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# RADON TRANSFORM ON SPHERES AND GENERALIZED BESSEL FUNCTION ASSOCIATED WITH DIHEDRAL GROUPS

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Abstract. Motivated by Dunkl operators theory, we consider a generating series involving a modified Bessel function and a Gegenbauer polynomial, that generalizes a known series already considered by L. Gegenbauer. We actually use inversion formulas for Fourier and Radon transforms to derive a closed formula for this series when the parameter of the Gegenbauer polynomial is a strictly positive integer. As a by-product, we get a relatively simple integral representation for the generalized Bessel function associated with even dihedral groups $D_2(2p), p \geq 1$ when both multiplicities sum to an integer. In particular, we recover a previous result obtained for $D_2(4)$ and we give a special interest to $D_2(6)$. The paper is closed with adapting our method to odd dihedral groups thereby exhausting the list of Weyl dihedral groups.

## 1. Introduction

The dihedral group $D_2(n)$ of order $n \geq 2$ is defined as the group of regular $n$-gone preserving-symmetries ([8]). It figures among reflections groups associated with root systems for which a spherical harmonics theory, generalizing the one of Harish-Chandra on semisimple Lie groups from a discrete to a continuous range of multiplicities, was introduced by C. F. Dunkl in the late eightees (see Ch.I in [3]). Since then, a huge amount of research papers on this new topic and on its stochastic side as well emerged yielding fascinating results (Ch. II, III in [3]). For instance, probabilistic considerations allowed the author to derive the so-called generalized Bessel function associated with dihedral groups ([4]). For even values $n = 2p, p \geq 1$, this function depending on two real variables, say $(x, y) \in \mathbb{R}^2$, is expressed in polar coordinates $x = \rho e^{i\phi}, y = re^{i\theta}, \rho, r \geq 0, \phi, \theta \in [0, \pi/2p]$ as

$$(1) \qquad D_k^W(\rho, \phi, r, \theta) = c_{p,k}\left(\frac{2}{r\rho}\right)^{\gamma} \sum_{j\geq 0} I_{2jp+\gamma}(\rho r) p_j^{l_1,l_0}(\cos(2p\phi)) p_j^{l_1,l_0}(\cos(2p\theta))$$

where

- $k = (k_0, k_1)$ is a positive-valued multiplicity function, $l_i = k_i - 1/2, i \in \{1, 2\}$, $\gamma = p(k_0 + k_1)$.
- $I_{2jp+\gamma}, p_j^{l_1,l_0}$ are the modified Bessel function of index $2jp + \gamma$ and the $j$-th orthonormal Jacobi polynomial of parameters $l_1, l_0$ respectively (the orthogonality (Beta) measure need not to be normalized here. In fact, the normalization only alters the constant $c_{p,k}$ below).

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- The constant $c_{p,k}$ depends on $p, k$ and is such that $D_k^W(0,y) = 1$ for all $y = (r,\theta) \in [0,\infty) \times [0, \pi/2p]$ (see [5])

$$c_{p,k} = 2^{k_0+k_1} \frac{\Gamma(p(k_1+k_0)+1)\Gamma(k_1+1/2)\Gamma(k_0+1/2)}{\Gamma(k_0+k_1+1)}.$$

In a subsequent paper ([5]), the special case $p = 2$ corresponding to the group of square-preserving symmetries was considered. The main ingredient used there was the famous Dijksma-Koornwinder's product formula for Jacobi polynomials ([7]) which may be written in the following way ([5]):

$$c(\alpha,\beta)p_j^{\alpha,\beta}(\cos 2\phi)p_j^{\alpha,\beta}(\cos 2\theta) = (2j+\alpha+\beta+1)\int\int C_{2j}^{\alpha+\beta+1}(z_{\phi,\theta}(u,v))\mu^\alpha(du)\mu^\beta(dv)$$

where $\alpha, \beta > -1/2$,

$$c(\alpha,\beta) = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$

$$z_{\phi,\theta}(u,v) = u\cos\theta\cos\phi + v\sin\theta\sin\phi,$$

and $\mu^\alpha$ is the symmetric Beta probability measure whose density is given by

$$\mu^\alpha(du) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1-u^2)^{\alpha-1/2}\mathbf{1}_{[-1,1]}(u)du, \quad \alpha > -1/2.$$

