COVERING NUMBERS FOR GRAPHS AND HYPERGRAPHS

BÉLA BOLLOBÁS[1] AND ALEX SCOTT

ABSTRACT. The *covering number* of a hypergraph H is the largest integer k such that the edges of H can be partitioned into k classes such that every vertex is incident with edges from every class. We prove a number of results on covering numbers, and state some open problems.

1. INTRODUCTION

Given a hypergraph H, we say that a set of edges $E' \subset E(H)$ is a *covering class* if every vertex of H is incident with at least one edge from E'. The *covering number* of a hypergraph H is the largest integer k such that the edges of H can be partitioned into k covering classes. For $k, r \geq 1$, we define $f(r, k)$ to be the smallest integer d such that every r-uniform hypergraph H with minimal degree at least d has covering number at least k. We will write $f_m(r, k)$ for the corresponding problem where we allow multiple edges.

Covering numbers for graphs (i.e. for $r = 2$) have been studied by a number of authors starting with Gupta [6] in the 1970s (see Section 3 for further discussion). There is also a substantial literature on the analogous problems for splitting covers of topological spaces, and for splitting covers by geometric objects (see, for instance, Tsaban [14] and Elekes, Mátrai and Soukup [2]). However, the hypergraph problem has received very little attention.

It is trivial that $f(r, k) \leq f_m(r, k)$, and it is easy to see that $f_m(r, k) \leq rk$: if H is an r-uniform multigraph with minimal degree at least rk, then an application of Hall's theorem allows us to assign pairwise disjoint sets of k edges to every vertex, which is clearly enough to obtain the required colouring. It is also easy to see that, for any fixed odd r and a positive density of values of k, $f(r, k) \geq 2k(r-1)/r$: given an integer $d < 2k(r-1)/r$, fix a set S of r vertices, and consider a d-regular, r-uniform hypergraph in which every edge is either disjoint

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from S or meets it in exactly $r-1$ vertices. Any covering set must use at least two edges to cover S, so the covering number is at most $|S|d/2(r-1) < k$, giving a lower bound on $f(r,k)$ whenever such a hypergraph exists.

The aim of this paper is to give better bounds on $f(r,k)$. We begin in Section 2 by defining a "levelling" operation that replaces a general hypergraph with a regular one; this allows us to connect the problem to hypergraph colouring. In Section 3, we consider the problem for graphs, giving a short proof of Gupta's bound for $f(2,k)$ and resolving the analogous problem for multigraphs; this answers a question of Xu and Liu [15]. In Section 4, we turn to hypergraphs: we determine $f(r,k)$ to within a constant factor for all r and k; while in Section 5 we determine $f(r,k)$ explicitly for a couple of small values of r and k. We conclude in Section 6 with some open questions.

2. Levelling

In this section, we define a useful "levelling" construction that will allow us to replace a general hypergraph with a regular one. Given a hypergraph $H_0 = (V_0, E_0)$ in which all edges have size at most r and all vertices have degree at least d, we say that an r-uniform hypergraph $H_1 = (V_1, E_1)$ is an (r,d)-*levelling* of H_0 (or, more simply, a d-levelling when r is obvious from context) if the following conditions are satisfied:

(1) $V_0 \subset V_1$
(2) There is an injective function $f : E_0 \to E_1$ such that:
  (a) for every $e \in E_0$ we have $f(e) \cap V_0 \subset e$; and
  (b) every edge of E_1 that meets V_0 is the image of some edge in E_0
(3) H_1 is d-regular.

It is easy to show that every H has an (r,d)-levelling: first delete vertices from the edges of H (possibly creating multiple edges or empty edges) until we obtain H' in which every vertex has degree d. Now take d copies of H', say $H'_1, \dots, H'_d$. For each edge $e \in E(H')$ we add $r - |e|$ new vertices, and extend the d copies of e in the H'_i by adding the new vertices to those edges. The result is a d-levelling of H.

