A completely positive map associated with a positive map

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Abstract

We show that each positive map from $B(K)$ to $B(H)$ is a scalar multiple of a map of the form $Tr - \psi$ with ψ completely positive. This is used to give necessary and sufficient conditions for maps to be $\mathcal{C}$-positive for a large class of mapping cones; in particular we apply the results to k-positive maps.

Introduction

In [6] we studied several norms on positive maps from $B(K)$ into $B(H)$, where K and H are finite dimensional Hilbert spaces. These norms were very useful in the study of maps of the form $Tr - \lambda\psi$, where Tr is the usual trace on $B(K), \lambda > 0$, and ψ a completely positive map of $B(K)$ into $B(H)$. In the present paper we shall see that every positive map is a positive scalar multiple of a map of the above form with $\lambda = 1$, hence the results in [6] are applicable to all positive maps. In particular they yield a simple criterion for some maps to be k-positive but not k+1-positive. As an illustration we give a new proof that the Choi map of $B(\mathcal{C}^3)$ into itself is atomic, i.e. not the sum of a 2-positive and a 2-copositive map.

$\mathcal{C}$-positive maps

Let K and H be finite dimensional Hilbert spaces. We denote by $B(B(K), B(H))$ (resp.$B(B(K), B(H))^+$) the linear (resp. positive linear) maps of $B(K)$ into $B(H)$. In the case $K = H$ we denote by $P(H) = B(B(H), B(H))^+$. Following [8] we say a closed cone $\mathcal{C} \subset P(H)$ is a *mapping cone* if $\alpha \circ \phi \circ \beta \in \mathcal{C}$ for all $\phi \in \mathcal{C}$ and $\alpha, \beta \in CP$ - the completely positive maps in $P(H)$. A map ϕ in $B(B(K), B(H))$ defines a linear functional $\tilde{\phi}$ on $B(K) \otimes B(H)$, identified with $B(K \otimes H)$ in the sequel, by $\tilde{\phi}(a \otimes b) = Tr(\phi(a)b^t)$, where Tr is the usual trace on $B(H)$ and t denotes the transpose. Let $P(B(K), \mathcal{C})$ denote the closed cone

$$P(B(K), \mathcal{C}) = \{a \in B(K \otimes H) : \iota \otimes \alpha(a) \geq 0 \ \ \forall \ \ \alpha \in \mathcal{C}\},$$

where ι denotes the identity map on $B(K)$. Then a map $\phi \in B((B(K), B(H))$ is said to be $\mathcal{C}$-*positive* if $\tilde{\phi}$ is positive on $P(B(K), \mathcal{C})$. We denote by $\mathcal{P}_\mathcal{C}$ the cone of $\mathcal{C}$-positive maps.

If (e_{ij}) is a complete set of matrix units for $B(K)$ then the *Choi matrix* for a map ϕ is

$$C_\phi = \sum e_{ij} \otimes \phi(e_{ij}) \in B(K \otimes H).$$

By [10] and [11] the transpose C_ϕ^t of C_ϕ is the density operator for $\tilde{\phi}$, and by [1] ϕ is completely positive if and only if $C_\phi \geq 0$ if and only if $\tilde{\phi} \geq 0$ as a linear functional on $B(K \otimes H)$. In the case $\mathcal{C} = CP$, $P(CP, B(K)) = B(K \otimes H)^+$, so ϕ is CP-positive if and only if ϕ is completely positive.

If $\mathcal{C}_1 \subset \mathcal{C}_2$ are two mapping cones on $B(H)$, then $P(B(K), \mathcal{C}_1) \supset P(B(K), \mathcal{C}_2)$, because if $\iota \otimes \alpha(a) \geq 0$ for all $\alpha \in \mathcal{C}_2$, then the same inequality holds for all $\alpha \in \mathcal{C}_1$. Thus $\tilde{\phi} \geq 0$ on $P(B(K), \mathcal{C}_1)$ implies $\tilde{\phi} \geq 0$ on $P(B(K), \mathcal{C}_2)$, so $\mathcal{P}_{\mathcal{C}_1} \subset \mathcal{P}_{\mathcal{C}_2}$.