Inverting the order of integration, we were in front of the following series

$$\left(\frac{2}{r\rho}\right)^\gamma \sum_{j\geq 0}(2j+k_0+k_1)I_{2jp+\gamma}(\rho r)C_{2j}^{k_0+k_1}(z_{p\phi,p\theta}(u,v)) \tag{2}$$

for $(u,v) \in ]-1,1[^2$, which specializes for $p = 2$ to

$$\frac{1}{2}\sum_{j\equiv 0[4]}(j+\gamma)I_{j+\gamma}(\rho r)C_{j/2}^{\gamma/2}(z_{2\phi,2\theta}(u,v)).$$

Using the identity noticed by Y. Xu ([13]):

$$C_j^\nu(\cos\zeta) = \int C_{2j}^{2\nu}\left(\sqrt{\frac{1+\cos\zeta}{2}}z\right)\mu^{\nu-1/2}(dz), \quad \nu > -1/2, \xi \in [0,\pi],$$

we were led to

$$\sum_{j\equiv 0[4]}(j+\gamma)I_{j+\gamma}(\rho r)C_j^\gamma(z_{2\phi,2\theta}(u,v))$$

which we wrote as

$$\frac{1}{4}\sum_{s=1}^{4}\sum_{j\geq 0}(j+\gamma)I_{j+\gamma}(\rho r)C_j^\gamma(z_{2\phi,2\theta}(u,v))e^{isj\pi/2}$$

after the use of the elementary identity

$$\frac{1}{n}\sum_{s=1}^{m}e^{2i\pi sj/m} = \begin{cases} 1 & \text{if} \quad j \equiv 0[m], \\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

valid for any integer $m \geq 1$. Accordingly (Corollary 1.2 in [5])

$$D_k^W(\rho,\phi,r,\theta) = \int\int i_{(\gamma-1)/2}\left(\rho r\sqrt{\frac{1+z_{2\phi,2\theta}(u,v)}{2}}\right)\mu^{l_1}(du)\mu^{l_0}(dv)$$

where

$$i_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{(\alpha+1)_m m!}\left(\frac{x}{2}\right)^{2m}$$

is the normalized modified Bessel function ([8]) and $\gamma = 2(k_0+k_1) \geq 2$ is even. This is a relatively simple integral representation of $D_k^W$ since the latter function may be expressed as a bivariate hypergeometric function of Bessel-type. Recall also that it follows essentially from closed formulas due to L. Gegenbauer (equations (4), (5), p.369 in [12]):

$$\left(\frac{2}{r\rho}\right)^{\gamma} \sum_{j\geq 0}(j+\gamma)I_{j+\gamma}(\rho r)C_j^{\gamma}(\cos\zeta)(\pm 1)^j = \frac{1}{\Gamma(\gamma)}e^{\pm\rho r\cos\zeta}.$$

In this paper, we shall see that a relatively simple integral representation of $D_k^W$ still exists for general integer $p \geq 2$ and integer $\nu := k_0 + k_1 \geq 1$[2]. In fact, with regard to (2), one has to derive closed formulas for both series below

$$f_{\nu,p}^{\pm}(R,\cos\zeta) := \left(\frac{2}{R}\right)^{p\nu} \sum_{j\geq 0}(j+\nu)I_{p(j+\nu)}(R)C_j^{\nu}(\cos\zeta)(\pm 1)^j \tag{4}$$

with $R = \rho r$ and $\cos\zeta := \cos\zeta(u,v) = z_{p\phi,p\theta}(u,v)$. The obtained formulas reduce to Gegenbauer's results when $p = 1$, $\nu \geq 1$ is an integer, and do not exist up to our knowledge. Moreover, our approach is somewhat geometric since we shall interpret the sequence:

$$(\pm 1)^j I_{p(j+\nu)}(R),\ j \geq 0$$

for fixed $R$ as the Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$, and since spherical functions on the sphere viewed as a homogeneous space are expressed by means of Gegenbauer polynomials ([1]). Then, following [1], solving the problem when $\nu$ is a strictly positive integer amounts to appropriately use inversion formulas for Fourier and Radon transforms. Our main result is stated as

