A HAM SANDWICH ANALOGUE FOR QUATERNIONIC MEASURES AND FINITE SUBGROUPS OF S^3

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ABSTRACT. A "ham sandwich" theorem is established for n quaternionic Borel measures on quaternionic space $\mathbb{H}^n$. For each finite subgroup G of S^3, it is shown that there is a quaternionic hyperplane H and a corresponding tiling of $\mathbb{H}^n$ into $|G|$ fundamental regions which are rotationally symmetric about H with respect to G, and satisfy the condition that for each of the n measures, the "G average" of the measures of these regions is zero. If each quaternionic measure is a 4-tuple of finite Borel measures on $\mathbb{R}^{4n}$, the original ham sandwich theorem on $\mathbb{R}^{4n}$ is recovered when $G = \mathbb{Z}_2$. The theorem applies to $\lfloor \frac{n}{4} \rfloor$ finite Borel measures on $\mathbb{R}^n$, and when G is the quaternion group Q_8 this gives a decomposition of $\mathbb{R}^n$ into 2 rings of 4 cubical "wedges" each, such that the measure any two opposite wedges is equal for each finite measure.

1. INTRODUCTION

The familiar ham sandwich theorem states that given n finite Borel measures $\mu_1, \ldots, \mu_n$ on $\mathbb{R}^n$, there exists a hyperplane $\mathrm{H} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{a} = b\}$ $(\mathbf{a} \neq 0, b \in \mathbb{R})$ bisecting each measure: $\mu_i(S^+) = \mu_i(S^-) = \frac{1}{2}\mu_i(\mathbb{R}^n)$ for each i, where $S^+ = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{a} \geq 0\}$ and $S^- = \{\mathbf{x} \mid \mathbf{x} \cdot \mathbf{a} \leq b\}$ are the half-spaces corresponding to H. Equivalently, one can say that for the group $\mathbb{Z}_2 = \{\pm 1\}$, the "$\mathbb{Z}_2$ average" $\mu_i(S^+) - \mu_i(S^-)$ of the measures of the half-spaces is zero for each measure μ_i. Thus there is a simultaneous $\mathbb{Z}_2$ symmetry (equality) of the measures of the two half-spaces, which corresponds to the $\mathbb{Z}_2$ symmetry on each pair of half-spaces given by the free and transitive action of reflection about each pair's common hyperplane.

Similarly, the $\mathbb{Z}_m$ ham sandwich theorem for complex measures [9] states that given n complex valued Borel measures $\mu_1, \ldots, \mu_n$ on $\mathbb{C}^n$ (see, e.g., [6] and [9]) and any integer $m \geq 2$, there exists a complex hyperplane (with respect to the standard hermitian inner product on $\mathbb{C}^n$) H$= \{\mathbf{z} \in \mathbb{C}^n \mid < \mathbf{z}, \mathbf{a} >= b\}$ $(\mathbf{a} \neq 0, b \in \mathbb{C})$ and m corresponding regular "$\frac{1}{m}$" sectors $\mathcal{S}_0, \ldots, \mathcal{S}_{m-1}$, $\mathcal{S}_k = \{\mathbf{z} \mid < \mathbf{z}, \mathbf{a} >= b + re^{i\theta}; r \geq 0, \theta \in [\frac{2\pi k}{m}, \frac{2\pi(k+1)}{m}]\}$, whose "$\mathbb{Z}_m$ average" $\sum_{k=0}^{m-1} \zeta_m^{-k} \mu_i(\mathcal{S}_k) \in \mathbb{C}$ is zero for each complex measure μ_i, $\zeta_m = e^{\frac{2\pi i}{m}}$. Again, the theorem shows a simultaneous $\mathbb{Z}_m$ symmetry of the measures of these regular sectors, which corresponds to the $\mathbb{Z}_m$ symmetry on each set of regular $\frac{1}{m}$ sectors given by the free and transitive action which rotates the sectors by multiples of $\frac{2\pi}{m}$ about their common complex hyperplane H.

In both theorems, one needs to assume that the measures are "proper". In the real case, this means that the measure of any hyperplane is zero, and in the complex case this means that any "real hyperplane" (one that is a hyperplane in $\mathbb{R}^{2n}$ under the canonical identification with $\mathbb{C}^n$) is a null set.

Note that $\mathbb{Z}_2$ is the the 0-dimensional sphere $S^0 \subseteq \mathbb{R}$, and the only finite subgroups of $S^1 \subseteq \mathbb{C}$ are precisely the subgroups $\mathbb{Z}_m = \{\zeta_m^k \mid 0 \leq k < m\}$.

We will find analogous results for n proper, quaternionic valued Borel measures (see section 6) $\mu_1, \ldots, \mu_n$ on quaternionic space $\mathbb{H}^n$ and finite subgroups of the unit sphere $S^3 \subseteq \mathbb{H}$. For each finite subgroup $G \leq S^3$ and each quaternionic hyperplane H (section 4), we partition $\mathbb{H}^n$ into $|G|$ fundamental regions $\mathcal{R}_g$, $g \in G$, which are symmetric about their common quaternionic hyperplane H via a free and transitive rotational G-action. When G is cyclic, these regions are regular $\frac{1}{m}$ sectors corresponding to a codimension 2 real affine space (complex hyperplane); for non-cyclic G, the regions are "polyhedral wedges", the sum of the codimension 4 affine space H and a cone in its orthogonal complement $\mathrm{H}^{\perp}$ on rotationally isometric uniform 3-dimensional polyhedra depending on the group G.

Taking the measure $\mu_\ell(\mathcal{R}_g)$ for each g, multiplying this on the left by g^{-1} and summing over G, we obtain a "G average" $\sum_{g\in G} g^{-1}\mu_\ell(\mathcal{R}_g) \in \mathbb{H}$ of the measures of the $\mathcal{R}_g$. Left multiplication of G on $\mathbb{H}$ has a rotational interpretation (see section 2), so the average can be seen as measuring a type of rotational symmetry of the measures of the $\mathcal{R}_g$. Similarly, one can multiply the measures of the $\mathcal{R}_g$ on the right by g^{-1}, thereby obtaining another G average $\sum_{g\in G} \mu_\ell(\mathcal{R}_g)g^{-1}$, which again can be seen as a rotational average of the measures of the regions.

It will be shown that for each G, there exists some quaternionic hyperplane and corresponding regions for which the "left" average above is zero for each measure μ_ℓ, thereby expressing a simultaneous quaternionic rotational symmetry of the $\mu_\ell(\mathcal{R}_g)$ for each μ_ℓ, and likewise that there exists some hyperplane and fundamental regions for which the "right" average is zero for each measure. In [9], it was assumed that $\mu_\ell(\mathbb{C}^n) \neq 0$ for at least one of the n complex measures, and we will make the analogous assumption that $\mu_\ell(\mathbb{H}^n) \neq 0$ for some μ_ℓ.