Proposition 1. *If H_1 is a d-levelling of H_0 and $E(H_1)$ can be split into k covering classes then $E(H_0)$ can be split into k covering classes.*

Proof. Just use the function f from the definition of d-levelling to induce a colouring of $E(H_0)$. □

The d-levelling construction shows that we lose nothing by allowing vertices of large degree or smaller edges. For $k, r \geq 1$, let $f_{\leq}(r,k)$ be the smallest integer d such that for every hypergraph H with minimal degree at least d and no edges of size greater than r there is a partition $E(H) = E_1 \cup \cdots \cup E_k$ such that every E_i is a covering class; let $f_=(r,k)$ be the minimum for d-regular r-uniform hypergraphs.

Lemma 2. *For $k, r \geq 1$, $f(r,k) = f_=(r,k) = f_\leq(r,k)$.*

It is easily seen that the same inequalities hold for the corresponding quantities for multigraphs.

Levelling also allows us to consider the dual problem. Indeed, for giving bounds on $f(r,k)$, Lemma 2 shows that it is enough to consider regular hypergraphs. So let $H = (V,E)$ be a d-regular, r-uniform hypergraph, and let $H^* = (E,V)$ be the dual hypergraph, with vertex set E and edges V, and the same edge-vertex incidence relation. Note that H^* is r-regular and d-uniform. An edge cover of H corresponds to a transversal of H^*, so a k-covering of H corresponds to a k-colouring of H^* in which every edge has a vertex of every colour. Thus $f(r,k)$ is the smallest d such that every d-uniform, r-regular hypergraph has a k-colouring in which every edge has a vertex of every colour.

For $k = 2$, this is the famous Property B or hypergraph 2-colouring, which has been extensively studied (see, for instance, [3, 4, 1, 10]). We will make explicit use of this connection in Section 5.

3. Graphs

The covering number for graphs has been examined by a number of authors. For a multigraph G, and $x \in V(G)$, let $m(x)$ be the maximum multiplicity of any edge incident with x. Gupta [6] announced the theorem that the covering number of G is at least

$$\min_{x \in V(G)} d(x) - m(x). \tag{1}$$

More generally, Gupta stated that if W is an independent set, then (1) can be weakened: G has covering number at least k, provided $d(x) \geq k$ for all $x \in W$ and $d(x) + m(x) \geq k$ for all $x \notin W$. A proof of Gupta's theorem was given by Fournier [5], as a consequence of a more general result on colourings of multigraphs. See also Hilton and de Werra [8], Hilton [7] and Schrijver [12] for related results and questions. The problem for infinite graphs and multigraphs has been considered by Elekes, Mátrai and Soukup [2].

It follows immediately from Gupta's Theorem that for graphs ($m(x) = 1$ for all x) we have

$$f(2,k) \leq k + 1. \tag{2}$$

For completeness, we give a short proof of this fact.

Lemma 3. *For $k \geq 2$, $f(2,k) = k+1$.*

Proof. Suppose G is a graph with $\delta(G) \geq k+1$. Let H be a $(k+1)$-levelling of G. It is enough for the upper bound to show that $E(H)$ can be partitioned into k covering classes.

Now H is $(k+1)$-regular, so by Vizing's Theorem it has a proper $(k+2)$-edge-colouring, with colours $[k+2] = \{1, \ldots, k+2\}$. Consider the edge set E' consisting of edges with colour $k+1$ or $k+2$. Every vertex of H is incident with 1 or 2 edges from E', so E' is a union of paths and cycles. Orient E' so that all paths and cycles are directed. Now every vertex v of H is missing exactly one colour $c(v)$ from $[k+2]$. If $c(v) \in \{k+1, k+2\}$ then v sees all colours in $[k]$; otherwise, v has degree 2 in E', so has an inedge which we recolour with colour $c(v)$ (which may produce a colouring that is not proper). We obtain a colouring in which every vertex sees all colours in $[k]$.

For the lower bound, when k is even we consider K_{k+1}. When k is odd we consider any graph that has all vertices of degree k, except for one of degree $k+1$, and (necessarily) an odd number of vertices. In both cases, there is no perfect matching and counting edges shows that the edges cannot be split into k covering classes. □

For multigraphs, the argument above will not work, as the multigraph analogue of Vizing's Theorem gives only $\chi'(G) \leq 3\Delta(G)/2$, and direct application of Gupta's Theorem similarly gives a relatively weak bound. The problem of determining lower bounds for the cover number of a multigraph with minimal degree δ was also investigated by Xu and Liu [15]. They gave sharp bounds for the cases $\delta = 2, 3, 4, 5$, and asked for a lower bound for general δ. We prove the following.