**Proposition 1.** *Assume $\nu \geq 1$ is a strictly positive integer, then*

$$\left(\frac{R}{2}\right)^{p\nu} f_{\nu,p}^{\pm}(R,\cos\zeta) = \frac{1}{2^{\nu}(\nu-1)!}\left[-\frac{1}{\sin\zeta}\frac{d}{d\zeta}\right]^{\nu} \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(\zeta+2\pi s)/p]}.$$

A first glance at the main result may be ambiguous for the reader since the LHS depends on $\cos\zeta$ while the RHS depends on $\cos(\zeta/p), p \geq 1$. But $\cos(\zeta/p), p \geq 1$ may be expressed, though in a very complicated way (inverses of linearization formulas), as a function of $\cos\zeta$. For instance, when $p = 2$,

$$\cos(\zeta/2) = \sqrt{\frac{1+\cos\zeta}{2}}, \quad \zeta \in [0,\pi].$$

One then recovers Corollary 1.2. in [5] after using appropriate formulas for modified Bessel functions. When $p = 3$, one has to solve a special cubic equation. To proceed, we rely on results from analytic function theory and the required solution is expressed by means of Gauss hypergeometric functions ([10]) in contrast to Cardan's solution. Therefore, we get a somewhat explicit formula for the series (2), though much more complicated than the one derived for $p = 2$. The paper is closed with adapting our method to odd dihedral groups, in particular to $D_2(3)$ thereby

[2] When $p = 2$, this condition is equivalent to $\gamma$ is even as stated in [5].

exhausting the list of dihedral groups that are Weyl groups ($p = 1$ corresponds to the product group $\mathbb{Z}_2^2$).

## 2. Proof of the main result

Recall the orthogonality relation for Gegenbauer polynomials ([8]):

$$\int_0^{\pi} C_j^{\nu}(\cos\zeta)C_m^{\nu}(\cos\zeta)(\sin\zeta)^{2\nu}d\zeta = \delta_{jm}\frac{\pi\Gamma(j+2\nu)2^{1-2\nu}}{\Gamma^2(\nu)(j+\nu)j!}$$
$$= \delta_{jm}\frac{\pi 2^{1-2\nu}\Gamma(2\nu)}{(j+\nu)\Gamma^2(\nu)}C_j^{\nu}(1)$$
$$= \delta_{jm}\nu\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{\Gamma(\nu+1)}\frac{C_j^{\nu}(1)}{(j+\nu)}$$

where we used $\Gamma(\nu+1) = \nu\Gamma(\nu)$, the Gauss duplication's formula ([8])

$$\sqrt{\pi}\Gamma(2\nu) = 2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+1/2),$$

and the special value ([8])

$$C_j^{\nu}(1) = \frac{(2\nu)_j}{j!}.$$

Equivalently, if $\mu^{\nu}(d\cos\zeta)$ is the image of $\mu^{\nu}(d\zeta)$ under the map $\zeta \mapsto \cos\zeta$, then

$$(j+\nu)\int C_j^{\nu}(\cos\zeta)C_m^{\nu}(\cos\zeta)\mu^{\nu}(d\cos\zeta) = \nu C_j^{\nu}(1)\delta_{jm}$$

so that (4) yields

$$\nu(\pm 1)^j\left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R) = \int W_j^{\nu}(\cos\zeta)f_{\nu,p}^{\pm}(R,\cos\zeta)\mu^{\nu}(d\cos\zeta) \tag{5}$$

where

$$W_j^{\nu}(\cos\zeta) := C_j^{\nu}(\cos\zeta)/C_j^{\nu}(1)$$

is the $j$-th normalized Gegenbauer polynomial. Thus, the $j$-th Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$ are given by

$$\nu(\pm 1)^j\left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R), \quad p \geq 2.$$