Theorem 1: Given a non-trivial finite subgroup G of S^3 and n proper, quaternionic Borel measures $\mu_1, \ldots, \mu_n$ on $\mathbb{H}^n$ as above, there exists a quaternionic hyperplane and corresponding fundamental G-regions $\mathcal{R}_g, g \in G$, satisfying

$$\sum_{g\in G} g^{-1}\mu_\ell(\mathcal{R}_g) = 0 \tag{1}$$

for each measure μ_ℓ. Likewise, there is a quaternionic hyperplane and corresponding G-regions satisfying

$$\sum_{g\in G} \mu_\ell(\mathcal{R}_g)g^{-1} = 0 \tag{2}$$

for each μ_ℓ.

We begin by providing a self-contained discussion of the geometry of the quaternions and the classification of finite subgroups of its unit sphere S^3. Corresponding to each subgroup, we find a canonical partition of $\mathbb{H}$ into fundamental regions which allow us to to tile $\mathbb{H}^n$ into the desired regions $\mathcal{R}_g$.

2. Quaternions and Finite Subgroups of S^3

Recall the Quaternions are the number system $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$, where i, j, and k satisfy the relations $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$, and $ij = -ji, jk = -kj, ki = -ik$. $\mathbb{H}$ is a non-commutative ring, where addition is defined component-wise and multiplication is defined by the distributive property and using the relations above. By analogy with $\mathbb{C}$, one can decompose each quaternion $u = a + bi + cj + dk$ into real and imaginary parts $Re(u) = a$ and $Im(u) = bi + cj + dk$, and likewise there is a notion of conjugates. To each $u \in \mathbb{H}$, one defines its conjugate $\bar{u} = a - bi - cj - dk$, so $\bar{u} = Re(u) - Im(u)$, and it is clear that $\overline{uv} = \bar{v}\bar{u}$. The conjugate affords the norm $|u| = \sqrt{u\bar{u}} = \sqrt{a^2 + b^2 + c^2 + d^2}$ on $\mathbb{H}$, so that each $u \neq 0$ has $u^{-1} = \frac{\bar{u}}{|u|^2}$ as its multiplicative inverse and $\mathbb{H}$ is a skew field. Viewing each $u \in \mathbb{H}$ as a 4-tuple of real numbers, $|u|$ is the Euclidian norm on $\mathbb{R}^4$ and the unit sphere S^3 (also called the unit quaternions) is precisely the set of elements of $\mathbb{H}$ of norm 1. The norm is multiplicative, i.e., $|uv| = |u||v|$ for each $u, v \in \mathbb{H}$, and restricting multiplication in $\mathbb{H}$ to S^3 shows that S^3 is a group.

In what follows, it will be useful to describe unit quaternions in terms of "polar coordinates" (see, e.g., [2] or [5]). To begin, the purely imaginary quaternions may identified with $\mathbb{R}^3$, with S^2 the imaginary quaternions of norm 1. For $x \in S^2$, $1 = x\bar{x} = x(-x)$ and $x^2 = -1$, so the quaternions $\{a + bx \mid a, b \in \mathbb{R}\}$, which we shall call the $(1, x)$-plane, form a field isomorphic to $\mathbb{C}$.

For each $x \in S^2$ and $\theta \in [0, 2\pi)$, one defines the "Euler formula" $e^{\theta x} = \cos\theta + \sin\theta x$. It is easily verified that each $u \in S^3$ can be put in this form, and this expression is unique when $u \neq \pm 1$. Just as in $\mathbb{C}$, it follows that multiplication on either the right or left by $e^{\theta x}$ rotates the $(1, x)$-plane by θ. We now describe multiplication in $\mathbb{H}$ geometrically.

For each $u = e^{\theta x} \in S^3$, let $r_u : \mathbb{H} \longrightarrow \mathbb{H}$ be the right multiplication map ("left screw") sending each $v \in \mathbb{H}$ to vu. Viewing $\mathbb{H}$ as $\mathbb{R}^4$, r_u is the rotation of $\mathbb{R}^4$ which rotates the $(1, x)$-plane by θ and rotates the plane orthogonal to it by $-\theta$: For each $x^\perp \in S^2$ orthogonal in $\mathbb{R}^3$ to x, $xx^\perp \in S^2$ is the usual cross-product in $\mathbb{R}^3$, hence is orthogonal in $\mathbb{R}^3$ to both x and $x^\perp$, so the plane generated by $x^\perp$ and $xx^\perp$ is orthogonal in $\mathbb{R}^4$ to the $(1,x)$-plane. As $xx^\perp = -x^\perp x$, $x^\perp e^{\theta x} = e^{-\theta x} x^\perp$, so multiplication on the right by $e^{\theta x}$ rotates the $(x^\perp, xx^\perp)$-plane by $-\theta$ (see, e.g., [5]). In a similar fashion, the left multiplication map ("right screw") l_u rotates both the $(1,x)$-plane and its orthogonal complement by θ.

In order to study the finite subgroups of S^3, we will need to examine the 2-fold covering group homomorphism of the special orthogonal group $SO(3)$ by S^3. By linear algebra, each element of $SO(3)$ is a rotation of $\mathbb{R}^3$ by some angle θ about some point $x \in S^2$ (equivalently, a rotation about $-x$ by $-\theta$), and the pair $\pm(x,\theta)$ representing a rotation is unique when the rotation is not the identity.

For each $u \in S^3$, the conjugation map $\varphi_u : \mathbb{H} \longrightarrow \mathbb{H}$ given by $\varphi_u(v) = uvu^{-1}$ is the composition $l_u \circ r_{u^{-1}}$. Hence for $u = e^{\theta x}$, the above discussion shows that φ_u fixes the $(1,x)$-plane and rotates its orthogonal complement by 2θ. Restricting φ_u to $\mathbb{R}^3$, $\varphi_u \in SO(3)$ is the rotation of $\mathbb{R}^3$ about x by 2θ (see, e.g., [5]).

By the properties of conjugation, the map $\varphi : S^3 \longrightarrow SO(3)$, $u \mapsto \varphi_u$, is a homomorphism, which is 2 to 1 because $u = e^{\theta x}$ and $-u = e^{(\theta+\pi)x}$ define the same element of $SO(3)$. By the characterization of φ_u, $ker\varphi = \{\pm 1\}$, and φ is surjective by the characterization of $SO(3)$, so $S^3/\{\pm 1\} \cong SO(3)$ by the first isomorphism theorem. In fact, φ is a smooth covering map, which shows that S^3 is the universal cover $Spin(3)$ of $SO(3)$, and that real projective space $\mathbb{R}P^3 = S^3/\{\pm 1\}$ and $SO(3)$ are diffeomorphic smooth manifolds.

