Müntz-type theorems on the half-line with weights

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Abstract

We consider the linear span S of the functions t^{a_k} (with some $a_k > 0$) in weighted L^2 spaces, with rather general weights. We give one necessary and one sufficient condition for S to be dense. Some comparisons are also made between the new results and those that can be deduced from older ones in the literature.

1 Introduction

The first "if and only if" solution to a problem of S. N. Bernstein [4] was given by Ch. H. Müntz [21]:

Theorem A
Let $0 = \lambda_0 < \lambda_1 <, \dots$ be an increasing sequence of real numbers. The linear subspace span$\{t^{\lambda_k} : k = 0, 1, \dots\}$ is dense in $C([0,1])$, if and only if $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.

This classical result was first proved in $L_2[0,1]$ and then extended to $C[0,1]$, as stated above. Also, it was stated only for increasing sequences λ_k. Subsequently, this theorem has had several different proofs and generalizations, and there are several surveys in this topic (see for instance the papers of J. Almira and A. Pinkus [1], [23]).

On $C[0,1]$ and $L_p(0,1)$, "full Müntz theorems", i.e. theorems with rather general exponents, were later proved by eg. P. Borwein, T. Erdélyi, W. B. Johnson and V. Operstein ([7], [13], [12], [22]). Versions of Müntz's theorem on

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compact subsets of positive measure [8], [9], and on countable compact sets [2] were also proved. Further results can be found for instance in the monographs of P. Borwein, T. Erdélyi [10], and B. N. Khabibullin [16].

In this paper we are interested in Müntz-type theorems on $(0,\infty)$. Several papers were written in the '40s on the completeness of the set $\{t^{\lambda_k}e^{-t}\}$ in $L_2(0,\infty)$ (see eg. [14], [5], [6]). In particular, we will use some ideas of W. Fuchs. His theorem is the following:

Theorem B

Let a_k be positive numbers, such that $a_{k+1} - a_k > c > 0$ $(k = 1, 2, \ldots)$, and let $\log\Psi(r) = 2\sum_{a_k<r}\frac{1}{a_k}$, if $r > a_1$, and $\log\Psi(r) = \frac{2}{a_1}$ if $r \le a_1$. Then $\{e^{-t}t^{a_k}\}$ is complete in $L_2(0,\infty)$, if and only if

$$\int_1^\infty \frac{\Psi(r)}{r^2} = \infty$$

A. F. Leontev [18] and G. V. Badalyan [3] proved similar theorems with more general weights (the weight being e^{-t} in the above theorem). In 1980, by the Hahn-Banach theorem technique, R. A. Zalik [27] proved a Müntz type theorem on the half-line with weights $|w| \le c\exp(-|\log t|^a)$ $(a > 0)$. In 1996 Kroó and Szabados [17] also had a related result on $(0,\infty)$.

In Theorem 1 and Theorem 2 below we will prove Müntz-type theorems on the half-line with more general weights, which generalize all the results mentioned above.

Closely related to our topic (by a $\log t$ substitution) are the results on the whole real line for exponential systems. The basic paper in this respect was written by P. Malliavin [20], and by this tool there are some nice generalizations of the above mentioned results, for instance by B. V. Vinnitskii, A. V. Shapovalovskii [25], by G. T. Deng [11], and by E. Zikkos [28].

2 Definitions, Results

Let us begin with a rather general definition. Some specific examples are given subsequently.

Definition 1 *We say that a weight function $w(t) = \nu(t)\mu(t)$ is admissible on $[0,\infty)$, if $\nu(t)$ and $\mu(t)$ are positive and continuous on $(0,\infty)$, w^2 has finite moments, and there is a function γ on $[0,\infty)$, such that*

$$\gamma(t) = \sum_{k=0}^{\infty} c_k t^{\gamma_k},$$

where $c_k > 0$ for all k, and $0 \le \gamma_0 < \gamma_1 < \gamma_2 < \ldots$, and there is a $C_0 > 0$ such that $\forall t > C_0$

$$\frac{1}{w^2(t)} \le \gamma(t) \tag{1}$$

and there is a $C > 1$, such that

$$\int_0^\infty \gamma\left(\frac{t}{C}\right) w^2(t)dt < \infty. \tag{2}$$

Furthermore we require that

$$\lim_{t\to 0+} \mu(t) \in (0, \infty), \tag{3}$$

and there is an $a > 0$ such that

$$\int_0^1 \left(\frac{t^{a-1}}{\nu(t)}\right)^2 < \infty. \tag{4}$$

Here and in the followings C, C_i and c are absolute constants, and the value of them will not be the same at each occurrence.

Remark:
If $\nu(t) \equiv 1$ (as in Theorem B) then we can choose $a = 1$. Also, it is easy to see that we can always assume $a \geq 1$.