Theorem 4. *For $k \geq 1$,*

$$f_m(2,k) = \left\lfloor \frac{4k+1}{3} \right\rfloor.$$

Proof. We write $k = 3t + i$, where $i \in \{0, 1, 2\}$. Note that we are claiming $f_m(2,k)$ equals $4t$ if $i = 0$, $4t+1$ if $i = 1$, and $4t+3$ if $i = 2$.

For the lower bound, consider a copy of K_3. If $i = 0$ we give two edges multiplicity $2t$ and one multiplicity $2t-1$ for a total weight of $6t-1$; if $i = 1$, we give all three edges multiplicity $2t$ for a total weight of $6t$; and if $i = 2$, we give all edges weight $2t+1$ for a total weight of $6t+3$. Since each cover uses at least two edges, and the total weight is less than $2k$ is each case, there are not k disjoint vertex-coverings.

Now for the upper bound. Let G be a multigraph with $\delta(G) \geq d = \lfloor (4k+1)/3 \rfloor$. We may assume G is d-regular, or replace it with a d-levelling of G. Now let $V(G) = V_0 \cup V_1$ be a partition with $e(V_0, V_1)$

maximal. Note that every vertex has at least $\lceil d/2 \rceil$ edges on the other side.

Let B be the bipartite graph with vertex classes V_0, V_1. We colour the edges of B with colours $\{1, \dots, k\}$ such that every vertex sees at least $\min\{k, d_B(c)\}$ colours. This is a standard result (see, for instance, Schrijver [12]), but we sketch a proof for completeness: if B is k-regular then the result follows easily from Hall's Theorem; otherwise, for each vertex v of degree greater than k, we replace v by one or more vertices of degree k and a vertex of degree at most k (thus splitting it into several vertices). The resulting bipartite multigraph has all degrees at most k and so has a proper k-edge-colouring (again using Hall); reuniting the split vertices gives the required colouring of B.

We let $H = G[V_0]$, and orient H so that every vertex v has outdegree at least $\lfloor d_H(v)/2 \rfloor$. Now consider $v \in V_0$ and suppose $|\Gamma(v) \cap V_1| = d - j$, so $j \le \lfloor d/2 \rfloor$. If $d - j \ge k$ then v is already adjacent to edges of all colours. Otherwise, v is adjacent to $d - j$ colours; since $d_H(v) = j$, the vertex v has outdegree at least $\lfloor j/2 \rfloor$ in the orientation of H. Now $d - j + \lfloor j/2 \rfloor = d - \lceil j/2 \rceil$, so if $d - \lceil j/2 \rceil \ge k$ then we can use the outedges in H to fill in the missing colours at v.

If $k = 3t$ then $d = 4t$, so $j \le 2t$ and $\lceil j/2 \rceil \le t$, giving $d - \lceil j/2 \rceil \ge 4t - t = k$. If $k = 3t + 1$ then $d = 4t + 1$, so $j \le 2t$ and $\lceil j/2 \rceil \le t$, giving $d - \lceil j/2 \rceil \ge 4t + 1 - t = k$. If $k = 3t + 2$ then $d = 4t + 3$, so $j \le 2t + 1$ and $\lceil j/2 \rceil \le t + 1$, giving $d - \lceil j/2 \rceil \ge (4t + 3) - (t + 1) = k$. We are done. □

Inverting this bound gives an optimal bound for k in terms of δ: if G is a multigraph with minimal degree $\delta \ge 1$ then the cover number of G is at least

$$\left\lfloor \frac{3\delta + 1}{4} \right\rfloor.$$

This answers the question of Xu and Liu [15], by giving an optimal bound for all δ.

4. Hypergraphs

While coverings of graphs have been extensively studied, much less is known about covering hypergraphs. Indeed, Elekes, Mátrai and Soukup leave this as an explicit open problem (Problem 8.2 in [2]). The aim of this section is to give bounds on $f(r, k)$ and $f_m(r, k)$. In particular, we determine $f(r, k)$ and $f_m(r, k)$ to within a constant factor for all r and k.