Using the double cover φ, one can classify the finite subgroups of S^3 as the pullbacks of those of $SO(3)$. It is a very classical result that the only finite subgroups of latter are of the following isomorphism type (see, e.g., [1]):

1) The cyclic groups C_m, consisting of the rotational symmetries of the regular m-gon $[j, \zeta_m j, \ldots, \zeta_m^{m-1} j]$ in the (j,k) plane of $\mathbb{R}^3$, i.e., rotations through i by multiples of $\frac{2\pi}{m}$.

2) The Dihedral groups D_m, consisting of all the $2m$ symmetries of the regular m-gon above: the rotations C_m, as well as rotations by π about each of the m lines ℓ in the (j,k)-plane bisecting the m-gon.

3) The rotational symmetries of the 5 Platonic Solids:

a) The tetrahedral group T, the 12 rotational symmetries of the regular tetrahedron $[\frac{1}{\sqrt{3}}(i+j+k), \frac{1}{\sqrt{3}}(-i-j+k), \frac{1}{\sqrt{3}}(-i+j-k), \frac{1}{\sqrt{3}}(i-j-k)]$: rotations by multiples of $\frac{2\pi}{3}$ about its vertices, and by π through the center of pairs of opposite edges (i.e., through i, j and k). D_2 consists of these last three rotations and the identity, so it is an index 3 subgroup of T.

b) The octahedral group O, the 24 rotational symmetries of the regular octahedron $[\pm i, \pm j, \pm k]$, and equivalently of its normalized dual cube $[\frac{1}{\sqrt{3}}(\pm i \pm j \pm k)]$: rotations by multiples of $\frac{\pi}{2}$ through each pair of opposite vertices of the octahedron, by multiples of $\frac{2\pi}{3}$ about pairs of opposite vertices of the cube, and by π through opposite pairs of edges. The tetrahedron above is contained in this cube, and looking at the elements listed shows that T is an index 2 subgroup of O.

c) The icosahedral group I, the 60 rotational symmetries of the regular icosahedron $[\frac{1}{\sqrt{2+\tau}}(\pm\tau i \pm j), \frac{1}{\sqrt{2+\tau}}(\pm\tau j \pm k), \frac{1}{\sqrt{2+\tau}}(\pm i \pm \tau k)]$, and equivalently of its normalized dual dodecahedron $[\frac{1}{\sqrt{3}}(\pm i \pm j \pm k), \frac{1}{\sqrt{3}}(\pm\tau^{-1} i \pm \tau j),$ $\frac{1}{\sqrt{3}}(\pm\tau^{-1} j \pm \tau k), \frac{1}{\sqrt{3}}(\pm\tau i \pm \tau^{-1} k)]$ (see, e.g., [3]), where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio: rotations by multiples of $\frac{2\pi}{5}$ about pairs of opposite vertices of the icosahedron, by multiples of $\frac{2\pi}{3}$ through pairs of opposite vertices of the dodecahedron, and by π through pairs of opposite edges. The cube above sits inside this dodecahedron, and correspondingly O is an index 5 subgroup of I.

We can now classify the finite subgroups G of S^3. [2], [4], and [5] give particularly nice accounts of these groups.

If $-1 \notin G$, then φ is injective and $G \cong \varphi(G)$. For $u = e^{\theta x}$, φ_u cannot be a rotation by π, for otherwise $\theta = \frac{\pi}{2}$, so that $u = e^{\frac{\pi}{2}x} = x \in S^2$ and $u^2 = -1 \in G$. Looking at 1) through 3), the only subgroups of $SO(3)$ which contain no rotations by π are the cyclic groups of order, so $\varphi(G) = C_m = \{a_m^p \mid 0 \le p < m\}$, where a_m is the rotation of i by $\frac{2\pi}{m}$. As m is odd, pulling back by φ gives $G = \{\zeta_m^p \mid 0 \le p < m\} = \mathbb{Z}_m$.

If $-1 \in G$, G consists of those $\pm e^{\frac{\theta}{2}x}$ for which the rotation by θ about x is in $\varphi(G)$, so $|G| = 2|\varphi(G)|$ and hence is "binary" to one of the groups in the above list. If $\varphi(G)$ is cyclic of order m, reasoning as above gives $G = \{\zeta_{2m}^p \mid 0 \le p < 2m\} = \mathbb{Z}_{2m}$. In the remaining cases, G is non-cyclic and is called a binary polyhedral group.

Assume that $\varphi(G) = D_m$. Each rotation by π about a line ℓ in (2) can be expressed as the composition of the rotation a_m^p and the rotation b about j by π, so $D_m = \{a_m^p b^q \mid 0 \le p < m, 0 \le q \le 1\}$. Pulling D_m back by φ, G is the binary dihedral group

$$D_m^* := \{\zeta_{2m}^p j^q \mid 0 \le p < 2m, 0 \le q \le 1\} = \mathbb{Z}_{2m} \cup \mathbb{Z}_{2m} j. \tag{3}$$

When $m = 2$, $D_2^* = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group Q_8, and for this reason the D_m^* are also called generalized quaternions and denoted Q_{4m}.

In the case that $\varphi(G)$ is one of the rotation groups of the Platonic solids, G is the binary tetrahedral group $T^* := \varphi^{-1}(T)$ of order 24, the binary octahedral group $O^* := \varphi^{-1}(O)$ of order 48, or the binary icosahedral group $I^* = \varphi^{-1}(I)$ of order 120.

Pulling the subgroup $D_2 \leq T$ back by φ reveals $D_2^* = Q_8$ as a subgroup of T^* of index 3, and the other 16 elements of T^* are the pullbacks of the 8 3-cycles of T. For instance, if a is the rotation of $\frac{1}{\sqrt{3}}(i+j+k)$ by $\frac{2\pi}{3}$, then $\varphi^{-1}(a) = \{\pm a^*\}$, where $a^* = \cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3})\frac{(i+j+k)}{\sqrt{3}} = \frac{1}{2}(1+i+j+k)$. Explicitly,

$$T^* = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\} = \cup_{r=0}^{2} Q_8 a^{*r}. \qquad (4)$$