Examples :
$w(t) = t^\beta e^{-Dt^\alpha}$, where $\beta > -\frac{1}{2}$ and $\alpha > 0$ is admissible, namely it has finite moments, and $\gamma(t) = e^{3Dt^\alpha}$ serves the purpose. When $\beta = 0$ and $D = \alpha = 1$, we get back the original case of Fuchs (Theorem B). When $\beta > -\frac{1}{2}$, $D = \frac{1}{2}$ and $\alpha = 1$, then w^2 is a Laguerre weight. When $\beta = 0$, $D > 0$ and $\alpha > 1$, then w is a Freud weight.

Let $w(t) = (4 + \sin t)t^\beta \prod_{k=1}^n e^{-D_k t^{\alpha_k}}$, and let us assume, that $\beta > -\frac{1}{2}$, and $0 \leq \alpha_1 < \alpha_2 \ldots < \alpha_n$, and $D_n > 0$. Then w is admissible, and $e^{Dt^{\alpha_n}}$ is a suitable choice for $\gamma(t)$, if D is large enough. In particular, if $w(t) = t(4 + \sin t)e^{-t}$ then the second derivative of $-\log(w(e^t))$ takes some negative values on (A, ∞) for any $A > 0$. This means that the results of [28] are not applicable in this case.

Definition 2 *Let w be a positive continuous weight function with w^2 having finite moments. Then define $\varphi(x)$ and $K(x)$ corresponding to $w(x)$ as*

$$\varphi(x) = \left(\int_0^\infty t^{2x} w^2(t)dt\right)^{\frac{1}{2x}} = (K(x))^{\frac{1}{2x}}, \quad x > 0. \tag{5}$$

Furthermore let us define another property of a weight function. The classical weight functions, and also our examples above, fulfil this "normality" condition, as we can see later.

Definition 3 *Let us call a weight function w^2 with finite moments "normal", if the largest zero of the n^{th} orthogonal polynomial $(x_{1,n})$ with respect to w^2, can be estimated as:*

$$x_{1,n} \leq e^{cn},$$

where $c = c(w)$ is a positive constant independent of n.

Remark:

In the cases of Laguerre and Freud weights $x_{1,n} \leq cn^{\lambda}$, where $\lambda = \lambda(w)$ is a positive constant depending on the weight function, moreover the same estimation is valid for a more general classes of weights on the real axis ([19] p. 313. Th. 11.1). As an application of the result of A. Markov ([24] p. 115. Th. 6.12.2), we can get a similar estimation for the examples above; for instance $w(x) = x^{\gamma} e^{-x^{\alpha}}$, there is a $\beta > 0$ such that with $W(x) = x^{\beta} e^{-x}$, the quotient $\frac{W}{w}$ is increasing on $(0, \infty)$; if $w(x) = x(4 + \sin x)e^{-x}$, then the corresponding W can be $W(x) = x^2 e^{-\frac{x}{2}}$.

Definition 4 *Let $\{a_k\}_{k=1}^{\infty}$ be positive numbers in increasing order. We define (as in [14] and Theorem B above)*

$$m(r) = \begin{cases} \frac{1}{a_1}, & \text{if } 0 \leq r \leq a_1 \\ \sum_{a_k < r} \frac{1}{a_k}, & \text{if } r > a_1 \end{cases} \tag{6}$$

and let

$$\Psi(r) = e^{2m(r)}. \tag{7}$$

Let us also introduce the following notations:

Notation:

Let w be a positive continuous weight function, and let us define the weighted L^2 space as $L_w^2(0, \infty) = \{f | fw \in L^2(0, \infty)\}$, and $\|f\|_{2,w} = \|fw\|_{2,(0,\infty)}$.

$$S = \text{span}\{t^{a_k} : k = 1, 2, \ldots\} \tag{8}$$

with $0 < a_1 < a_2 < \ldots$.

We are now in position to state the main results of this note (the proofs will follow in the next Section).

Theorem 1 *Let w be an admissible and normal weight function on $[0, \infty)$. If there exists a monotone increasing function f on $[0, \infty)$, such that for all $0 < x \leq r$*

$$x \log \frac{\Psi(r)}{\varphi(x)} \leq f(r), \tag{9}$$

and

$$\int_1^{\infty} \frac{f(r)}{r^2} < \infty, \tag{10}$$

then S is incomplete in $L_w^2(0, \infty)$.

This result is then nicely complemented by the following positive result.