Theorem 5. *There are constants $C > c > 0$ such that, for $r, k \geq 2$,*

$$ck \log r \leq f(r,k) \leq f_m(r,k) \leq Ck \log r.$$

The lower bound can be found below at (7) and the upper bound in Theorem 6. The lower bound follows by explicit construction. The upper bound follows by a fairly straightforward application of the Lovász Local Lemma. In light of the connection with hypergraph colouring noted in Section 2, this is not surprising; however the fact that k can be much larger than r requires some additional argument.

We turn first to the lower bound. We will give two explicit constructions that give lower bounds. The first construction, based on projective spaces, works for $k \geq r$; the second, using subsets of the cube, works for all k and r. Both give bounds of form $\Omega(k \log r)$.

For $k \geq r$, we have the following construction. For q a prime power and $d \geq 1$, let $P(t,q)$ be the projective space of dimension t over $\mathbb{F}_q$ which has $(q^{t+1}-1)/(q-1)$ points. Let $H(t,q)$ be the hypergraph with vertex set $P(t,q)$ and edges consisting of all 1-codimensional subspaces of $P(t,q)$. Thus every edge in $H(t,q)$ has $r(t,q) = (q^t-1)/(q-1)$ points. The number of s-dimensional subspaces going through any vertex of $P(t,q)$ is

$$\frac{\prod_{i=n-s+1}^{n}(q^i-1)}{\prod_{i=1}^{s}(q^i-1)},$$

so in particular the number of 1-codimensional subspaces through any point is $d(t,q) = (q^t-1)/(q-1)$. Note that $d(t,q) = r(t,q)$, and $H(t,q)$ has

$$|H(t,q)| \cdot \frac{d(t,q)}{r(t,q)} = \frac{q^{t+1}-1}{q-1}$$

edges. Now at least $t+1$ edges are needed to cover the vertices of $P(t,q)$, and so the edges of $H(t,q)$ can be split into no more than

$$\frac{q^{t+1}-1}{(q-1)(t+1)} \leq \frac{qd(t,q)}{t+1}$$

coverings.

Now fix $q = 3$. For $t \geq 1$, we get $r = d = (3^t-1)/2$, and the covering number is at most $3d/(t+1)$. Setting $k = \lfloor 3d/(t+1) \rfloor + 1$, we see that there is a constant $c_1 > 0$ such that for these values of r and k (and all t),

$$f(r,k) \geq c_1 k \log r.$$

We can extend this example to general r and $k \geq r$ as follows. Note first that, given an r-uniform hypergraph H we can easily generate an

$(r+1)$-uniform example by creating a new vertex v and adding v to every edge of H: we shall refer to this as *extending H by v*. Thus

$$f(r+1,k) \geq f(r,k). \tag{3}$$

We would like to be able to extend constructions giving lower bounds for $f(r,k)$ to obtain constructions giving lower bounds for $f(r,sk)$. One situation where we will be able to do this is the following: suppose H is an r-uniform hypergraph with minimum degree at least d, and $S \subset V(H)$ is such that no cover of S uses fewer than T edges. Let us write $d(S)$ for the number of edges of H incident with S. Then clearly the covering number of H is at most $d(S)/T$, and so $f(r,k) > d$ for $k > e(H)/T$. Now for $s \geq 2$, we can consider the multigraph obtained by taking s copies of each edge of H: by the same argument, the covering number of this multigraph is at most

$$sd(S)/T,$$

and so $f_m(r,k) > sd$ for $k > se(H)/T$.

A slight modification of this argument allows us to construct an $(r+1)$-uniform hypergraph (without repeated edges), which can be used to obtain a bound on $f(r+1,k)$. Given an r-uniform hypergraph H with minimal degree at least d, we construct a *(s,d)-expansion* $H^{(s,d)}$ of H as follows: for each edge e of H we delete e and replace it by s new edges of size $r+1$, each obtained by adding a newly created vertex to e. Let H' be the resulting hypergraph: H' is $(r+1)$-uniform, and the old vertices now have degree at least sd, while the newly created vertices all have degree 1. We now take a large number of copies of H' and add edges among the vertices of degree 1 (in any canonical way) to create an $(r+1)$-uniform hypergraph $H^{(s,d)}$ in which all vertices have degree at least sd. Let S be the vertices of H' corresponding to vertices of H. If no cover of H has fewer than T edges, then the covering number of $H^{(s,d)}$ is at most

$$d_{H^{(s,d)}}(S)/T = sd_H(S)/T. \tag{4}$$

Now if we start with $H = H(t,3)$ and consider $H^{(s,d)}$, we see that, for $s \geq 1$ and r, k as above,

$$f(r+1,ks) = \Omega(ks\log r) \tag{5}$$

uniformly in $s \geq 1$ (and t). Since $k(t+1,3)/k(t,3)$ and $r(t+1,3)/r(t,3)$ are both at most 3, it follows from (3) and (5) that

$$f(r,k) = \Omega(k\log r) \tag{6}$$

uniformly in r and $k \geq r$.