As T is an index 2 subgroup of O, T^* is an index 2 subgroup of O^*. Letting $b^* = \frac{1}{\sqrt{2}}(1+i)$, $\varphi(b^*)$ is the rotation about i by $\frac{\pi}{2}$, an element of $O - T$, so $O^* - T^* = T^* b^*$ is a rotated copy of T^* and

$$O^* = T^* \cup T^* b^*. \qquad (5)$$

Let σ be the 5-cycle of I rotating about $\frac{1}{\sqrt{2+\tau}}(\tau i + j)$ by $\frac{2\pi}{5}$. As $\cos(\frac{\pi}{5}) = \frac{1}{2}\tau$ and $1 + \tau^{-1} = \tau$, $\sigma^* = \cos(\frac{\pi}{5}) + \sin(\frac{\pi}{5})(\frac{1}{\sqrt{2+\tau}}(\tau i + j)) = \frac{1}{2}(\tau + i + \tau^{-1} j)$ is the pullback (along with $-\sigma^*$) of σ, and since I is the union of the cosets $T\sigma^r$, $0 \leq r \leq 4$,

$$I^* = \cup_{r=0}^{4} T^* \sigma^{*r}. \qquad (6)$$

3. Tiling $\mathbb{H}$ by Finite Subgroups of S^3

For each finite subgroup $G \leq S^3$, we will partition of $\mathbb{H}$ into non-overlapping, contractible regions R_g, $g \in G$. In each case, the boundaries of the R_g will be contained in a union of hyperplanes of $\mathbb{R}^4$, and G will act freely and transitively on the R_g by multiplication on the right and left (equivalently, as left or right screws): $R_{g_1} g_2 = R_{g_1 g_2}$ and $g_1 R_{g_2} = R_{g_1 g_2}$ for $g_1, g_2 \in G$, and in particular each $R_g = R_1 g = g R_1$ will be a rotationally isometric copy of R_1. Such a decomposition will be called a G-tiling, and the R_g will be called fundamental G-regions. Each $R_g = cone(0, C_g) = \cup_{r \geq 0} r C_g$ will be the cone on a region C_g, with the C_g forming a G-tiling of a topological S^3.

3.1. $G = \mathbb{Z}_m$. As $ij = k$, $\mathbb{H} = \mathbb{C} \times \mathbb{C}j$, and multiplication of $\mathbb{H}$ on the right (left) by ζ_m^p corresponds to rotating the first coordinate by $\frac{2\pi p}{m}$ ($\frac{2\pi p}{m}$) and the second by $-\frac{2\pi p}{m}$ ($\frac{2\pi p}{m}$). Fixing the circle $0 \times S^1$ and dividing $S^1 \times 0$ into the closed arcs a_p from $(\zeta_m^p, 0)$ to $(\zeta_m^{p+1}, 0)$, $0 \leq p < m$, let $C_{\zeta_m^p}$ be the 3-dimensional disk ("lens") formed by taking the union of the great circle arcs $\alpha(u,v)(t) = (\cos(t)u, \sin(t)v), t \in [0, \frac{\pi}{2}]$, from $(u, 0) \in a^p$ to $(0, v) \in 0 \times S^1$. Fixing u, these arcs form a disk D_u^2 with boundary $0 \times S^1$, and $C_{\zeta_m^p} = \cup_u D_u^2$. In particular, $\partial C_{\zeta_m^p}$ is the topological S^2 that is the union of the two caps $D^2_{\zeta_m^p}$ and $D^2_{\zeta_m^{p+1}}$ which form the "top" and "bottom" of the lens. As S^3 is the union of great circles from $S^1 \times 0$ to $0 \times S^1$, the $C_{\zeta_m^p}$ cover S^3, and their interiors are disjoint. Multiplying $C_{\zeta_m^p}$ on either the right or left by ζ_m^q rotates $C_{\zeta_m^p}$ to $C_{\zeta_m^{p+q}}$, so the $C_{\zeta_m^p}$ constitute a $\mathbb{Z}_m$-tiling of S^3 (see, e.g., [4], [5] and [7]).

Equivalently, the $C_{\zeta_m^p}$ are the points of S^3 whose argument in the first coordinate lies in $[\frac{2\pi p}{m}, \frac{2\pi(p+1)}{m}]$, so that using the standard hermitian inner product on $\mathbb{C}^2$, $R_{\zeta_m^p} = \cup_{r\geq 0} rC_{\zeta_m^p} = S_p \times \mathbb{C}j = < e_1 >^\perp + S_p e_1$, where S_p is the closed sector $\{re^{i\theta} \in \mathbb{C} \mid r \geq 0, \theta \in [\frac{2\pi p}{m}, \frac{2\pi(p+1)}{m}]\}$ in $\mathbb{C}$ and $e_1 = (1, 0)$. Thus the $R_{\zeta_m^p} = \{(z, w) \mid < (z, w), e_1 > \in S_p\} = \mathcal{S}_p$ are the regular $\frac{1}{m}$ sectors corresponding to the complex hyperplane $< e_1 >^\perp$. The boundary $\partial S_p = \{re^{\frac{2\pi p i}{m}} \mid r \geq 0\} \cup \{re^{\frac{2\pi(p+1)i}{m}} \mid r \geq 0\}$ of S_p is the union of half lines in $\mathbb{C}$, so $\partial R_{\zeta_m^p} = < e_1 >^\perp + \partial S_p e_1$ is the union of two half-hyperplanes in $\mathbb{R}^4$. In particular, R_1 and R_{-1} are the half-spaces corresponding to the hyperplane $(0, 1, 0, 0)^\perp$ in $\mathbb{R}^4$ when $m = 2$.

3.2. **Binary Polyhedral Groups.** For each binary polyhedral group G, there is a canonical 4-dimensional-polytope P_G whose boundary triangulates S^3. In each case, $P_G = Conv(G)^*$, the dual of the convex hull $Conv(G)$, whose vertices are the elements of the group G. Thus for each $g \in G$, we look at the translated tangent space $H_g = T_g S^3 + g = < g >^\perp + g = \{w \in \mathbb{R}^4 \mid w \cdot g = 1\}$ centered at g and the corresponding closed half-space $S_g^- = \{w \mid w \cdot g \leq 1\}$. $P_G = \cap_{g\in G} S_g^-$, and the boundary ∂P_G provides a triangulation of S^3 into $|G|$ interior disjoint (as we shall see, uniform) 3-dimensional polyhedra $C_g = H_g \cap P_G$ with center g. For a more detailed exposition of the P_G than given below, as well as for a discussion of $Conv(G)$, [2], [4], and [5] are recommended reading.

As the norm on $\mathbb{H}$ is multiplicative, multiplication on the left or right by a fixed $u \in S^3$ preserves the inner product on $\mathbb{R}^4$, and since G is a group, it follows easily that $H_{g_1} g_2 = H_{g_1 g_2}$ and $g_1 H_{g_2} = H_{g_1 g_2}$ for $g_1, g_2 \in G$, and in particular that $H_1 g = H_g = gH_1$ for each g. $C_g = H_g \cap P_G$, so G acts freely and transitively on the C_g by right or left multiplication, and since each r_g (and l_g) is a rotation, each C_g is a rotated copy of the polyhedron C_1.