Theorem 2 *Let w be positive and continuous on $(0,\infty)$, such that w^2 has finite moments. Let us suppose that there is a constant $d>0$ such that*

$$a_{k+1}-a_k>d \tag{11}$$

If there exists a monotone increasing function h on $[0,\infty)$, for which

$$C<\frac{h(r)}{h(r_1)}<D, \quad \text{for} \quad \frac{1}{2}\leq\frac{r}{r_1}\leq 2 \tag{12}$$

with some $0<C,D$, and there are $\alpha, C, c>0$, such that for all $0<x\leq r$

$$0<h(r)\leq C^{\frac{1}{x}}\frac{cx}{\varphi^{\alpha}(x)}\Psi^{\alpha}(r), \tag{13}$$

and

$$\int_1^{\infty}\frac{h(r)}{r^2}=\infty, \tag{14}$$

then S is complete in $L^2_w(0,\infty)$.

Comparing the conditions of the above theorems we conclude the following:

Corollary:
If w is admissible and normal on $(0,\infty)$, and there is a d such that $a_{k+1}-a_k>d>0$, and $f(r)=ch(r)$, where h has the same properties as in Theorem 2, then S is dense in $L^2_w(0,\infty)$ if and only if $\int_1^{\infty}\frac{h(r)}{r^2}=\infty$.

Remark:

(1) Let

$$B_{\alpha}(r)=\inf_{x\in(0,r)}C^{\frac{1}{x}}\frac{x}{\varphi^{\alpha}(x)}.$$

Then assuming (11) and (12), if there exists a $0\leq h(r)\leq cB_{\alpha}(r)\Psi^{\alpha}(r)$, for which (14) is valid, then S is dense in $L^2_w(0,\infty)$.

(2) Theorem 2 can be stated also in $L^p_w(0,\infty)$, with $1\leq p<\infty$, and in $C_{w,(0,\infty)}$ with the same proof. That is, let us define

$$\varphi_p(x)=\left(\int_0^{\infty}t^{px}w^p(t)dt\right)^{\frac{1}{px}}, \quad x>0, \quad 1\leq p<\infty$$

and

$$\varphi_c(x)=\left(\sup_{t>0}t^x w(t)dt\right)^{\frac{1}{x}}, \quad x>0.$$

Using the standard the notations $L^p_w(0,\infty)=\{f:\|fw\|_{p,(0,\infty)}<\infty\}$, and $C_{w,(0,\infty)}=\{f\in C(0,\infty):\lim_{\substack{t\to 0+\\ t\to\infty}}f(t)w(t)=0\}$, we can formulate the following theorem:

Theorem 3 *Let w be positive and continuous on $(0,\infty)$, and let us assume that $t^x w(t) \in L^p_{(0,\infty)}$ in the L^p_w-case, and that for all $a > 0$ $\lim_{\substack{t\to 0+\\ t\to\infty}} t^a w(t) = 0$ in the C_w-case. Furthermore let $\{a_k\}$ be a sequence of positive numbers for which (11) is satisfied. If there is a monotone increasing function h on $(0,\infty)$ with the properties (12) and (14), and for which there are $\alpha, C, c > 0$, such that for all $0 < x \le r$*

$$0 < h(r) \le C^{\frac{1}{x}} \frac{cx}{\varphi^\alpha_{p/c}(x)} \Psi^\alpha(r),$$

then S is complete in $L^p_w(0,\infty)$/in $C_{w,(0,\infty)}$.

(3) If $B_\alpha(r) > B > 0$ $(\forall r \ge 1)$, then $h(r) = B\Psi^\alpha(r)$. This is the situation when $w(t) = e^{-Dt^\alpha}$. Furthermore with suitable D and α $\inf_{x\in(0,r)} \frac{x}{\varphi^\alpha(x)} > B > 0$. In this case

$$K(x) = \int_0^\infty t^{2x} e^{-2Dt^\alpha} dt = \frac{1}{\alpha (2D)^{\frac{2x+1}{\alpha}}} \Gamma\left(\frac{2x+1}{\alpha}\right)$$

By Stirling's formula (see eg. [15])

$$\frac{x}{\varphi^\alpha(x)} = \frac{x}{\left(\frac{\sqrt{2\pi}\left(\frac{2x+1}{\alpha}\right)^{\frac{2x+1}{\alpha}-\frac{1}{2}} e^{-\frac{2x+1}{\alpha}} e^{J\left(\frac{2x+1}{\alpha}\right)}}{\alpha(2D)^{\frac{2x+1}{\alpha}}}\right)^{\frac{\alpha}{2x}}} = (*),$$

where J is the Binet function. For $x > 0$ we have $0 < J(x) < \frac{1}{12x}$. That is,