Following [1] p.356, the Mehler's integral representation of $W_j^{\nu}$ ([9], p.177)

$$W_j^{\nu}(\cos\zeta) = 2^{\nu}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu)\sqrt{\pi}}(\sin\zeta)^{1-2\nu}\int_0^{\zeta}[\cos(j+\nu)t](\cos t - \cos\zeta)^{\nu-1}dt$$

valid for real $\nu > 0$, transforms (5) to

$$\left(\frac{2}{R}\right)^{p\nu}(\pm 1)^j I_{p(j+\nu)}(R) = \frac{2^{\nu}}{\pi}\int_0^{\pi} f_{\nu,p}^{\pm}(R,\cos\zeta)\sin\zeta\int_0^{\zeta}[\cos(j+\nu)t](\cos t - \cos\zeta)^{\nu-1}dtd\zeta$$
$$= \frac{2^{\nu}}{\pi}\int_0^{\pi}[\cos(j+\nu)t]\int_t^{\pi} f_{\nu,p}^{\pm}(R,\cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta dt. \tag{6}$$

The second integral displayed in the RHS of the second equality is known as the Radon transform of $\zeta \mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$ and inversion formulas already exist ([1]). As a matter of fact, we firstly need to express $(\pm 1)^{j+\nu}I_{p(j+\nu)}$, when $\nu \geq 1$ is an integer, as the Fourier-cosine coefficient of order $j+\nu$ of some function. This is a consequence of the Lemma below. Secondly, we shall use the appropriate inversion formula for the Radon transform.

**Lemma.** *For any integer $p \geq 1$ and any $t \in [0, \pi]$:*

$$2\sum_{j\geq 0}(\pm 1)^j I_{pj}(R)\cos(jt) = I_0(R) + \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}.$$

*Proof of the Lemma*: we will prove the $(+)$ part, the proof of the $(-)$ part follows the same lines with minor modifications. Write

$$\begin{aligned} 2\sum_{j\geq 0} I_{pj}(R)\cos(jt) &= \sum_{j\geq 0} I_{pj}(R)[e^{ijt} + e^{-ijt}] \\ &= I_0(R) + \sum_{j\in\mathbb{Z}} I_{pj}(R)e^{ijt} \end{aligned}$$

where used the fact that $I_j(r) = I_{-j}(r), j \geq 0$. Using the identity (3), one obviously gets

$$\sum_{j\in\mathbb{Z}} I_{pj}(R)e^{ijt} = \frac{1}{p}\sum_{s=1}^{p}\sum_{j\in\mathbb{Z}} I_j(R)e^{ij(t+2\pi s)/p}.$$

The $(+)$ part of the Lemma then follows from the generating series for modified Bessel functions ([12]):

$$e^{(z+1/z)R/2} = \sum_{j\in\mathbb{Z}} I_j(R)z^j,\ z \in \mathbb{C}.$$

The Lemma yields

$$I_{pj}(R) = I_0(R)\delta_{j0} + \frac{1}{\pi}\int_0^{\pi}\cos(jt)\frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}dt$$

for any integer $j \geq 0$. Assuming that $\nu$ is a stricltly positive integer, one has

$$I_{p(j+\nu)}(R) = \frac{1}{\pi}\int_0^{\pi}\cos((j+\nu)t)\frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}dt. \tag{7}$$