The G-regions R_g are the cones on these three-dimensional polyhedra C_g whose apex is the origin and whose cross-section bases are scaled copies of the C_g. The boundary ∂C_g of each uniform polyhedra C_g is a union of regular polygons, so ∂R_g is the union of the cones of these planar polygons and hence is contained in a union of hyperplanes in $\mathbb{R}^4$.

3.2.1. $G = D_m^*$. The elements of D_m^* (3) form the vertices of two mutually orthogonal regular $2m$-gons $P_{2m}{=}[1, \zeta_{2m}, \ldots, \zeta_{2m}^{2m-1}]$ and $P_{2m}j{=}[j, \zeta_{2m}j, \ldots, \zeta_{2m}^{2m-1}j]$. Let P_{2m}^* be the regular $2m$-gon dual to P_{2m}, so in particular $P_{2m}^* j$ is dual to $P_{2m}j$. Solving the equations defining the dual of $Conv(D_m^*)$, one sees that $P_{D_m^*}{=}P_{2m}^* \times P_{2m}^* j$ is the product of these $2m$-gons. Therefore, $\partial P_{D_m^*}$ is composed of $2m$ regular prisms C_g, each of which is the product of one of these dual regular $2m$-gons with the edge of the other, and S^3 is triangulated as $T_{2m} \cup T_{2m}j$, the union of two solid tori $T_{2m} = \partial P^*_{2m} \times P^*_{2m}j$ and $T_{2m}j = P^*_{2m} \times \partial P^*_{2m}j$ with common boundary $\partial P^*_{2m} \times \partial P^*_{2m}j$. The first solid torus $T_{2m} = \cup_{p=0}^{2m-1} C_{\zeta^p_{2m}}$ is formed by stacking each prism $C_{\zeta^{p+1}_{2m}}$ onto the adjacent prism $C_{\zeta^p_{2m}}$ along their common $2m$-gon face, thereby forming a ring of regular prisms, and the second solid torus $T_{2m}j = \cup_{p=0}^{2m-1} C_{\zeta^p_{2m}j}$ is formed in the same way.

When $m = 2$, $P_{Q_8} = [\pm 1 \pm i \pm j \pm k]$ is the hypercube dual to the cross-polytope $[\pm 1, \pm i, \pm j, \pm k] = Conv(Q_8)$, and each polyhedra of P_{Q_8} is a 3-dimensional cube dual to an element of Q_8. For example, $C_1 = [1 \pm i \pm j \pm k]$ is the cube dual to 1.

3.2.2. *$G = T^*, O^*, I^*$.* We describe C_1; each C_g is a rotationally isometric copy. By (4), T^* is the union of the vertices of the cross-polytope and the vertices of its dual hypercube $[\frac{1}{2}(\pm 1 \pm i \pm j \pm k)]$, normalized so that its vertices lie in S^3. The points closest to 1 are the vertices of the cube $[\frac{1}{2}(1 \pm i \pm j \pm k)]$, and C_1 is the regular octahedra $[1 \pm i, 1 \pm j, 1 \pm k]$ in the hyperplane $a = 1$ formed by intersecting H_1 with the half-spaces determined by the hyperplanes tangent to the vertices of this cube.

As O^* is the union of T^* and its translate T^*b^*, it follows that C_1 is the intersection of H_1 with the half-spaces determined by the hyperplanes tangent to the vertices of the cube $\frac{1}{2}[1 \pm i \pm j \pm k]$ above (the points of T^* closest to 1) and the hyperplanes tangent to the vertices of the octahedron $\frac{1}{\sqrt{2}}[1 \pm i, 1 \pm j, 1 \pm k]$, (the points of T^*b^* (and of O^*) closest to 1). Therefore, C_1 is the intersection of the octahedra $[1 \pm i, 1 \pm j, 1 \pm k]$ and the cube $[1 \pm (\sqrt{2}-1)i \pm (\sqrt{2}-1)j \pm (\sqrt{2}-1)k]$, the truncated cube lying in the hyperplane $a = 1$ whose faces are 8 equilateral triangles in the planes $\pm b \pm c \pm d = 1$ and 6 regular octagons in the planes $\pm b = \sqrt{2}-1$, $\pm c = \sqrt{2}-1$, and $\pm d = \sqrt{2}-1$.

For I^*, the points closest to 1 are the vertices of the regular icosahedron $[\frac{1}{2}(\tau \pm j \pm \tau^{-1}k), \frac{1}{2}(\tau \pm \tau^{-1}i \pm k), \frac{1}{2}(\tau \pm i \pm \tau^{-1}j)]$ in the hyperplane $a = \frac{\tau}{2}$. Reasoning as above, C_1 is the regular dodecahedron $[1 \pm i \pm j \pm k, 1 \pm (\tau - 1)i \pm (1 - \tau^{-1})j, 1 \pm (1 - \tau^{-1})i \pm (\tau - 1)k, 1 \pm (1 - \tau^{-1})j \pm (\tau - 1)k]$.

4. G-Tiling $\mathbb{H}^n$

Scalar multiplication of $\mathbb{H}^n$ on the left or right by elements of $\mathbb{H}$ makes $\mathbb{H}^n$ into a left or right module over the ring $\mathbb{H}$, with corresponding canonical "left" and "right" Hermitian inner products. For $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, these are defined by $< \mathbf{w}, \mathbf{v} >_l = w_1\bar{v}_1 + \ldots + w_n\bar{v}_n$, and $< \mathbf{w}, \mathbf{v} >_r = < \bar{\mathbf{w}}, \bar{\mathbf{v}} >_l = \bar{w}_1 v_1 + \ldots + \bar{w}_n v_n$, respectively, where $\bar{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_n)$ is the conjugate of $\bar{\mathbf{v}}$. Note that the left product is "conjugate-linear" in the second variable: $< \mathbf{w}, a\mathbf{v} >_l = < \mathbf{w}, \mathbf{v} >_l \bar{a}$ for each $a \in \mathbb{H}$, while the right product is "conjugate-linear" in the first-variable: $< \mathbf{w}a, \mathbf{v} >_r = \bar{a} < \mathbf{w}, \mathbf{v} >_r$.

With respect to these inner products, one can define "left" and "right" quaternionic hyperplanes by $\mathrm{H}_l := \{\mathbf{u} \in \mathbb{H}^n \mid < \mathbf{u}, \mathbf{a} >_l = b\}$ and $\mathrm{H}_r := \{\mathbf{u} \mid < \mathbf{a}, \mathbf{u} >_r = b\}$, respectively, for any $\mathbf{a} \neq 0$ and $b \in \mathbb{H}$.