For a more general construction (for all r and k), we proceed as follows. Let $d \geq 1$, and let $X = \{A \subset [d] : |A| \geq d/2\}$. For $i = 1, \ldots, d$, let $F_i = \{A \in X : i \in A\}$, and let H be the hypergraph with vertex set X and edge set $\{F_i : i \in [d]\}$. Note that every vertex in H has degree at least $\lceil d/2 \rceil$ (since a vertex $A \in X$ is covered $|A|$ times); on the other hand, no set $\{F_i : i \in B\}$ of at most $d/2$ edges covers X, as $[d] \setminus B \in X$ remains uncovered. Therefore, H has no edge-splitting into two vertex-covers.

Now H has edges of size $r = 2^{d+O(1)}$ and minimal degree $\lceil d/2 \rceil = \Omega(\log r)$. It follows that we obtain a sequence $r_1 < r_2 < \cdots$ with $r_i = 2^{i+O(1)}$ and $\sup(r_{i+1}/r_i) < \infty$ such that $f(r_i, 2) = \Omega(\log r_i)$. It follows from (3) that, for $r \geq 2$,

$$f(r, 2) = \Omega(\log r).$$

But now we can argue as in (4). It follows that, for $k \geq r \geq 2$,

$$f(r, k) = \Omega(k \log r). \tag{7}$$

From the other side, we have the following result (note that the case $r = 2$ is covered by Theorem 4).

Theorem 6. *For some $c > 0$ and every $r \geq 3$ and $k \geq 2$,*

$$f_m(r, k) \leq k \log r + ck \log \log r. \tag{8}$$

Proof. Recall (from the Introduction) that it follows easily from Hall's Theorem that $f_m(r, k) \leq rk$ for every r and k. The error term in (8) therefore allows us to assume $r > R$ for any fixed R. We set $M = \log^2 r / \log \log r$ and $\alpha = \alpha(r) = 5 \log \log r / \log r$. We will consider k and r in two ranges.

Case 1: $k \leq M$: Let $d = \lceil (1 + \alpha) k \log r \rceil$. We show that $f(k, r) \leq d$. Given an r-uniform hypergraph H_0 with minimal degree at least d, we first find an (r, d)-levelling H_1 of H_0. By Proposition 1 it is sufficient to show that $E(H_1)$ can be partitioned into k covering sets. Let $c : E(H_1) \to [k]$ be a random k-colouring, where each edge is coloured independently and uniformly at random. We will apply the Lovász Local Lemma to show that, with positive probability, every colour class is a covering set.

For each vertex v of H_1, let A_v be the event that v is not incident with a vertex of every colour. Then

$$p := \mathbb{P}(A_v) \leq k(1 - 1/k)^d \leq k e^{-d/k} \leq k / r^{1+\alpha}.$$

Now let G be the graph with vertex set $V(H_1)$ and edges between every pair of vertices that belong to a common edge of H_1. Then

$\Delta(G) \le d(r-1) < 2kr\log r$, and G is a dependency graph for the events A_v, so the Lovász Local Lemma gives our result immediately, as, for sufficiently large r,

$$ep(\Delta+1) \le \frac{ek}{r^{1+\alpha}} \cdot 2kr\log r \le \frac{6k^2\log r}{r^\alpha} \le \frac{6M^2\log r}{\log^5 r} < 1,$$

since $r^\alpha = (\log r)^5$.

Case 2: $k > M$: Let $d = \lceil (1+\lambda\alpha)k\log r\rceil$, where λ is a large constant. Let H_0 be an r-uniform hypergraph with minimal degree at least d. As before, we first find a (r,d)-levelling H_1 of H_0. We now repeatedly (and recursively) use the Lovász Local Lemma to split our problem into two subproblems, until we obtain problems small enough to apply Case 1.