Note that

$$t \mapsto \int_t^{\pi} f(R,\cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta$$

as well as

$$t \mapsto \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}$$

are even functions. This is true since

$$\zeta \mapsto f(R,\cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}$$

is an odd function so that

$$\int_{-t}^{t} f(R,\cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta = 0,$$

and since

$$\cos[(-t + 2s\pi)/p] = \cos[(t + 2(p-s)\pi)/p]$$

so that one performs the index change $s \to p - s$ and notes that the terms corresponding to $s = 0$ and $s = p$ are equal. Similar arguments yield the $2\pi$-periodicity of these functions, therefore, the Fourier-cosine transforms of their restrictions on $(-\pi, \pi)$ coincide with their Fourier transforms on that interval. By injectivity of the Fourier transform and $2\pi$-periodicity,

$$\left(\frac{R}{2}\right)^{p\nu} \int_t^{\pi} f_{\nu,p}(R, \cos\zeta) \sin\zeta (\cos t - \cos\zeta)^{\nu-1} d\zeta = \frac{1}{2^\nu p} \sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}$$

for all $t$ since both functions are continuous. Finally, the Proposition follows from Theorem 3.1. p.363 in [1]. ■

**Remark.** *When $\nu = (d-1)/2$ for some integer $d \geq 1$, the Gegenbauer-Fourier transform is interpreted as the Fourier Transform on the sphere $S^{d+1}$ considered as a homogenous space $SO(d+1)/SO(d)$. More precisely, the spherical functions of this space are given by ([1] p.356):*

$$W_j^\nu(\langle z, N\rangle),\ z \in S^{d+1},$$

*where $N = (0, \cdots, 0, 1) \in S^{d+1}$ is the north pole and $\langle\cdot,\cdot\rangle$ denotes the Euclidian inner product on $\mathbb{R}^{d+1}$.*

**Corollary 1.** *For any integer $\nu \geq 1$*

$$\sum_{j\geq 0}(2j+\nu) I_{p(2j+\nu)}(R) C_{2j}^\nu(\cos\zeta) = \frac{1}{2^\nu \Gamma(\nu)} \left[-\frac{1}{\sin\zeta}\frac{d}{d\zeta}\right]^\nu \frac{1}{p}\sum_{s=1}^{p} \cosh\left(R\cos[(\zeta+2\pi s)/p]\right).$$

## 3. Weyl group settings $p = 2, 3$: explicit formulas

3.1. **p=2.** Letting $p = 2$ and using the fact that $u \mapsto \cosh u$ is an even function, our main result yields

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j\geq 0}(2j+\nu) I_{2(2j+\nu)}(R) C_{2j}^\nu(\cos\zeta) = \frac{1}{2^\nu\Gamma(\nu)}\left[-\frac{4}{R^2\sin\zeta}\frac{d}{d\zeta}\right]^\nu \cosh\left(R\cos(\cdot/2)\right)(\zeta).$$

Noting that

$$-\frac{4}{R^2\sin\zeta}\frac{d}{d\zeta}\cosh\left(R\cos(\cdot/2)\right)(\zeta) = \frac{1}{R\cos t/2}\frac{d}{dt}\left(u \mapsto \cosh u\right)_{|u=R\cos(\zeta/2)},$$

after the use of the identity $\sin\zeta = 2\sin\zeta/2\cos\zeta/2$, it follows that

$$\begin{aligned}
\left[-\frac{4}{R^2\sin\zeta}\frac{d}{d\zeta}\right]^\nu \cosh\left(R\cos(\cdot/2)\right)(\zeta) &= \left[\frac{1}{u}\frac{d}{du}\right]^\nu (u \mapsto \cosh u)_{|u=R\cos(\zeta/2)} \\
&= \left[\frac{1}{u}\frac{d}{du}\right]^{\nu-1} (u \mapsto \frac{\sinh u}{u})_{|u=R\cos(\zeta/2)} \\
&= \sqrt{\frac{\pi}{2}}\left[\frac{d}{du}\right]^{\nu-1}\left(u \mapsto \frac{I_{1/2}(u)}{\sqrt{u}}\right)_{|u=R\cos(\zeta/2)} \\
&= \sqrt{\frac{\pi}{2}}\frac{1}{u^{\nu-1/2}} I_{\nu-1/2}(u)_{|u=R\cos(\zeta/2)} \\
&= \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+1/2)} i_{\nu-1/2}(R\cos(\zeta/2))
\end{aligned}$$