$$(*) \ge \frac{2De\alpha x}{2x+1} \left(\frac{2De\alpha}{2x+1} \left(\frac{\alpha(2x+1)}{2\pi}\right)^{\frac{\alpha}{2}} \frac{1}{e^{\frac{\alpha^2}{12(2x+1)}}} \right)^{\frac{1}{2x}} = b(D,\alpha,x)$$

and $b(D,\alpha,x)$ tends to $De\alpha$ when x tends to infinity, and if $C(\alpha,D) = \frac{\sqrt{2De\alpha}}{e^{\frac{\alpha^2}{12}} \left(\frac{2\pi}{\alpha}\right)^{\frac{\alpha}{4}}} > 1$ then $\lim_{x\to 0+} \frac{x}{\varphi^\alpha(x)} = \infty$. (In the case of Fuchs, $\alpha = D = 1$, $C(\alpha, D) > 1$.)

(4) For $\alpha = D = 1$, $h(r) = f(r) = \Psi(r), r \ge 0$. By the substitution $t = Du^\alpha$ (without any further restrictions on the exponents a_k for $\alpha \ge 1$, and with the restriction $a_k \ne \frac{1}{2}(\frac{1}{\alpha} - 1)$ for $0 < \alpha < 1$), after some obvious estimations one can deduce from the result of Fuchs (Theorem B), that $\{t^{a_k} e^{-Dt^\alpha}\}$ is dense if and only if $\int_1^\infty \frac{\Psi^\alpha(r)}{r^2} = \infty$. We get the same from Theorems 1 and 2. After the third remark we need to check the assumptions of Theorem 1. Now $\varphi^x(x) = \sqrt{K(x)} \le (cx)^{\frac{x}{\alpha}}$, and so

$$\left(\frac{\varphi(x)}{\Psi(r)}\right)^x \le \left(\frac{cx}{\Psi^\alpha(r)}\right)^{\frac{x}{\alpha}}.$$

Theorem 4 *With the notations of Theorem 1*

$$f(r) = C + r \max\left\{\frac{1}{2}\frac{K^{'}}{K}(r), 2m(r)\right\} - \frac{1}{2}\log K(r) \tag{15}$$

is a good choice for $f(r)$ with a suitable C. That is, if w is admissible on $[0,\infty)$, and

$$\int_1^\infty \frac{r\max\left\{\frac{1}{2}\frac{K^{'}}{K}(r), 2m(r)\right\} - \frac{1}{2}\log K(r)}{r^2} < \infty \tag{16}$$

then S is incomplete in $L^2_w(0,\infty)$.

Remark:

If $w(t) = e^{-Dt^\alpha}$, and $\frac{1}{2}\frac{K^{'}}{K}(r) > 2m(r)$ on a set H, then on H

$$f(r) = r\frac{1}{2}\frac{K^{'}}{K}(r) - \frac{1}{2}\log K(r)$$

$$= \frac{r}{\alpha}\left(\frac{\Gamma^{'}}{\Gamma}\left(\frac{2r+1}{\alpha}\right) - \log(2D)\right) - \frac{1}{2}\log\left(\frac{1}{\alpha(2D)^{\frac{2x+1}{\alpha}}}\Gamma\left(\frac{2x+1}{\alpha}\right)\right)$$

$$= \frac{r}{\alpha}\log\frac{2r+1}{\alpha} - \frac{r}{2(2r+1)} - \frac{r}{\alpha}I\left(\frac{2r+1}{\alpha}\right) - \frac{r}{\alpha}\log(2D)$$

$$-\frac{1}{2}\log\frac{\sqrt{2\pi}\left(\frac{2x+1}{\alpha}\right)^{\frac{2x+1}{\alpha}-\frac{1}{2}} e^{-\frac{2x+1}{\alpha}} e^{J\left(\frac{2x+1}{\alpha}\right)}}{\alpha(2D)^{\frac{2x+1}{\alpha}}}$$

$$= \frac{r}{\alpha} - \frac{2-\alpha}{4\alpha}\log\frac{2r+1}{\alpha} + O(1)$$

(In the last step we used that $\frac{\Gamma^{'}}{\Gamma}(z) = \log z - \frac{1}{2z} - I(z)$ where $I(z) = 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi t}-1)}dt$ (see eg [26]).) That is, if H is large then the integral in (10) is divergent.

3 Proofs

For the proof of the first theorem, at first we need a lemma:

Lemma 1 *Let $a = m$ be a positive integer. If w^2 is a continuous, positive, normal weight function on $(0,\infty)$ with finite moments, then there is a function $b(z)$ such that $\frac{1}{b(z)}$ is regular on $\Re z \geq -a$, and it fulfils the inequality on $\Re z \geq -\frac{1}{2}$:*

$$\sqrt{\frac{K(x+a)}{K(x)}} \leq |b(z)|,$$

where $z = x + iy$.

