

# HOLOMORPHIC CARTAN GEOMETRIES, CALABI–YAU MANIFOLDS AND RATIONAL CURVES

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ABSTRACT. We prove that if a Calabi–Yau manifold  $M$  admits a holomorphic Cartan geometry, then  $M$  is covered by a complex torus. This is done by establishing the Bogomolov inequality for semistable sheaves on compact Kähler manifolds. We also classify all holomorphic Cartan geometries on rationally connected complex projective manifolds.

## 1. INTRODUCTION

In [16], the second author conjectured that the only Calabi–Yau manifolds that bear holomorphic Cartan geometries are those covered by the complex tori. Our first theorem confirms this conjecture. More precisely, we prove:

**Theorem 1.1.** *Let  $M$  be a compact connected Kähler manifold with  $c_1(M) = 0$ . If  $M$  is equipped with a holomorphic Cartan geometry, then there is an étale Galois holomorphic covering map*

$$A \longrightarrow M,$$

where  $A$  is a complex torus.

This generalizes our earlier result in [6]; in [6], Theorem 1.1 was proved under the assumption that  $M$  is a complex projective manifold.

We then consider Cartan geometries on rationally connected projective varieties.

Let  $G$  be a complex Lie group and  $H \subset G$  a closed complex subgroup. The following theorem is proved in Section 4.

**Theorem 1.2.** *Let  $M$  be a rationally connected complex projective manifold and  $(E_H, \theta)$  a holomorphic Cartan geometry of type  $G/H$  on  $M$ . Then the homogeneous space  $G/H$  is a rational projective variety, and  $M \cong G/H$ . Furthermore,  $(M, E_H, \theta)$  is holomorphically isomorphic to the tautological Cartan geometry of type  $G/H$  on  $G/H$ .*

While we were finalizing this work, a preprint [8] appeared where Theorem 1.1 is proved under the assumption that the image of  $H$  in  $\text{Aut}(\text{Lie}(G))$  for the adjoint representation is algebraic (see [8, p. 1]). Theorem 1.2 is known for parabolic geometries [15], [5].

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## 2. BOGOMOLOV INEQUALITY

Let  $M$  be a compact connected Kähler manifold equipped with a Kähler form  $\omega$ .

For a torsionfree coherent analytic sheaf  $V \neq 0$  on  $M$ , define

$$\text{degree}(V) := (c_1(V) \wedge \omega^{d-1}) \cap [M] \in \mathbb{R},$$

where  $d = \dim_{\mathbb{C}} M$ . We recall that  $V$  is called *semistable* (respectively, *stable*) if for all torsionfree coherent analytic sheaves  $V' \subset V$  with  $\text{rank}(V') < \text{rank}(V)$ , the inequality

$$\frac{\text{degree}(V')}{\text{rank}(V')} \leq \frac{\text{degree}(V)}{\text{rank}(V)} \quad (\text{respectively, } \frac{\text{degree}(V')}{\text{rank}(V')} < \frac{\text{degree}(V)}{\text{rank}(V)})$$

holds. A semistable sheaf  $V$  is called *polystable* if it is a direct sum of stable sheaves.

**Lemma 2.1.** *Let  $V$  be a semistable sheaf on  $M$ . Then*

$$((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [M] \geq 0,$$

where  $r$  is the rank of  $V$ .

*Proof.* If  $V$  is polystable and reflexive, then this is proved in [2] (see [2, p. 40, Corollary 3]).

First assume that  $V$  is polystable.

The double dual  $V^{**}$  is reflexive. Since  $V$  is torsionfree, the natural homomorphism  $V \rightarrow V^{**}$  is injective. Hence we have a short exact sequence of coherent analytic sheaves

$$(2.1) \quad 0 \rightarrow V \rightarrow V^{**} \rightarrow T \rightarrow 0$$

on  $M$ , where  $T$  is a torsion sheaf, and the (complex) codimension of the support of  $T$  is at least two (see [11, p. 154, Proposition (4.14)]).

Since  $V$  is polystable, we know that  $V^{**}$  is also polystable (follows from the fact that the double dual of a stable sheaf is stable). Hence

$$(2.2) \quad ((2r \cdot c_2(V^{**}) - (r-1)c_1(V^{**})^2) \cup \omega^{d-2}) \cap [M] \geq 0$$

[2, p. 40, Corollary 3].