where the fourth equality is a consequence of the differentiation formula (6) p.79 in [12]. With the help of Gauss duplication's formula, one easily gets:

$$\left(\frac{4}{R^2}\right)^{\nu}\sum_{j\geq 0}(2j+\nu)I_{2(2j+\nu)}(R)C_{2j}^{\nu}(\cos\zeta) = \frac{1}{2\Gamma(2\nu)}i_{\nu-1/2}(R\cos(\zeta/2))$$

and finally recovers Corollary 1.2 in [5] since $c_{2,k}/c(k_1-1/2, k_0-1/2) = \Gamma(2\nu+1)/\nu$.

3.2. **p=3.** The corresponding dihedral group $D_2(6)$ is isomorphic to the Weyl group of type $G_2$ ([2]). Let $\zeta \in ]0,\pi[$ and start with the linearization formula:

$$4\cos^3(\zeta/3) = \cos\zeta + 3\cos(\zeta/3).$$

Thus, we are led to find a root lying in $[-1,1]$ of the cubic equation

$$Z^3 - (3/4)Z - (\cos\zeta)/4 = 0$$

for $|Z| < 1$. Set $Z = (\sqrt{-1}/2)T, |T| < 2$, the above cubic equation transforms to

$$T^3 + 3T - 2\sqrt{-1}\cos\zeta = 0.$$

The obtained cubic equation already showed up in analytic function theory in relation to the local inversion Theorem ([10] p.265-266). Amazingly (compared to Cardan's formulas), its real and both complex roots are expressed through the Gauss Hypergeometric function ${}_2F_1$. Since we are looking for real $Z = (\sqrt{-1}/2)T$, we shall only consider the complex roots (see the bottom of p. 266 in [10]):

$$T^{\pm} = \pm\sqrt{-1}\left[\sqrt{3}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{3}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right]$$

so that

$$Z^{\pm} = \pm\left[\frac{\sqrt{3}}{2}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{6}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right].$$

Since for $\zeta = \pi/2$, $\cos\zeta/3 = \cos\pi/6 = \sqrt{3}/2$, it follows that

$$\cos(\zeta/3) = \left[\frac{\sqrt{3}}{2}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{6}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right]$$

for all $\zeta \in (0,\pi)$. Now, write $Z = Z(\cos\zeta)$ so that

$$\begin{aligned}\cos[(\zeta+2s\pi)/3] &= \cos(2s\pi/3)\cos(\zeta/3) - \sin(2s\pi/3)\sqrt{1-\cos^2(\zeta/3)} \\ &= \cos(2s\pi/3)Z(\cos\zeta) - \sin(2s\pi/3)\sqrt{1-Z^2(\cos\zeta)}\end{aligned}$$

for any $1 \leq s \leq 3$. It follows that

$$f_{\nu,3}(R,\cos\zeta) = \frac{1}{3\Gamma(\nu)}\left[-\frac{4}{R^3\sin\zeta}\frac{d}{d\zeta}\right]^{\nu}\sum_{s=1}^{3}g_s(RZ(\cos\zeta))$$

where

$$g_s(u) = \cosh\left[\left(\cos(2s\pi/3)u - \sin(2s\pi/3)\sqrt{R^2-u^2}\right)\right], u \in (-1,1).$$