Proof:

At first let $x = n$ be also a positive integer. Then, using the Gaussian quadrature formula on the zeros of the N^{th} orthogonal polynomials ($x_{1,N} > \ldots > x_{k,N} > \ldots > x_{N,N}$) with respect to w^2, where $N = n + m + 1$, we get, that

$$\frac{K(n+m)}{K(n)} = \frac{\int_0^\infty t^{2(n+m)} w^2}{\int_0^\infty t^{2n} w^2} = \frac{\sum_{k=1}^N \lambda_{k,N} x_{k,N}^{2(n+m)}}{\sum_{k=1}^N \lambda_{k,N} x_{k,N}^{2n}} \leq x_{1,N}^{2m},$$

that is, by the condition of "normality"

$$\sqrt{\frac{K(n+m)}{K(n)}} \leq e^{cNm}.$$

Now we can consider, that $\frac{K(x+a)}{K(x)}$ is increasing on $\Re z > -\frac{1}{2}$, namely

$$\left(\frac{K(x+a)}{K(x)}\right)' = \frac{K(x+a)}{K(x)} \left(\frac{K'}{K}(x+a) - \frac{K'}{K}(x)\right),$$

which is nonnegative, because $\frac{K'}{K}$ is increasing. The last statement can be seen by the Cauchy-Schwarz inequality, that is the derivative of $\frac{K'}{K}$ is nonnegative:

$$\left(2 \int_0^\infty t^{2x} |\log t| w^2(t) dt\right)^2 \leq \int_0^\infty t^{2x} w^2(t) dt \int_0^\infty t^{2x} 4 \log^2 t w^2(t) dt.$$

So with $a = m$ and $x > 0$,

$$\sqrt{\frac{K(x+a)}{K(x)}} \leq e^{ca(a+1+\lceil x \rceil)} \leq e^{ca(a+2+x)} = C(a) \left|e^{caz}\right|.$$

Remark:

(1) If $\left(\frac{\varphi(2x)}{\varphi(x)}\right)^x$ does not grow too quickly, then one can choose $b(z) = c(a) b_1(z)$, where $b_1(z)$ is independent of a, because
$\sqrt{\frac{K(x+a)}{K(x)}} \leq K^{\frac{1}{4}}(2a) \frac{K^{\frac{1}{4}}(2x)}{K^{\frac{1}{2}}(x)} = c(a) \left(\frac{\varphi(2x)}{\varphi(x)}\right)^x$

(2) Usually we can give $b(z)$ in polynomial form, for instance if $w(t) = e^{-Dt^\alpha}$ then $\sqrt{\frac{K(x+a)}{K(x)}} = \frac{1}{(2D)^{\frac{a}{\alpha}}} \sqrt{\frac{\Gamma\left(\frac{2x+1}{\alpha} + \frac{2a}{\alpha}\right)}{\Gamma\left(\frac{2x+1}{\alpha}\right)}} \leq c(2x+1+2a)^{\frac{a}{\alpha}}$, and so $b(z) = c(z + 2a)^n$, where $n > \frac{a}{\alpha}$ is an integer.

Proof: of Theorem 1.

Let us extend $f(r)$ to $\mathbb{R}$ as $f(-r) = f(r)$. Let $a \geq 1$ be as in (4). Furthermore let a be an integer. Because $\int_1^\infty \frac{f(r)}{r^2} < \infty$, the function

$$p(z) = p(x+iy) = p(re^{i\vartheta}) = \frac{2}{\pi}(x+a) \int_{-\infty}^\infty \frac{f(t)}{(x+a)^2 + (t-y)^2} dt \qquad (17)$$

is harmonic on $\Re z > -a$. Since $f(t)$ is increasing, and $x^2 + y^2 = r^2$

$$p(z) \geq \frac{2}{\pi} f(r) \int_{|t|>r} \frac{x+a}{(x+a)^2 + (t-y)^2} dt$$

$$= f(r) \frac{2}{\pi} \left(\pi - \left(\arctan \frac{r-y}{x+a} + \arctan \frac{r+y}{x+a} \right) \right) > f(r).$$

(In the last inequality we applied the height theorem of a triangle.) Let us choose q so that $-p + iq$, and hence $g(z) = g_a(z) = e^{-p+iq}$, be regular on $\Re z > -a$. According to the assumptions of Theorem 1, for this $g(z) \not\equiv 0$ on $\Re z \geq -a$ we have that

$$|g(z)| \leq e^{-f(r)} \leq \left(\frac{\varphi(x)}{\Psi(r)} \right)^x \qquad \Re z \geq 0. \tag{18}$$