Consider the quotient  $T$  in (2.1). Let  $S_1, \dots, S_\ell$  be the components of the support of  $T$  of codimension two. Let  $r_i$  be the rank of  $T|_{S_i}$ . Using (2.1) we conclude that

$$(2.3) \quad c_1(V) = c_1(V^{**}) \quad \text{and} \quad c_2(V) = c_2(V^{**}) + \sum_{i=1}^{\ell} r_i \cdot \psi([S_i]),$$

where  $\psi : H_{2d-4}(M, \mathbb{Q}) \rightarrow H^4(M, \mathbb{Q})$  is the isomorphism given by the Poincaré duality.

Since  $\text{Volume}_\omega(S_i) = (\omega^{d-2} \cup \psi([S_i])) \cap [M] \geq 0$ , from (2.2) and (2.3) we conclude that

$$(2.4) \quad ((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Finally, let  $V$  be any semistable sheaf. There is a filtration of coherent analytic subsheaves of  $V$

$$0 =: V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that each successive quotient  $V_i/V_{i-1}$ ,  $i \in [1, n]$ , is torsionfree and stable, and furthermore,

$$\frac{\text{degree}(V_i/V_{i-1})}{\text{rank}(V_i/V_{i-1})} = \frac{\text{degree}(V)}{\text{rank}(V)}$$

(see [11, p. 175, Theorem (7.18)]). Therefore,

$$\widehat{V} := \bigoplus_{i=1}^n V_i/V_{i-1}$$

is a polystable sheaf. Since

$$c_i(V) = c_i(\widehat{V})$$

for all  $i \geq 0$ , from (2.4) we conclude that the inequality in the lemma holds. This completes the proof of the lemma.  $\square$

### 3. CARTAN GEOMETRIES AND CALABI–YAU MANIFOLDS

Let  $G$  be a complex Lie group and  $H \subset G$  a closed complex subgroup. The Lie algebra of  $G$  (respectively,  $H$ ) will be denoted by  $\mathfrak{g}$  (respectively,  $\mathfrak{h}$ ). As before,  $M$  is a compact connected Kähler manifold.

A *holomorphic Cartan geometry* on  $M$  of type  $G/H$  is a holomorphic principal  $H$ -bundle

$$(3.1) \quad E_H \longrightarrow M$$

together with a holomorphic one-form on  $E_H$  with values in  $\mathfrak{g}$

$$(3.2) \quad \theta \in H^0(E_H, \Omega_{E_H}^1 \otimes_{\mathbb{C}} \mathfrak{g})$$

satisfying the following three conditions:

- $\tau_h^* \theta = \text{Ad}(h^{-1}) \circ \theta$  for all  $h \in H$ , where  $\tau_h : E_H \longrightarrow E_H$  is the translation by  $h$ ,
- $\theta(z)(\zeta_v(z)) = v$  for all  $v \in \mathfrak{h}$  and  $z \in E_H$ , where  $\zeta_v$  is the vector field on  $E_H$  defined by  $v$  using the action of  $H$  on  $E_H$ , and
- for each point  $z \in E_H$ , the homomorphism from the holomorphic tangent space

$$(3.3) \quad \theta(z) : T_z E_H \longrightarrow \mathfrak{g}$$

is an isomorphism of vector spaces.

(See [17].)

Let  $(E_H, \theta)$  be a holomorphic Cartan geometry of type  $G/H$  on  $M$ . Let

$$(3.4) \quad E_G := (E_H \times G)/H \longrightarrow M$$

be the holomorphic principal  $G$ -bundle obtained by extending the structure group of  $E_H$  using the inclusion of  $H$  in  $G$  (the action of  $h \in H$  sends any  $(z, g) \in E_H \times G$  to  $(zh, h^{-1}g)$ ). Let

$$\theta_{\text{MC}} : TG \longrightarrow G \times \mathfrak{g}$$

be the  $\mathfrak{g}$ -valued Maurer–Cartan one-form on  $G$  constructed using the left invariant vector fields. Consider the  $\mathfrak{g}$ -valued holomorphic one-form

$$\tilde{\theta} := p_1^* \theta + p_2^* \theta_{\text{MC}}$$

on  $E_H \times G$ , where  $p_1$  (respectively,  $p_2$ ) is the projection of  $E_H \times G$  to  $E_H$  (respectively,  $G$ ). This form  $\tilde{\theta}$  descends to a  $\mathfrak{g}$ -valued holomorphic one-form on the quotient space  $E_G$  in (3.4). This descended one-form defines a holomorphic connection on  $E_G$ .