Let $\mathcal{C}$ be a mapping cone on $B(H)$. Let $\mathcal{P}_C^o$ denote the *dual cone of* $\mathcal{P}_\mathcal{C}$ defined as

$$\mathcal{P}_C^o = \{\phi \in B(B(K), B(H)) : Tr(C_\phi C_\psi) \geq 0 \ \ \forall \psi \in \mathcal{P}_\mathcal{C}\}.$$

Thus if $\mathcal{C}_1 \subset \mathcal{C}_2$ then $\mathcal{P}_{\mathcal{C}_1}^o \supset \mathcal{P}_{\mathcal{C}_2}^o$. In the particular case when $\mathcal{C} \supset CP$ we thus get $\mathcal{P}_\mathcal{C}^o \subset \mathcal{P}_{CP}^o = CP(K, H)$ - the completely positive maps of $B(K)$ into $B(H)$.

Following [6] $\mathcal{C}$ defines a norm on $B(B(K), B(H))$ by

$$\| \phi \|_\mathcal{C} = sup\{| Tr(C_\phi C_\psi) | : \psi \in \mathcal{P}_C^o, Tr(C_\psi) = 1\}.$$

In the special case when $\mathcal{C} \supset CP$ it follows from the above that

$$\| \phi \|_\mathcal{C} = sup \mid \rho(C_\phi) \mid,$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathcal{P}_C^o$. Let $\phi \in B(B(K), B(H))$ be a self-adjoint map, i.e. $\phi(a)$ is self-adjoint for a self-adjoint. Then C_ϕ is a self-adjoint operator, so is a difference $C_\phi^+ - C_\phi^-$ of two positive operators with orthogonal supports. Let $c \geq 0$ be the smallest positive number such that $c1 \geq C_\phi$. Then $c = \| C_\phi^+ \|$. Hence, if $c \neq 0$ there exists a map $\phi_{cp} \in B(B(K), B(H))$ such that the Choi matrix for ϕ_{cp} equals $1 - c^{-1}C_\phi$, which is a positive operator. Thus, if we let Tr denote the map $x \mapsto Tr(x)1, \phi_{cp}$ is completely positive, and $c^{-1}\phi = Tr - \phi_{cp}$, since $C_{Tr} = 1$, as is easily shown. Combining the above discussion with [6], Prop. 2, we thus have.

Theorem 1 *Let ϕ be a self-adjoint map of $B(K)$ into $B(H)$. Then if $-\phi$ is not completely positive, we have*

(i) *There exists a completely positive map $\phi_{cp} \in B(B(K), B(H))$ such that $\| C_\phi^+ \|^{-1} \phi = Tr - \phi_{cp}$.*

(ii) *If $\mathcal{C}$ is a mapping cone on $B(H)$ containing CP then ϕ is $\mathcal{C}$-positive if and only if*

$$1 \geq \| \phi \|_\mathcal{C} = sup \ \rho(C_{\phi_{cp}}),$$

where the sup is taken over all states ρ on $B(K \otimes H)$ with density operator C_ψ with $\psi \in \mathcal{P}_{\mathcal{C}}^o$.

Note that we did not need to take the absolute value of $\rho(C_{\phi_{cp}})$ because $C_{\phi_{cp}} \geq 0$ and $\psi \in \mathcal{P}_{\mathcal{C}}^o \subset CP$.

We next spell out the theorem for some well known mapping cones. Recall that a map ϕ is *decomposable* if $\phi = \phi_1 + \phi_2$ with ϕ_1 completely positive and ϕ_2 copositive, i.e. $\phi_2 = t \circ \psi$ with ψ completely positive. Also recall that a state ρ on $B(K \otimes H)$ is a *PPT -state* if $\rho \circ (\iota \otimes t)$ is also a state.