We will show that in this case S is not dense. For this let us define a regular function on the half plane $\Re z \geq 0$ by

$$H(z) = \prod_{k=1}^{\infty} \frac{a_k - z}{a_k + z} e^{\frac{2z}{a_k}} \tag{19}$$

According to a Lemma of Fuchs ([14] L.5)

$$|H(z)| \leq (C\Psi(r))^x \quad \text{on} \quad \Re z \geq 0. \tag{20}$$

Let us replace the a_k-s in the definition of $H(z)$ by $a_k + a$, and let us denote the new function by $H^*(z)$. Now, with the help of g and H^* we can define a function $G(z) = G_a(z)$ which is regular on $\Re z \geq -a$:

$$G(z) = \frac{g(z+a)H^*(z+a)}{b(z)C_1^{z+a}}, \tag{21}$$

where, according to Lemma 1, $\frac{1}{b(z)}$ is regular on $\Re z \geq -a$, and on $\Re z \geq -\frac{1}{2}$ we have

$$\sqrt{\frac{K(x+a)}{K(x)}} \leq |b(z)| \tag{22}$$

Because for an $a > 0$ $\frac{K(x+a)}{K(x)}$ is positive, and it tends to zero, when x tends to $-\frac{1}{2}$, according to Lemma 1, we can suppose that $|b(z)| > \delta > 0$ on $\Re z \in \left[-a, -\frac{1}{2}\right]$.

Now, because $|H^*(z+a)| \leq (C\Psi(r))^{x+a}$ $(x \geq -a)$ (see (20)), we have that if C_1 is large enough, than according to (22)

$$|G(z)| \leq (\varphi(x))^x \quad \text{on} \quad \Re z > -\frac{1}{2}, \tag{23}$$

and because $a > \frac{1}{2}$, on $\Re z \in \left[-a, -\frac{1}{2}\right]$:

$$|G(z)| \leq \frac{(\varphi(x+a))^{x+a}}{|b(z)|} \leq \frac{1}{\delta} \max_{x \in \left[-a, -\frac{1}{2}\right]} \sqrt{K(x+a)} = M \tag{24}$$

In the followings we will show that if there exists a function G which is not identically zero, and is regular on $\Re z \geq -a$, and fulfils the equations $G(a_k) = 0$ $(k = 1, 2, \ldots)$, and the inequalities (23) and (24) are valid, then S is not complete.

For the purpose of showing this, we need to construct a function $0 \not\equiv k(t) \in L^2_w(0,\infty)$ such that $\int_0^\infty t^{a_k} k(t) w^2(t) = 0$ for $k = 1, 2, \ldots$. We give $k(t)$ by the inversion formula for the Mellin transform of $\frac{G(z)}{(1+a+z)^2}$: on $\Re z \geq -a$ let us define the function $u(t)$ by an integral along a line parallel with the imaginary axis

$$t\nu(t)u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(z)}{(1+a+z)^2} t^{-z} dy \tag{25}$$

It can be easily seen (by taking the integral round a rectangle $x_k \pm iL$ $k = 1, 2$, where $L \to \infty$) that the integral is independent of x. Let us choose

$$k(t) = \frac{\nu(Ct)u(Ct)}{w^2(t)}, \tag{26}$$

where C is the same as in (2). Using that

$$\frac{G(z)}{(1+a+z)^2} = \int_0^\infty \nu(t)u(t)t^z dt,$$

we have that

$$\begin{aligned} \int_0^\infty t^{a_k} k(t) w^2(t) dt &= \frac{1}{C_1^{a_k+a}} \int_0^\infty v^{a_k-1} v u(v) \nu(v) dv \\ &= \frac{1}{C_1^{a_k+a}} \frac{G(a_k)}{(1+a+a_k)^2} = 0 \end{aligned} \tag{27}$$

We have to show, that $k(t) \in L^2_w(0,\infty)$.

$$\|k\|^2_{2,w} = \int_0^\infty \frac{u^2(Ct)\nu^2(Ct)}{w^2(t)} dt = \int_0^{\frac{A}{C}} (\cdot) + \int_{\frac{A}{C}}^\infty (\cdot) = I + II, \tag{28}$$

where $A = \max\{1, CC_0\}$.