For each finite subgroup $G \leq S^3$, we associate to each hyperplane H_l a partition of $\mathbb{H}^n$ into non-overlapping fundamental "G-regions" $\mathcal{R}_{l,g} := \{\mathbf{u} \mid < \mathbf{u}, \mathbf{a} >_l = b + v; v \in R_g\}$ for each $g \in G$. There is a natural free and transitive G right action on the $\mathcal{R}_{l,g}$ defined by setting $\mathcal{R}_{l,g_1} \cdot g_2 = \mathcal{R}_{l,g_1 g_2}$ for $g_1, g_2 \in G$, so that each $\mathcal{R}_{l,g} = \mathcal{R}_{l,1} \cdot g$ is a G-translate of $\mathcal{R}_{l,1}$. For each $w \in S^3$, conjugate linearity of the inner product shows that multiplying $\mathbf{a}$ on the left by w and b on the right by w^{-1} gives the same left hyperplane. However, each $v \in R_g$ is rotated by the left screw w, so restricting w to C_1 gives all the possible G-regions $\{\mathbf{u} \mid < \mathbf{u}, \mathbf{v} >_l = b + vw; v \in R_g\}$ corresponding to H_l.

Geometrically, $\mathrm{H}_l = < \mathbf{a} >_l^\perp + \frac{b\mathbf{a}}{||\mathbf{a}||^2}$ is a $(4n-4)$-dimensional affine space in $\mathbb{R}^{4n}$ corresponding to the linear subspace $< \mathbf{a} >_l^\perp = \{\mathbf{u} \mid < \mathbf{u}, \mathbf{a} >_l = 0\}$ of $\mathbb{R}^{4n}$. Recalling that $R_g = Cone(0, C_g)$, $\mathcal{R}_{l,g}$ is the "wedge" region $\mathrm{H}_l + R_g \mathbf{a}$. Each $g \in G$ acts as the rotation of $\mathbb{H}^n$ about H_l which rotates the orthogonal complement $\mathrm{H}_l^\perp = < \mathbf{a} >_l = \mathbb{H}\mathbf{a}$ as a left screw; since $R_{g_1 g_2} = R_{g_1} g_2$, g_2 rotates $\mathcal{R}_{l,g_1}$ to $\mathcal{R}_{l,g_1 g_2}$ and the $\mathcal{R}_{l,g}$ are rotationally symmetric about H_l by the action $\cdot$ above. The $\mathcal{R}_{l,g}$ are contractible with disjoint interiors, so the $\mathcal{R}_{l,g}$ form a "left" G-tiling of $\mathbb{H}^n$.

With this view, the left G average $\sum_g g^{-1} \mu_\ell(\mathcal{R}_{l,g})$ of the introduction is obtained by rotating $\mathcal{R}_{l,1}$ by the left-screw g onto $\mathcal{R}_{l,g}$, taking the measure $\mu_\ell(\mathcal{R}_{l,g})$, and rotating $\mu_\ell(\mathcal{R}_{l,g})$ back by the right screw g^{-1}. Summing over G gives a rotational average of the measures of the G-translates of $\mathcal{R}_{l,1}$, and the conclusion of Theorem 1 is a type of rotational symmetry of the measures of the G-translates of $\mathcal{R}_{l,1}$ with respect to each of the n measures.

Similarly, one has a "right" G-tiling of $\mathbb{H}^n$ with respect to any right hyperplane H_r by defining fundamental G-regions $\mathcal{R}_{r,g} = \{\mathbf{u} \mid < \mathbf{a}, \mathbf{u} >_r = b + v; v \in R_g\} = \mathrm{H}_r + \mathbf{a} R_g$, where $\mathrm{H}_r = < \mathbf{a} >^\perp + \frac{\mathbf{a} b}{||\mathbf{a}||^2}$ is the $(4n-4)$-dimensional affine space corresponding to the linear subspace $< \mathbf{a} >_r^\perp = \{\mathbf{u} \mid < \mathbf{a}, \mathbf{u} >_r = 0\}$ of $\mathbb{R}^{4n}$. G acts freely and transitively on these G-regions on the left by setting $g_1 \cdot \mathcal{R}_{r,g_2} = \mathcal{R}_{r,g_1 g_2}$, i.e., by rotating $\mathcal{R}_{l,g_2}$ about H_r via the right screw g_1, and there is an analogous interpretation of the right average $\sum_g \mu_\ell(\mathcal{R}_{r,g}) g^{-1}$.

5. Proof of Theorem 1

By way of motivation, we recall proofs of the ham sandwich theorem in the real and complex cases. For each real (or complex) hyperplane H, there is a corresponding G-tiling of $\mathbb{R}^n$ (or $\mathbb{C}^n$) into G-regions, and one connects the free and transitive G action on these regions to a free G action on the unit sphere in $\mathbb{R}^n$ (or $\mathbb{C}^n$) in a continuous fashion. In the real case, $\mathbb{Z}_2$ acts freely and transitively on pairs of half-spaces by reflecting the half-spaces about H, and acts freely on the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ by the antipodal action sending each point x to its antipode $-x$. In the complex case, $\mathbb{Z}_m$ acts freely and transitively on each set of regular $\frac{1}{m}$ sectors by rotating each sector by multiples of ζ_m about H, and acts freely on $S^{2n-1} \subseteq \mathbb{C}^n$ by coordinate wise multiplication by powers of ζ_m, so that each coordinate is rotated successively by $\frac{2\pi}{m}$.

To each $x \in S^n - (S^0 \times 0)$ (respectively, $S^{2n+1} - (S^1 \times 0)$), one assigns a real (complex) hyperplane $\mathrm{H}(x)$ and half-spaces $S^+(x)$ and $S^-(x)$ (regular $\frac{1}{m}$ sectors $\mathcal{S}_k(x)$, $0 \le k < m$) in a way that respects the given G actions: $\mathrm{H}(-x) = \mathrm{H}(x)$ and $S^+(-x) = S^-(x)$, and $\mathrm{H}(\zeta_m^k x) = \mathrm{H}(x)$ and $\mathcal{S}_0(\zeta_m^k x) = \mathcal{S}_k(x)$ for each k. The association $x \mapsto \mu_i(S^+(x))$ $(x \mapsto \mu_i(\mathcal{S}_0(x)))$ is continuous for each measure μ_i, and both theorems follow from "Borsuk-Ulam" theorems applied to the their respective groups.