Corollary 2 *Let $\phi \in B(B(K), B(H))$ be a self-adjoint map. Then we have.*

(i) *ϕ is positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all separable states ρ on $B(K \otimes H)$.*

(ii) *ϕ is decomposable if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all PPT-states ρ on $B(K \otimes H)$.*

(iii) *ϕ is completely positive if and only if $\rho(C_{\phi_{cp}}) \leq 1$ for all states ρ on $B(K \otimes H)$.*

Proof. (i) That ϕ is positive is the same as saying that ϕ is $P(H)-$positive. Since the dual cone of $P(H)$ is the cone of separable states (i) follows.

(ii) A state ρ is PPT if and only if its density operator is of the form C_ψ with ψ a map which is both positive copositive, see e.g. [10],Prop.4. But the dual of those maps is the cone of decomposable maps, see e.g. [7]. Thus (ii) follows from the theorem.

(iii) This follows since the dual cone of the completely positive maps is the cone of completely positive maps, and that the density operator for a state is positive, hence the corresponding map ψ is completely positive.

k-positive maps

A map $\phi \in B(B(K), B(H))$ is said to be *k-positive* if $\phi \otimes \iota \in B(B(K \otimes L), B(H \otimes L))^+$ whenever L is a k-dimensional Hilbert space. The k-positive maps in $P(H)$ form a mapping cone P_k containing CP. Denote by $P_k(K, H)$ the cone of k-positive maps in $B(B(K), B(H))$. Then we have ,

Lemma 3 *With the above notation we have $\mathcal{P}_{P_k} = P_k(K, H)$.*

Proof. We have $P_k^o = SP_k$, the k-superpositive maps in $P(H)$, which is the mapping cone generated by maps of the form AdV defined by $AdV(a) = VaV^*$, where $V \in B(H), rankV \leq k$, see e.g. [7]. By [11] the dual cone of $\mathcal{P}_{P_k^o}$ is given by

$$\mathcal{P}_{P_k^o}^o = \{\phi \in B(B(K), B(H)) : AdV \circ \phi \in CP(K, H) \ \forall V \in B(H), rankV \leq k\}.$$

By [5],Theorem 3, or [6],Theorem 2, it follows that $\mathcal{P}_{P_k^o}^o = P_k(K, H)$. By[8], Theorem 3.6, $\mathcal{P}_{P_k}$ is generated by maps of the form $\alpha \circ \beta$ with $\alpha \in P_k, \beta \in$

$CP(K,H)$. Let $AdV \circ \gamma, AdV \in SP_k, \gamma \in CP(K,H)$ be a generator for $\mathcal{P}_{P_k^o}$. Then

$$Tr(C_{\alpha\circ\beta}C_{AdV\circ\gamma}) = Tr(C_{AdV^*\circ\alpha\circ\beta}C_\gamma) \geq 0,$$

since $AdV^* \circ \alpha$ is completely positive since $\alpha \in P_k$ and $rankV \leq k$. Since the above inequality holds for the generators of the two cones, it follows that $\mathcal{P}_{P_k} = \mathcal{P}^o_{P_k^o} = P_k(K,H)$, completing the proof of the lemma.

It follows from the above description of $\mathcal{P}^o_{P_k}$ that the states with density operators $C_\psi, \psi \in \mathcal{P}^o_{P_k}$, are the same as the vector states generated by vectors in the Schmidt class $S(k)$, i.e. the vectors $y = \sum_{i=1}^k x_i \otimes y_i, x_i \in K, y_i \in H$, where the x_i and y_i are not necessarily all $\neq 0$.

Theorem 4 *Let $\phi \in B(B(K),B(H))^+$. Then we have.*

(i) *ϕ is k-positive if and only if $sup_{x\in S(k),\|x\|=1}(C_{\phi_{cp}}x,x) \leq 1$.*

(ii) *Suppose $k < min(dimK, dimH)$, and that there exists a unit vector $y = \sum_{i=1}^k x_i \otimes y_i \in S(k)$ such that $y \perp C_\phi y \notin X \otimes Y$, where $X = span(x_i), Y = span(y_i)$. Then ϕ is not k+1-positive.*

. In order to prove the theorem we first prove a lemma.