Finally,

$$f_{\nu,3}(R,\cos\zeta) = \frac{1}{3\Gamma(\nu)}\left[\frac{4}{R^3}\frac{d}{du}\right]^{\nu}\sum_{s=1}^{3}h_s(u)_{|u=\cos\zeta}$$

where $h_s(u) := g_s(RZ(u)), 1 \leq s \leq 3$. For instance, let $\nu = 1$, then it is not difficult to see that

$$\frac{d}{du}h_s(u)_{|u=\cos\zeta} = \frac{R}{\sin\zeta/3}\frac{dZ}{du}_{|u=\cos\zeta}\sin\left(\frac{\xi+2\pi s}{3}\right)\sinh\left[\sin\left(\frac{\xi+2\pi s}{3}\right)\right]$$

for any $s \in \{1,2,3\}$ and the derivative of $u \mapsto Z(u)$ is computed using the differentiation formula for ${}_2F_1$:

$$\frac{d}{du}{}_2F_1(a,b,c;u) = \frac{ab}{c}{}_2F_1(a+1,b+1,c+1;u),\ |u|<1, c\neq 0.$$

As the reader may conclude, formulas are cumbersome compared to the ones derived for $p = 2$.

## 4. Odd Dihedral groups

Let $n \geq 3$ be an odd integer. For odd dihedral groups $D_2(n)$, the generalized Bessel function reads ([4] p.157):

$$D_k^W(\rho,\phi,r,\theta) = c_{n,k}\left(\frac{2}{r\rho}\right)^{nk}\sum_{j\geq 0} I_{n(2j+k)}(\rho r)p_j^{-1/2,l_0}(\cos(2n\phi))p_j^{-1/2,l_0}(\cos(2n\theta))$$

where $k \geq 0, \rho, r \geq 0, \theta, \phi \in [0, \pi/n]$, and

$$c_{n,k} = 2^k\Gamma(nk+1)\frac{\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k+1)}.$$

In order to adapt our method to those groups, we need to write down the product formula for orthonormal Jacobi polynomials in the limiting case $\alpha = -1/2$ or equivalently $k_1 = 0$. But note that, from an analytic point of view, this generalized Bessel function is obtained from the one associated with even dihedral groups via the substitutions $k_1 = 0, p = n$. Hence one expects the product formula for orthonormal Jacobi polynomials still holds in the limiting case. Indeed, the required limiting formula was derived in [7] p.194 using implicitely the fact that the Beta distribution $\mu^\alpha$ converges weakly to the dirac mass $\delta_1$. In order to fit it into our normalizations, we proceed as follows: use the well-known quadratic transformation ([8]):

$$P_j^{-1/2,k-1/2}(1-2\sin^2(n\theta)) = (-1)^jP_j^{k-1/2,-1/2}(2\sin^2(n\theta)-1) = (-1)^j\frac{(1/2)_j}{(k)_j}C_{2j}^k(\sin(n\theta))$$

where $P_j^{\alpha,\beta}$ is the (non orthonormal) $j$-th Jacobi polynomial, together with $\cos(2n\theta) = 1-2\sin^2(n\theta)$ to obtain

$$P_j^{-1/2,k-1/2}(\cos(2n\theta))P_j^{-1/2,k-1/2}(\cos(2n\phi)) = \left[\frac{(1/2)_j}{(k)_j}\right]^2 C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)).$$

Now, let $k > 0$ and recall that the squared $L^2$-norm of $P_j^{-1/2,k-1/2}$ is given by ([8])

$$\frac{2^k}{2j+k}\frac{\Gamma(j+1/2)\Gamma(j+k+1/2)}{j!\Gamma(j+k)} = \frac{2^k\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}\frac{(1/2)_j}{(k)_j}\frac{(k+1/2)_j}{(2j+k)j!}.$$

Recall also the special value

$$C_{2j}^k(1) = \frac{(2k)_{2j}}{(2j)!} = \frac{2^{2k}}{\Gamma(2k)}\frac{\Gamma(k+j)\Gamma(k+j+1/2)}{\Gamma(j+1/2)j!} = 2\frac{(k)_j(k+1/2)_j}{(1/2)_j j!}$$

where we use Gauss duplication formula twice to derive both the second and the third equalities. It follows that

$$\begin{aligned} c(k)p_j^{-1/2,k-1/2}(\cos(2n\theta))p_j^{-1/2,k-1/2}(\cos(2n\phi)) &= \frac{(1/2)_j}{(k)_j}\frac{(2j+k)j!}{(k+1/2)_j}C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= \frac{(2j+k)}{C_{2j}^k(1)}C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= (2j+k)\int C_{2j}^k\left(z_{n\phi,n\theta}(u,1)\right)\mu^k(du), \end{aligned}$$

according to [7] p.194, where

$$c(k) := \frac{2^{k+1}\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}.$$