In the real case (see [8]), one extends the map to all of S^n and applies the standard ($\mathbb{Z}_2$) Borsuk-Ulam Theorem: For a continuous map $f : S^n \longrightarrow \mathbb{R}^n$, there exists some $x \in S^n$ such that $f(x) = f(-x)$, i.e., such that the $\mathbb{Z}_2$ average $f(x) - f(-x)$ is 0. In the complex case, one extends the map to $S^{2n+1} - (\mathbb{Z}_m \times 0)$ and uses a $\mathbb{Z}_m$-variant of the Borsuk-Ulam Theorem: For a continuous map $f : S^{2n+1} - (\mathbb{Z}_m \times 0) \longrightarrow \mathbb{C}^n$, there exists some z for which the $\mathbb{Z}_m$ average $\sum_{k=0}^{m-1} \zeta_m^{-k} f(\zeta_m^k z)$ is 0. (see [9]).

In the quaternionic case, $G \le S^3$ acts on $\mathbb{H}^n$ by restricting left (or right) scalar multiplication by $\mathbb{H}$ to G, so that each $g \in G$ rotates each coordinate of $\mathbb{H}^n$ as a right (or left) screw; these actions are free when $\mathbb{H}^n$ is restricted to S^{4n-1}. The proof of Theorem 1 will proceed as in the cases above, by connecting this free left (right) G action on quaternionic spheres to the free and transitive G right (left) action on G-regions. We concentrate on proving (1) of Theorem 1; the proof of (2) is similar. Thus for each $w \in S^{4n+3} - (S^3 \times 0)$, we will assign quaternionic hyperplanes $\mathrm{H}_l(w)$, and a corresponding set of G-regions $\mathcal{R}_{l,g}(w)$, in such a way as to preserve the group actions: $\mathrm{H}_l(gw) = \mathrm{H}_l(w)$ and $\mathcal{R}_{l,g_1}(g_2 w) = \mathcal{R}_{l,g_1 g_2}(w)$.

As above, we will need to extend this map. Let $X := \cup_{g \in G} \partial \bar{C}'_g \times 0$, where $C'_g = S^3 \cap R_g$ and $\bar{C}'_g = \{u^{-1} \mid u \in C'_g\}$ is its conjugate. Thus C'_g is the spherical image of the polyhedron C_g on S^3 when G is binary dihedral, $C'_g = C_g$ when G is cyclic, and each $\partial \bar{C}'_g$ is a topological S^2 . Theorem 1 will follow from the following "Borsuk-Ulam" Theorem:

Theorem 2: For any continuous map $f : S^{4n+3} - X \longrightarrow \mathbb{H}^n$, there exists some $w \in S^{4n+3} - X$ such that

$$\sum_{g \in G} g^{-1} f(gw) = 0. \tag{7}$$

Proof. If no such w exists, then $h(w) := \sum_{g \in G} g^{-1} f(gw)$ never vanishes, yielding a continuous map $h' : S^{4n+3} - X \longrightarrow S^{4n-1}$ given by $h'(w) = \frac{h(w)}{||h(w)||}$. This map is left G-equivariant, i.e., $h'(gw) = gh'(w)$ for each $g \in G$ and

$w \in S^{4n+3} - X$. In particular, h' is $\mathbb{Z}_m$-equivariant for any $\mathbb{Z}_m = \{\zeta_m^p \mid 0 \leq p < m\}$ which is a subgroup of G, $m \geq 2$.

For each $u \in S^3$, the union of the great circle arcs $\alpha(u,w)(t) = (\cos(t)u, \sin(t)w)$, $0 \leq t \leq \frac{\pi}{2}$, $w \in S^{4n-1}$, forms a $4n$-dimensional disk D_u^{4n} whose boundary is $0 \times S^{4n-1}$, and $D_u^{4n} \subseteq S^{4n+3} - X$ when $u \notin \cup_g \partial \bar{C}_g'$. Thus the inclusion map $i : S^{4n-1} \hookrightarrow S^{4n+3} - X$, $w \mapsto (0,w)$, is nullhomotopic, as it is "filled in" by the extension $D_u^{4n} \hookrightarrow S^{4n+3} - X$. The composition $k := h' \circ i : S^{4n-1} \longrightarrow S^{4n-1}$ is therefore $\mathbb{Z}_m$-equivariant and nullhomotopic as well. It follows by a standard topological argument that we no describe that no such map k can exist.

Let $L^{4n-1}(m)$ be the Lens Space $S^{4n-1}/\mathbb{Z}_m$, the quotient space whose elements are the equivalence class (orbits) $[w] = \mathbb{Z}_m w$ of each $w \in S^{4n-1}$ under left multiplication. Since $\mathbb{Z}_m$ acts freely and is finite, the quotient map $q : S^{4n-1} \longrightarrow L^{4n-1}(m)$ sending each w to its orbit is a covering map. This implies that $L^{4n-1}(m)$ is a manifold, and in fact an orientable manifold because $\mathbb{Z}_m$ acts by rotations, which are orientation preserving.

S^{4n-1} is simply-connected, so $\pi_1(L^{4n-1}(m)) \cong \mathbb{Z}_m$ by covering space theory. Explicitly, if $s_0 \in S^{4n-1}$ and α is any path from s_0 to $\zeta_m s_0$, then $\bar{\alpha} := q \circ \alpha$ is a loop in $L^{4n-1}(m)$ at $x_0 := [s_0]$ whose homotopy class $[\bar{\alpha}]$ is a generator of $\pi(L^{4n-1}(m); x_0)$.

By the $\mathbb{Z}_m$-equivariance, k induces a well-defined continuous map $\bar{k} : L^{4n-1}(m) \longrightarrow L^{4n-1}(m)$ given by $\bar{k}[w] = [k(w)]$. Thus $\bar{k} \circ q = q \circ k$, and we have the following commutative diagram:

$$\begin{array}{ccc} S^{4n-1} & \xrightarrow{k} & S^{4n-1} \\ \downarrow q & & \downarrow q \\ L^{4n-1}(m) & \xrightarrow{\bar{k}} & L^{4n-1}(m) \end{array}$$

It follows that $\bar{k} : \pi_1(L^{4n-1}(m); x_0) \longrightarrow \pi_1(L^{4n-1}(m); \bar{k}(x_0))$ represents the identity homomorphism from $\mathbb{Z}_m$ to itself. The unique lift of $\bar{k} \circ \bar{\alpha}$ beginning at $k(s_0)$ is $k \circ \alpha$ (i.e., $q \circ k \circ \alpha = \bar{k} \circ \bar{\alpha}$), which is a path from $k(s_0)$ to $k(\zeta_m s_0) = \zeta_m k(s_0)$, and as $\bar{k}_*([\bar{\alpha}]) = [\bar{k} \circ \bar{\alpha}]$, $\bar{k}_*$ is the identity on $\mathbb{Z}_m$.