Let $\Lambda = 4\sqrt{d\log(rd)}$, and consider a random 2-colouring of $E(H_1)$ with colours red and blue, with each edge independently given each colour with probability 1/2. For each vertex v, Chernoff's Inequality implies that the number of red edges incident with v is at least $d/2-\Lambda$ with failure probability at most $\exp(-\Lambda^2/2d)$. Let A_v be the event that v is incident with at least $d/2-\Lambda$ edges of each colour. We can define a dependency graph G as above, with maximum degree at most $(r-1)d$. Since $\exp(-\Lambda^2/2d) < 1/4rd$, it follows from the Lovász Local Lemma that there is some colouring such that every vertex is incident with at least $d/2-\Lambda$ edges of each colour. We now split $[k]$ into two sets of size $\lfloor k/2\rfloor$ and $\lceil k/2\rceil$, and assign one set to each colour: this gives two subproblems.

We now repeat the argument, recursively splitting each problem into two subproblems. Let $d_0 = d$ and, for $i \ge 0$, $d_{i+1} = d_i - 4\sqrt{d_i\log(rd_i)}$; let I be minimal such that $k/2^I < 2M/3$, and let $D = D/2^I$. Then, for $i \le I$, at the ith stage we have 2^i subproblems where in each case we have an r-uniform hypergraph of minimal degree at least d_i, and want to split into (at most) $\lceil k/2^i\rceil$ disjoint coverings. If $i < I$, we take a d_i-levelling of each, split each subproblem into two as above, and repeat.

At the ith stage we have

$$d_i \ge \frac{d}{2^i}\prod_{j=0}^{i-1}\left(1-4\sqrt{\frac{\log(rd_j)}{d_j}}\right) \ge \frac{d}{2^i}\left(1-\sum_{j=0}^{i-1}4\sqrt{\frac{\log(rd_j)}{d_j}}\right).$$

Now $d_{i+1}/d_i > 3/2$ for each j and $\sqrt{\log(rx)/x}$ is decreasing in x, so provided $d_i \geq M$, we have

$$\sum_{j=0}^{i-1} 4\sqrt{\log(rd_i)/d_i} = O(\sqrt{\log M/M}) = O(\log\log r/\log r) = O(\alpha).$$

Since $k_{i-1} > M/4$ for each i, it follows by induction that $d_i > M$ for each i, and so $d_i > (1-\alpha)d/2^i$ for each i. Thus we have

$$d_I/k_I \geq (d/k)(1-O(\alpha))(1+\lambda\alpha)\log r \geq (1+\alpha)\log r,$$

provided r is large enough. We can now apply the argument from Case 1 to each subproblem. □

Finally, we note that any bounds on $f(r,k)$ or $f_m(r,k)$ extend immediately to the infinite case. Indeed, given any positive integer d, and an infinite r-uniform hypergraph H, we can construct an (r,d)-levelling by well-ordering the edges and then proceeding as before; bounds on the covering number then extend easily to the infinite case by a compactness argument.

5. Small numbers

For $r = 2$, we know from Gupta's theorem (or Lemma 3 above) that $f(2,k) = k+1$. For $r \geq 3$, determining $f(r,k)$ exactly appears considerably more difficult. In this section we determine values for $f(r,k)$ in a couple of small cases.

We use explicitly the connection with hypergraph 2-colouring noted in section 2. Hypergraph 2-colouring for small values of k and r has been considered by several authors, including McDiarmid [9].

Lemma 7. $f(3,2) = 4$.

Proof. The Fano plane shows that 3-regular, 3-uniform hypergraphs need not be 2-colourable, and its dual therefore shows that $f(3,2) > 3$. On the other hand, Seymour [11] showed that a minimal non-2-colourable hypergraph $G = (V,E)$ must have $|E| \geq |V|$, and so every 3-regular, 4-uniform hypergraph is 2-colourable: it follows by duality that $f(3,2) \leq 4$. □

Lemma 8. $f(4,2) = 4$.