According to (1), and by the positivity of the coefficients in γ,

$$\begin{aligned} II \leq \int_{\frac{A}{C}}^\infty \nu^2(Ct)u^2(Ct) \sum_{k=0}^\infty c_k t^{\gamma_k} dt &\leq \sum_{k=0}^\infty \frac{c_k}{C^{\gamma_k+1}} \int_A^\infty t^{\gamma_k} \nu^2(t) u^2(t) dt \\ &= \sum_{\substack{k \\ \gamma_k < \frac{1}{3}}} (\cdot) + \sum_{\substack{k \\ \gamma_k \geq \frac{1}{3}}} (\cdot) = S_1 + S_2 \end{aligned} \tag{29}$$

Using Parseval's formula for the Mellin transform (see e.g. [15])

$$\int_0^\infty t^{2x+1} \nu^2(t) u^2(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \left| \frac{G(z)}{(1+a+z)^2} \right|^2 dy$$

$$\leq c\,(\varphi(x))^{2x} \leq c\left(\varphi(x+\frac{1}{2})\right)^{2x+1}, \tag{30}$$

where the equality is valid on $\Re z \geq -a$, the first inequality is on $\Re z > -\frac{1}{2}$, and the last inequality is on $\Re z \geq -\frac{1}{3}$ say, where we used again that $\frac{K(x+a)}{K(x)}$ is increasing, that is

$$0 < c \leq \frac{K(\frac{1}{6})}{K(-\frac{1}{3})} \leq \frac{K(x+\frac{1}{2})}{K(x)}$$

Therefore, by (30)

$$S_2 \leq c \sum_{\substack{\gamma_k \\ k \geq \frac{1}{3}}} \frac{c_k}{C^{\gamma_k+1}} \int_0^\infty t^{\gamma_k} \nu^2(t) u^2(t) dt \leq c \sum_{k=0}^\infty \frac{c_k}{C^{\gamma_k+1}} \left(\varphi\left(\frac{\gamma_k}{2}\right)\right)^{\gamma_k}$$

$$\leq c \sum_{k=0}^\infty \frac{c_k}{C^{\gamma_k+1}} \int_0^\infty t^{\gamma_k} w^2(t) \leq c \sum_{k=0}^\infty c_k \int_0^\infty \left(\frac{t}{C}\right)^{\gamma_k} w^2(t)$$

$$= c \int_0^\infty \gamma\left(\frac{t}{C}\right) w^2(t) < \infty \tag{31}$$

To estimate S_1 and I, we will use that by (25), with $x = -\frac{1}{3}$

$$\nu^2(t) u^2(t) \leq c t^{-\frac{4}{3}} \varphi\left(-\frac{1}{3}\right)^{-\frac{2}{3}} \int_{-\infty}^\infty \frac{1}{\left|\left(\frac{2}{3}+a+iy\right)^2\right|^2} dy = c t^{-\frac{4}{3}} \tag{32}$$

That is

$$t^{\gamma_k} \nu^2(t) u^2(t) \leq c t^{\beta_k}, \quad \text{where} \quad \beta_k < -1,$$

and therefore S_1 is bounded. Similarly, if instead of $x = -\frac{1}{3}$ we use $x = -a$ in (32), we obtain by (25) that $\nu^2(t) u^2(t) \leq c M^2 t^{2a-2}$, and so by (4)

$$I = \int_0^{\frac{A}{C}} \frac{u^2(Ct)\nu^2(Ct)}{\nu^2(t)\mu^2(t)} dt \leq c \int_0^{\frac{A}{C}} \frac{t^{2(a-1)}}{\nu^2(t)} < \infty \tag{33}$$

This proves Theorem 1.

We now turn to the proof of Theorem 2. We will need a technical lemma. Following carefully the proof of Lemma 7 – Lemma 11 in [14], actually W. Fuchs proved the following:

Lemma 2 *[14] If there is a nonnegative, monotone increasing function h on $(0, \infty)$, which fulfils (12), and*

$$\int_1^\infty \frac{h(r)}{r^2} = \infty, \tag{34}$$

and if there is a function g regular on $\Re z \geq 0$ such that there are $C, c > 0$, $\alpha > 0$

$$|g(z)| \leq C \left(\frac{cx}{h(r)} \right)^{\frac{x}{\alpha}}, \tag{35}$$

then

$$g \equiv 0 \quad on \quad \Re z \geq 0 \tag{36}$$

Remark:

In Lemma 2 C and c means that instead of a regular function g another regular function: $bA^z g(z)$ can be considered (A, b are positive constants). It means that $\Psi(r)$ can be replaced by a function $\Psi_1(r)$ such that $\frac{\Psi}{\Psi_1}$ lies between finite positive bounds, and $\Psi_1(r)$ has a continuous derivative. Therefore in the followings we will assume that $\Psi(r)$, that is $m(r)$, is continuously differentiable, if it is necessary. Furthermore since $m(r)$ is increasing, we will assume that the derivative of m is nonnegative. If it is necessary, we can assume the same on h.

Proof: of Theorem 2.

