i]$ is order isomorphic (i.e., as ordered rings) to R_{P_i} with $P_i = 1 + \sum_{j=1}^{n(i)} X_{ji}$, via the map $\beta_j \mapsto X_{ji}/P$ and β_0 (corresponding to the facet that misses the origin) is sent to $1/P_i$ (see the discussion at the end of [H88]), and is $R = \mathbf{R}[\prod K_i]$, then $R \cong R_P$ where $P = \prod P_i$.

Proof. By [H88, Lemma II.1(b)], any principal ideal(β^w) (a product of the irredundant linear forms β_i) is an order ideal. Thus (b) implies (c), and now only (c) implies (d) requires proof.

We adapt the arguments of [H88, II.2 and II.3], in order to verify the criteria of II.5 therein. First, if $\{F_\alpha\}$ is a family of facets with $\cap F_\alpha = \emptyset$, then we show the intersection of the corresponding affine spans, $\tilde{F}_\alpha$, is also empty. Let β_α be the corresponding (irredundant) linear forms exposing F_α (and for which $\beta_\alpha | K \geq 0$). Then $\beta := \sum \beta_\alpha$ vanishes nowhere on K so is an order unit. Therefore the order ideal generated by $\{\beta_\alpha\}$ is improper. Since sums of order ideals are order ideals and each $\beta_\alpha R$ is itself an order ideal (a consequence of [H88,II.1(b)]), it follows that $\{\beta_\alpha\}$ generates the improper ideal (as an ideal, not just as an order ideal). In particular, β_α have no common zero in all of $\mathbf{R}^n$ (where $K \subset \mathbf{R}^n$). This implies $\cap \tilde{F}_\alpha$ is empty.

If in the preceding, we replace facets by faces, we observe that every face is an intersection of facets, so we obtain, if $\{F_\alpha\}$ is a collection of faces with empty intersection, then $\cap \tilde{F}_\alpha$ is also empty. This verifies condition (ai) of [H88, II.5].

Now suppose that there is a vertex lying in more than n facets, $F_1, F_2, \ldots, F_{n+1}$ abutting the vertex v; let β_i be the corresponding linear forms. Without loss of generality, we may assume that v is origin, so that β_i have zero constant terms, and are thus linear. Since the intersection of any n of the facets must consist only of the vertex, we find that any n-element subset of $\{\beta_i\}_{i=1}^{n+1}$ is linearly independent, and therefore spans $\mathbf{R}^n$. Hence we may find reals a_i such that $\beta_{n+1} = \sum_{1 \leq i \leq n} a_i \beta_i$. All of the a_i must be nonzero (else linear independence of any subset of size n or less is violated). If all the a_i are positive, then $\beta_{n+1}^{-1} 0 \cap K$ can only be a singleton, rather than a facet, a contradiction; if all the a_i are negative, β_{n+1} is negative on K, again a contradiction.

Hence $\{1, 2, \ldots, n\}$ decomposes as $S \cup T$, where $S = \{i \mid a_i > 0\}$ and $T = \{i \mid a_i < 0\}$, and both S and T are nonempty. We thus have $\sum_T |a_i| \beta_i \leq \sum_S a_i \beta_i$. From [H88] again, each $\beta_i R$ is an order ideal, and the hypothesis asserts that $\sum_S \beta_i R$ is an order ideal. Thus (as $|a_i| > 0$ for i in T), $\{\beta_i\}_T \subset \sum_S \beta_i R$; in particular, the ideal of R generated by $\{\beta_i\}_{1 \leq i \leq n}$ is generated by $\{\beta_i\}_S$, which has fewer than n elements. This is impossible—$M = (\beta_i)$ is a maximal ideal, and since M/M^2 is n-dimensional, M cannot be generated by fewer than n elements. •

In a dimension group, finite intersections of order ideals are order ideals. I do not know whether this holds for all $\mathbf{R}[K]$, or whether it imposes constraints on K. The rings of the form R_P also have the property that (finite) products of order ideals are order ideals (not all ordered rings that are also dimension groups satisfy this property). Again, it would be worth investigating what $\mathbf{R}[K]$ having this property says about K.

It also follows from the argument of (c) implies (d), that $U^{-1}\mathbf{R}[K]$ satisfies interpolation (or its weaker forms) implies K is simplicial (exactly n facets hitting each vertex). Whether the converse holds is unknown.

References

[EHS] Edward G Effros, David E Handelman, and Chao-Liang Shen, *Dimension Groups and Their Affine Representations,* American Journal of Mathematics 102 (1980) 385–407.

[GH] K Goodearl and D Handelman, *Rank functions and K_0 of regular rings,* J Pure Appl. Alg 7 (1976) 195–216.

[H81] David Handelman, *Rings with involution as partially ordered abelian groups,* Rocky Mountain J Math, Volume 11 (1981) 337–382.

[H85] David Handelman, *Positive polynomials and product type actions of compact groups,* Mem Amer Math Soc, 320 (1985) 79 p+ xi.

[H86] David Handelman, *Deciding eventual positivity of polynomials,* Ergodic Theory and Dynamical Systems (1986) 57–79.

[H87] David Handelman, *Extending traces on fixed point C^* algebras under xerox product type actions of compact Lie groups,* Journal of functional analysis 72 (1987) 44–57.

[H87A] David Handelman, *Positive polynomials, convex integral polytopes and a random walk problem,* (1987), Springer-Verlag 1282, 142p.

[H88] David Handelman, *Representing polynomials by positive linear functions on compact convex polyhedra,* Pacific J Math Volume 132 (1988) 35–62.

[H90] David Handelman, *Effectiveness of an affine invariant for indecomposable integral polytopes,* Int J Math 4 (1993) 59–88.

[H93] David Handelman, *Representation rings as invariants for compact groups and ratio limit theorems for them,* J Pure Appl Algebra 66 (1990) 165–184.

[H95] David Handelman, *Iterated multiplication of characters of compact connected Lie groups,* J Algebra 173 (1995) 67–96.

[H96] David Handelman, *Eventual positivity for analytic functions,* Math Ann, 304 (1996) 315–338.

[H03] David Handelman, *More eventual positivity for analytic functions*, Canad J Math 55 (2003) 1019–1079

[H09] David Handelman, *Matrices of positive polynomials*, Electron. J Linear Algebra 19 (2009) 2–89.

[O'B] Ellen O'Brien, *A saturation property for powers of irreducible characters,* Crelle's J 456 (1994) 151–171.

Mathematics Department, University of Ottawa, Ottawa ON K1N 6N5, Canada; dehsg@uottawa.ca