As a matter of fact, we are led again to series of the form

$$\left(\frac{2}{R}\right)^{nk}\sum_{j\geq 0}(2j+k)I_{n(2j+k)}(R)C_{2j}^k(\cos\zeta) = \frac{1}{2}[f_{k,n}^+ + f_{k,n}^-](R,\cos\zeta).$$

## 5. Two Remarks

The first remark is concerned with $D_2(4)$ which coincides with the $B_2$-type Weyl group ([8]). Recall from ([6]) that $D_k^W$ may be expressed through a bivariate hypergeometric function as

$$D_k^W(x,y) = {}_1F_0^{(1/k_1)}\left(\frac{\gamma+1}{2},\frac{x^2}{2},\frac{y^2}{2}\right),$$

where we set $x^2 := (x_1^2, x_2^2) = (\rho^2\cos^2\phi, \rho^2\sin^2\phi)$ and similarly for $y^2$. This series is defined via Jack polynomials:

$${}_1F_0^{(1/r)}(a,x,y) = \sum_{\tau}(a)_\tau \frac{J_\tau^{1/r}(x)J_\tau^{1/r}(y)}{J_\tau^{1/r}(\mathbf{1})|\tau|!}$$

where $\mathbf{1} = (1,1)$, $\tau = (\tau_1,\tau_2)$ is a partition of length 2, $|\tau| = \tau_1+\tau_2$ is its weight and $(a)_\tau$ is the generalized Pochhammer symbol (see [6] for definitions). But those polynomials, known also as Jack polynomials of type $A_1$, may be expressed through Gegenbauer polynomials, a result due to M. Lassalle (see for instance formula 4.10 in [11]):

$$J_\tau^{1/r}(x^2) = \frac{(\tau_1-\tau_2)!}{2^{|\tau|}(r)_{\tau_1-\tau_2}}\sin^{|\tau|}(2\phi)C_{\tau_1-\tau_2}^r\left(\frac{1}{\sin(2\phi)}\right)$$

where $(r)_{\tau_1-\tau_2}$ is the (usual) Pochammer symbol. As a matter fact, one wonders if it is possible to come from the hypergeometric series to Corollary 1.2 in [5] and vice-versa.

The second remark comes in the same spirit of the first one. Consider the odd dihedral system $I_2(3) = \{\pm e^{-i\pi/2}e^{i\pi l/3},\ 1\leq l\leq 3\}$ ([8]). It is isomorphic to the $A_2$-type root system defined by

$$\{\pm(1,-1,0), \pm(1,0,-1), \pm(0,1,-1)\} \subset \mathbb{R}^3$$

which spans the hyperplane $(1,1,1)^{\perp}$. The isomorphism is given by

$$(z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{2}}\left(\sqrt{\frac{3}{2}}z_2, \frac{z_3 - z_1}{\sqrt{2}}\right)$$

subject to $z_1 + z_2 + z_3 = 0$ and for the $A_2$-type root system, the generalized Bessel function is given by the trivariate hypergeometric series ${}_0F_0^{(1/k)}$ (see [6] for the definition). Is it possible to relate this function to

$$\frac{c_{3,k}}{c(k)}\int [f_{k,3}^{+} + f_{k,3}^{-}](\rho r, z_{3\phi,3\theta}(u,1))\mu^k(du) = \frac{3\Gamma(3k)}{4}\int [f_{k,3}^{+} + f_{k,3}^{-}](\rho r, z_{3\phi,3\theta}(u,1))\mu^k(du)$$

in the same way the ${}_0F_1^{1/k_1}$ is related to the integral representation derived for $p = 2$?

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