Therefore, the principal  $G$ -bundle  $E_G$  in (3.4) is equipped with a holomorphic connection.

Let  $\text{ad}(E_G)$  (respectively,  $\text{ad}(E_H)$ ) be the adjoint bundle of  $E_G$  (respectively,  $E_H$ ). So  $\text{ad}(E_G)$  (respectively,  $\text{ad}(E_H)$ ) is associated to  $E_G$  for the adjoint action of  $G$  (respectively,  $H$ ) on  $\mathfrak{g}$  (respectively,  $\mathfrak{h}$ ). There is a natural inclusion

$$\text{ad}(E_H) \hookrightarrow \text{ad}(E_G).$$

Using the form  $\theta$ , the quotient  $\text{ad}(E_G)/\text{ad}(E_H)$  gets identified with the holomorphic tangent bundle  $TM$ . Therefore, we get a short exact sequence of holomorphic vector bundle on  $M$

$$(3.5) \quad 0 \longrightarrow \text{ad}(E_H) \longrightarrow \text{ad}(E_G) \longrightarrow TM \longrightarrow 0.$$

**Theorem 3.1.** *Assume that*

- $c_1(M) = 0$ , and
- $M$  is equipped with a holomorphic Cartan geometry  $(E_H, \theta)$  of type  $G/H$ .

*Then there is an étale Galois holomorphic covering map*

$$\gamma : A \longrightarrow M,$$

*where  $A$  is a complex torus.*

*Proof.* Consider the principal  $G$ -bundle  $E_G$  in (3.4). Recall that  $\theta$  defines a holomorphic connection on  $E_G$ . A holomorphic connection on  $E_G$  induces a holomorphic connection on the associated vector bundle  $\text{ad}(E_G)$ .

Since  $c_1(M) = 0$ , a theorem due to Yau says that the holomorphic tangent bundle  $TM$  admits a Kähler–Einstein metric [18]. Fix a Kähler–Einstein Kähler form  $\omega$  on  $M$ . The

vector bundle  $TM$  is polystable because  $\omega$  is Kähler–Einstein; see [11, p. 177, Theorem (8.3)] Hence from Lemma 2.1,

$$(3.6) \quad (c_2(TM) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Since  $\text{ad}(E_G)$  admits a holomorphic connection, and  $TM$  is polystable with

$$\text{degree}(TM) = 0,$$

we conclude that the vector bundle  $\text{ad}(E_G)$  is semistable (see [3, p. 2830]). Also, we have

$$(3.7) \quad c_j(\text{ad}(E_G)) = 0$$

for all  $j \geq 1$  because the vector bundle  $\text{ad}(E_G)$  admits a holomorphic connection [1, p. 192–193, Theorem 4].

Consider the short exact sequence of vector bundles in (3.5). Since  $c_1(M) = 0$ , from (3.5) and (3.7) it follows that

$$(3.8) \quad c_1(\text{ad}(E_H)) = 0.$$

Since  $\text{ad}(E_H)$  is a subbundle of the semistable vector bundle  $\text{ad}(E_G)$  of degree zero, we conclude that  $\text{ad}(E_H)$  is also semistable (any subsheaf of  $\text{ad}(E_H)$  violating the semistability condition of  $\text{ad}(E_H)$  would also violate the semistability condition of  $\text{ad}(E_G)$ ). Now from Lemma 2.1 and (3.8) we conclude that

$$(3.9) \quad (c_2(\text{ad}(E_H)) \cup \omega^{d-2}) \cap [M] \geq 0.$$

In view of (3.7), (3.8) and (3.9), from the short exact sequence in (3.5) it follows that

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] = -(c_2(\text{ad}(E_H)) \cup \omega^{d-2}) \cap [M] \leq 0.$$

On the other hand, from Lemma 2.1,

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Hence

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] = 0.$$

Consequently,

$$((2d \cdot c_2(TM) - (d-1)c_1(TM)^2) \cup \omega^{d-2}) \cap [M] = 0.$$

Therefore, from Corollary 3 in [2, p. 40] we conclude that  $TM$  is projectively flat. In particular,

$$\text{End}(TM) = TM \otimes \Omega_M^1$$

is a flat vector bundle. Hence  $c_2(\text{End}(TM)) = 0$ . But

$$c_2(\text{End}(TM)) = 2d \cdot c_2(TM) - (d-1)c_1(TM)^2.$$

Since  $c_1(TM) = 0$ , we conclude that  $c_2(TM) = 0$ . Now from [10, p. 248, Corollary 2.2] (see also Lemma 1 of [16]) we know that  $M$  is holomorphically covered by a complex torus. This completes the proof of the theorem.  $\square$