From (13), and the previous lemma it follows that if a function $g(z)$ is regular on $\Re z \geq 0$, and it satisfies the inequality

$$|g(z)| \leq \left(c \frac{\varphi(x)}{\Psi(r)} \right)^{x}, \tag{37}$$

then $g \equiv 0$. Namely, if $r \geq x > 0$, then (13) and (37) together gives (35), and by the definition of φ and Ψ, $\lim_{x \to 0+} \left(c \frac{\varphi(x)}{\Psi(r)} \right)^{x} = \|w\|_{2,(0,\infty)}$, so we can choose a constant C, such that $\left(\frac{\varphi(x)}{\Psi(r)} \right)^{x} \leq C \left(\frac{cx}{h(r)} \right)^{\frac{x}{\alpha}}$ on $\Re z \geq 0$.

Now let us assume, by contradiction, that S is not dense in L^2_w. In this case there is a function $f \not\equiv 0$ in L^2_w, such that the function

$$G(z) = \int_0^\infty t^z f(t) w^2(t) dt \tag{38}$$

defined on $\Re z \geq 0$, satisfies the equalities

$$G(a_k) = 0 \quad k = 1, 2, \ldots \tag{39}$$

and we can estimate its modulus by

$$|G(z)| \leq \|f\|_{2,w} \left(\varphi(x) \right)^{x} \tag{40}$$

Let us define now on $\Re z \geq 0$

$$g(z) = \frac{G(z)}{H(z) C_1^{z+1}}, \tag{41}$$

where H is as in (19). By Lemma 4 [14]

$$|H(z)| \geq \left(C_2 \Psi(r) \right)^{x} \tag{42}$$

on $\mathbb{C}\setminus\cup_{k=1}^{\infty}B\left(a_k,\frac{d}{3}\right)$, where $B\left(a_k,\frac{d}{3}\right)$ are balls around a_k with radius depending on d (see (11)), and on the imaginary axis without exception. This implies that

$$|g(z)|\leq\left(c\frac{\varphi(x)}{\Psi(r)}\right)^x$$

on $\Re z\geq 0\setminus\cup_{k=1}^{\infty}B\left(a_k,\frac{d}{3}\right)$ (and on the imaginary axis). But g is regular on $\Re z\geq 0$, so this inequality holds on the whole half-plane, and thus by Lemma 2 $g\equiv 0$, and hence $G\equiv 0$, a contradiction.

Proof: of Theorem 4.

Let us introduce the following notation on $0\leq x\leq r$, where $r\geq 0$ is fixed:

$$v_r(x)=2xm(r)-\frac{1}{2}\log K(x) \tag{43}$$

Because $\frac{K'}{K}$ is increasing (see the first remark after the proof of Theorem 1), $v_r(x)$ is concave on $[0,r]$. That is, we need to distinguish three cases: (a) $v_r(x)$ is strictly decreasing on $(0,r]$, (b) $v_r(x)$ has a maximum on $(0,r]$, (c) $v_r(x)$ is strictly increasing on $[0,r]$.

In case (a) the first derivative of $v_r(x)$ is negative on $[0,r]$, that is

$$2m(r)<\frac{K'}{2K}(x)\quad 0\leq x\leq r. \tag{44}$$

Furthermore $\frac{K'}{K}$ is increasing, and it means that

$$2m(r)\leq\frac{K'}{2K}(0) \tag{45}$$

Since the right-hand side is constant, and the left-hand side is increasing, there is an r_0, such that for all $r>r_0$ (45) must be wrong.

In case (b) there is an $0<x_0=x_0(r)\leq r$, where $v_r'(x_0)=0$. That is, for all $0\leq x\leq r$

$$v_r(x)\leq v_r(x_0)=\frac{x_0}{2}\left(\frac{K'}{K}(x_0)-\frac{\log K(x_0)}{x_0}\right)\leq\frac{r}{2}\left(\frac{K'}{K}(r)-\frac{\log K(r)}{r}\right) \tag{46}$$

because $\frac{x}{2}\left(\frac{K'}{K}(x)-\frac{\log K(x)}{x}\right)$ is increasing, since it's derivative is $\frac{1}{2}x\left(\frac{K'}{K}\right)'(x)$, which is nonnegative.

In case (c) $2m(r)>\frac{K'}{2K}(x)$ if $0\leq x\leq r$. That is, $2m(r)>\frac{K'}{2K}(r)$. In this case $v_r(x)\leq v_r(r)$, $\frac{r}{2}\left(\frac{K'}{K}(r)-\frac{\log K(r)}{r}\right)\leq v_r(r)$, and $v_r(r)$ itself is increasing, because using the remark after Lemma 1

$$v_r(r)'=\left(2m(r)-\frac{1}{2}\frac{K'}{K}(r)\right)+2rm'(r)\geq 0$$

That is, we can find a constant C, such that $v_r(x)\leq f(r)$ even in case (a), and f is increasing.

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