Lemma 5 *Let A be a self-adjoint operator in $B(K\otimes H)$. Suppose $y = \sum_{i=1}^k x_i \otimes y_i$ satisfies $(Ay,y) = 1$, and $Ay \notin X \otimes Y$ with X, Y as in Theorem 4. Then there exist a unit product vector $x \perp X \otimes Y$ and $s \in (0,1)$ such that $(A(sx + (1-s^2)^{1/2})y), sx + (1-s^2)^{1/2}y) > 1$.*

Proof. Since $Ay \notin X \otimes Y$ there exists a product vector $x \perp X \otimes Y$ such that $Re(x, Ay) > 0$. Let $s \in (-1,1)$ and $t = t(s) = (1-s^2)^{1/2}$, and let f denote the function

$$f(s) = (A(sx+ty), st+ty) = s^2(Ax,x) + t^2(Ay,y) + 2stRe(Ax,y).$$

Since $(Ay,y) = 1$ we get

$$f'(0) = 2(1-s^2)^{1/2}Re(Ax,y) > 0.$$

Therefore, for $s > 0$ and near 0 we have $(A(sx+ty), st+ty) > f(0) = 1$, proving the lemma.

Proof of Theorem 4.

(i) is a direct consequence of Theorem 1, since, as noted in the proof of Lemma 3, the vector states ω_x with $x \in S(k)$ generate the set of states with density operators C_ψ with $\psi \in \mathcal{P}^o_{P_k}$.

(ii) By Theorem 1 $C_{\phi_{cp}} = 1 - \| C_\phi^+ \|^{-1} C_\phi$, so that $(C_{\phi_{cp}}y,y) = 1$, using the assumption that $C_\phi y \perp y$. Furthermore $C_{\phi_{cp}}y = y - \| C_\phi^+ \|^{-1} C_\phi y$. Since $C_\phi y \notin X\otimes Y$ $C_{\phi_{cp}}y \notin X\otimes Y$. Thus by Lemma 5 there exist a unit product vector $x \in X\otimes Y$ and $s, t = (1-s^2)^{1/2} > 0$ such that $(C_{\phi_{cp}}(sx+ty), sx+ty) > 1$. Since

$sx + ty$ is a unit vector in S(k+1), ϕ is not k+1-positive by part (i), completing the proof of the theorem.

Example We illustrate the above results by an application to the Choi map $\phi \in B(B(C^3), B(C^3))$ defined by

$$\phi((x_{ij})) = \begin{bmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{11} + x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{22} + x_{33} \end{bmatrix}$$

We have $C_{t\circ\phi} = (\iota \otimes t)C_\phi$. So if $y = x \otimes x$ with $x = 3^{-1/2}(1,1,1) \in C^3$, then $(C_\phi y, y) = (C_{t\circ\phi} y, y) = 0$, and $C_\phi y \neq 0 \neq C_{t\circ\phi} y$. Hence, by Theorem 4, neither ϕ nor $t \circ \phi$ is 2-positive, i.e. ϕ is neither 2-positive nor 2-copositive. Since ϕ is an extremal positive map of $B(C^3)$ into itself by [2], ϕ cannot be the sum of a 2-positive and a 2-copositive map, hence ϕ is atomic, a result first proved by Tanahashi and Tomiyama [12], and then extended to more general maps by others, see [3] for references.

ϕ can also be shown to be a positive map by a straightforward argument using Corollary 2.

It should be remarked that the Choi map ϕ also yields an example of a PPT-state on $B(C^3) \otimes B(C^3)$ which is not separable. Indeed, in [9] we gave an example of a positive matrix in A in $B(C^3) \otimes B(C^3)$ such that its partial transpose $t \otimes \iota(A)$ is also positive, and that $\phi \otimes \iota(A)$ is not positive. Then A cannot be of the form $\sum A_i \otimes B_i$ with A_i and B_i positive, hence the state $\rho(x) = Tr(A)^{-1}Tr(Ax)$ is PPT but not separable. An example of a PPT state on $B(C^3) \otimes B(C^3)$ which is not separable was later exhibited by P. Horodecki [4].

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