**Remark 3.2.** The holomorphic tangent bundle  $TM$  of a compact connected Kähler manifold  $M$  admits a holomorphic connection if and only if  $M$  admits an étale covering by a complex torus. Indeed, if  $A \rightarrow M$  is a covering map, where  $A$  is a complex torus, then the flat connection on  $TA$  given by a trivialization of  $TA$  descends to a holomorphic flat connection on  $TM$ . To prove the converse, assume that  $TM$  admits a holomorphic connection. So  $c_1(TM) = 0 = c_2(TM)$  [1, p. 192–193, Theorem 4]. Since  $TM$  is polystable [18], we know that  $TM$  is projectively flat [2, p. 40, Corollary 3]. Since  $c_1(TM) = 0$ , and  $TM$  is projectively flat, it follows that a Kähler–Einstein metric on  $M$  is flat. Therefore,  $M$  is covered by a complex torus.

It should be pointed out that even when  $TM$  admits a holomorphic connection, the exact sequence in (3.5) is not compatible with the holomorphic connection on  $\text{ad}(E_G)$  (unless  $\dim M = 0$ ) in the sense that the connection on  $TM$  is not a quotient of the connection on  $\text{ad}(E_G)$ .

#### 4. CARTAN GEOMETRIES AND RATIONAL CURVES

A smooth complex projective variety  $Z$  is said to be rationally connected if for any two given points of  $Z$ , there exists an irreducible rational curve on  $Z$  that contains them; see [13, p. 433, Theorem 2.1] and [13, p. 434, Definition–Remark 2.2] for other equivalent conditions. Note that a rationally connected projective manifold is connected.

As before,  $G$  is a complex Lie group and  $H \subset G$  a closed complex subgroup. The quotient map  $G \rightarrow G/H$  defines a holomorphic principal  $H$ -bundle over  $G/H$ . There is a canonical holomorphic Cartan connection  $\theta_0$  on this  $H$ -bundle  $G \rightarrow G/H$  given by the identification of  $\mathfrak{g}$  with the left-invariant vector fields on  $G$ . We will call  $(G/H, G, \theta_0)$  the *tautological Cartan geometry*.

**Theorem 4.1.** *Let  $(M, E_H, \theta)$  be a holomorphic Cartan geometry of type  $G/H$  such that  $M$  is rationally connected. Then the homogeneous space  $G/H$  is a rational projective variety, and  $M \cong G/H$ . Furthermore,  $(M, E_H, \theta)$  is holomorphically isomorphic to the tautological Cartan geometry  $(G/H, G, \theta_0)$ .*

*Proof.* Consider the principal  $G$ -bundle  $E_G$  defined in (3.4). Let  $D$  denote the holomorphic connection on the principal  $G$ -bundle  $E_G$  induced by  $\theta$  (see Section 3). Since  $M$  is rationally connected, the curvature of  $D$  vanishes (see [4, p. 160, Theorem 3.1]). From the fact that  $M$  is rationally connected it also follows that  $M$  is simply connected [7, p. 545, Theorem 3.5], [12, p. 362, Proposition 2.3].

Since  $D$  is flat, and  $M$  is simply connected, we have the developing map for the Cartan geometry  $(M, E_H, \theta)$

$$\gamma : M \rightarrow G/H$$

which is a submersion [17], [14, p. 3, Theorem 2.7]. Hence  $\gamma$  is a covering map because  $M$  is compact. Since  $G/H$  is covered by the rationally connected complex projective

manifold  $M$ , we conclude that  $G/H$  is also a rationally connected complex projective manifold. We noted above that this implies that  $G/H$  is simply connected [7, p. 545, Theorem 3.5], [12, p. 362, Proposition 2.3]. Since  $\gamma$  is a covering map, and  $G/H$  is simply connected, it follows that  $\gamma$  is an isomorphism.

Since  $G/H$  is a simply connected homogeneous projective variety, we know that  $G/H$  is rational. This completes the proof of the theorem.  $\square$

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