Proof. The lower bound $f(4,2) \geq 4$ follows by monotonicity. On the other hand, Thomassen ([13]; see also [9]) showed that every 4-regular, 4-uniform hypergaph is 2-colourable; it follows by duality that $f(4,2) \leq 4$. □

It turns out that we can also determine the corresponding quantities for multigraphs. Recall that, for graphs, we know from section 3 that f and f_m may take different values. However, for the special case $k = 2$, it makes no difference if we allow repeated edges.

Lemma 9. *For $r \geq 1$, $f(r, 2) = f_m(r, 2)$.*

Proof. Let H be a multihypergraph with minimal degree at least $d = f(2, k)$. If any edge is repeated, we give one copy colour red and and other copy colour blue. Note that all vertices belonging to repeated edges are now covered in both colours. Let V_1 be the set of all vertices that are not yet covered. Let H_1 be the hypergraph with vertex set V' and one edge $e \cap V_1$ for each edge of H that meets V_1. If H_1 has repeated edges, we repeat the process (colouring red and blue, and restricting to a smaller vertex set of uncovered vertices) to obtain H_2. Repeating, we eventually obtain a hypergraph H_k with no repeated edges. Note that H_k has minimal degree at least d, so let H' be a (r, d)-levelling of H_k. Then H' is a simple d-regular hypergraph and so the edges can be split into k covers, as $d = f(2, k)$. It follows that the edges of H_k and therefore also of H can be split into k covers, and so $f_m(2, k) = f(2, k)$. □

We immediately have the following corollary to Lemmas 7 and 8.

Corollary 10. $f_m(3, 2) = f_m(4, 2) = 4$.

It would be interesting to determine f or f_m exactly for other pairs (r, k) with $r \geq 3$.

6. Conclusion

The quantities $f(r, k)$ and $f_m(r, k)$ are not in general the same. Indeed, for graphs, we have seen in section 3 that $f_m(2, k) - f(2, k) \sim k/3$, so in general f and f_m take different values. However, for the special case $k = 2$, Lemma 9 shows that f and f_m are identical. It would be interesting to know more generally how the two parameters differ. For instance, is it true that for every $r \geq 3$, we have $f_m(r, k) - f(r, k) \to \infty$ as $k \to \infty$? Or could it be the case that $f_m(r, k) = f(r, k)$ for sufficiently large r and k (or for fixed $r \geq 3$ and sufficiently large k)?

It is also natural to consider multicoloured versions of the problem. In order to simplify the formulation, we first reformulate the definition of $f_m(r, k)$ in as follows: let $\mathcal{G}(r, d)$ be the collection of bipartite graphs $G = G(A, B; E)$ with vertex partition (A, B) such that every vertex in A has degree at least d and every vertex in B has degree at most r. Then $f_m(r, k)$ is the smallest d such that, for every graph in $\mathcal{G}(r, d)$

there is a partition of B into k sets, each of which covers A (i.e. every vertex in A has a neighbour in each of the sets).

We define multicolour versions of this as follows. For $r, k, t \geq 1$, let $\mathcal{G}_1(r, d, t)$ be the collection of bipartite graphs $G = G(A, B; E)$ with vertex partition (A, B) and edge-colouring $c : E \to [t]$ such that every vertex in A is incident with at least d edges in every colour and every vertex in B has degree at most r. Let $\mathcal{G}_2(r, d, t)$ be the collection of bipartite graphs $G = G(A, B; E)$ with vertex partition (A, B) and edge-colouring $c : E \to [t]$ such that every vertex in A is incident with at least d edges in every colour and every vertex in B is incident with at most r edges in each colour.

For integers $r, k, t \geq 1$, let $g(r, k, t)$ be the smallest integer such that for every (coloured) graph $G = G(A, B; E)$ in $\mathcal{G}_1(r, d, t)$ there is a partition of B into k sets, such that every vertex of A sends an edge of every colour to every set; similarly let $h(r, k, t)$ be the same for $\mathcal{G}_2(r, d, t)$. It would be interesting to determine bounds on g and h. Closely related to the case $r = t = 2$ is the following question. Given $k \geq 2$, what is the smallest d such that every directed graph with minimal indegree and outdegree at least d has an edge-partition into k sets, each of which is the edge set of a spanning digraph with no sources or sinks?

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Béla Bollobás, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK; *and* Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA

Alex Scott, Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, UK

E-mail address: `scott@maths.ox.ac.uk`