Next, we examine the homology groups $H_i(L^{4n-1}(m))$ and cohomology ring $H^*(L^{4n-1}(m))$, where coefficients will be taken in the ring $\mathbb{Z}_m$ of integers *modulo m*. Using the standard CW-structure of $L^{4n-1}(m)$, it follows that $H_i(L^{4n-1}(m)) \cong \mathbb{Z}_m$ for $i \leq 4n-1$ and is 0 otherwise; the same is true for the cohomology groups $H^i(L^{4n-1}(m))$ with $\mathbb{Z}_m$ coefficents. By Poincaré Duality applied to the oriented manifold $L^{4n-1}(m)$, it follows that $\{x, y, xy, y^2, \ldots, y^{2n}, xy^{2n}\}$ is a basis for the ring $H^*(L^{4n-1}(m))$, where x is a generator of H^1, $y = \beta(x)$ is a generator of H^2, and β is the Bockstein homomorphism $\beta : H^1(L^{4n-1}(m)) \xrightarrow{\cong} H^2(L^{4n-1}(m))$ (see, e.g., [7], for a discussion of Lens Spaces and the Bockstein homomorphism).

As the induced map $\bar{k}_*$ on π_1 represents the identity map on $\mathbb{Z}_m$, it follows by the universal coefficients theorem (see, e.g., [7]) that $\bar{k}^* : H^1(L^{4n-1}(m)) \longrightarrow H^1(L^{4n-1}(m))$ also represents the identity on $\mathbb{Z}_m$, so $\bar{k}^*(x) = x$, and by the naturality of the Bockstein we have $\bar{k}^*(y) = \beta\bar{k}^*(x) = y$. As $\bar{k}^*$ is a ring homomorphism, we conclude that $\bar{k}^*$ is the identity on H^*, and in particular on H^{4n-1}. By the universal coefficients theorem, it follows that $\bar{k}_* : \mathbb{Z}_m \longrightarrow \mathbb{Z}_m$ is the identity on H_{4n-1}, so the $\mathbb{Z}_m$ degree of $\bar{k}$, $deg(\bar{k}) \in \mathbb{Z}_m$, is 1. On the other hand, h is nullhomotopic, so $\bar{h}$ is as well, so that $\bar{h}_*$ on H_{4n-1} is the 0 map and $deg(\bar{k}) = 0$. □

We now prove Theorem 1.

Proof. For each $w = (w_0, w_1, \ldots, w_n) \in S^{4n+3} - X$, define $\mathrm{H}_l(w) = \{u \in \mathbb{H}^n \mid < u, (w_1, \ldots, w_n) >_l = -\bar{w}_0\}$ and $\mathcal{R}_{l,g}(w) = \{u \mid < u, (w_1, \ldots, w_n) >_l = -\bar{w}_0 + v; v \in R_g\}$. The left Hermitian inner product is "conjugate-linear" in the second variable, so $\mathrm{H}_l(gw) = \mathrm{H}(w)$ for each $g \in G$, and for the same reason $\mathcal{R}_{l,g_1}(g_2 w) = \mathcal{R}_{l,g_1 g_2}(w)$ for $g_1, g_2 \in G$. When $w \notin S^3 \times 0$, $\mathrm{H}_l(w)$ is a quaternionic hyperplane, the $\mathcal{R}_{l,g}(w)$ are G-regions, and the association $w \mapsto \mathcal{R}_{l,1}(w)$ preserves the G actions. For $w = (w_0, 0)$, $\mathrm{H}_l(w) = \emptyset$ and $\mathcal{R}_{l,g}(w) = \{u \mid w_0 \in \bar{C}'_g\}$. Hence $\mathcal{R}_g(w) = \mathbb{H}^n$ if $w_0 \in \bar{C}'_g$ and is empty otherwise.

For each measure μ_ℓ, define $f_\ell : S^{4n+3} - X \longrightarrow \mathbb{H}$ by $f_\ell(w) = \mu_\ell(\mathcal{R}_{l,1}(w))$. Each f_ℓ is continuous (Lemma 1), so $f := (f_1, \ldots, f_n) : S^{4n+3} - X \longrightarrow \mathbb{H}^n$ is continuous as well. By (7), there must be some $w \in S^{4n+3} - X$ such that $\sum_{g \in G} g^{-1} \mu_\ell(\mathcal{R}_{l,1}(gw)) = 0$ for each ℓ, and as $\mathcal{R}_{l,1}(gw) = \mathcal{R}_{l,g}(w)$ for each g,

$$\sum_{g \in G} g^{-1} \mu_\ell(\mathcal{R}_{l,g}(w)) = 0 \tag{8}$$

for each μ_ℓ.

To finish the proof, we only need to show that $w \notin S^3 \times 0$, so that $\mathrm{H}_l(w)$ is a quaternionic hyperplane and the $\mathcal{R}_{l,g}(w)$ are G-regions. Assume for a contradiction that $w = (w_0, 0)$, $w_0 \notin \cup_{g \in G} \partial \bar{C}'_g$. As the C_g have disjoint interiors, so do the $\bar{C}'_g$, and there is some unique g_0 for which $w_0 \in \bar{C}'_{g_0}$. By the above discussion, $\mu_\ell(\mathcal{R}_{l,g}) = \mu_\ell(\mathbb{H}^n)$ if $g = g_0$ and $\mu_\ell(\mathcal{R}_{l,g}) = 0$ otherwise, so by (8), $g_0^{-1} \mu_\ell(\mathbb{H}^n) = 0$ for each ℓ. Hence $\mu_\ell(\mathbb{H}^n) = 0$ for each ℓ, contrary to the assumption on the μ_ℓ.

For (2), let $\mathrm{H}_r(w) = \{u \mid < (w_1, \ldots, w_n), u >_r = -\bar{w}_0\}$ and $\mathcal{R}_{r,g}(w) = \{u \mid < (w_1, \ldots, w_n)_r, u >_r = -\bar{w}_0 + v; v \in R_g\}$ for each $w \in S^{4n+3} - X$. $\mathrm{H}_r(wg) = \mathrm{H}_r(w)$ and $\mathcal{R}_{r,g_2}(wg_1) = \mathcal{R}_{r,g_1 g_2}(w)$ by conjugate linearity, and (2) follows as above by using a right multiplication version of Theorem 2: $\sum_g f(wg) g^{-1} = 0$ for some w, which is proved in the analogous way as for left multiplication. □

6. Quaternionic Measures

Let $\mathfrak{B}(\mathbb{H}^n)$ denote the Borel sets of $\mathbb{H}^n$. By direct analogy with the definition of a complex Borel measure (see, e.g., [6] or [9]), a function $\mu : \mathfrak{B}(\mathbb{H}^n) \longrightarrow \mathbb{H}$ is called a quaternionic Borel measure on $\mathbb{H}^n$ if

1) $\mu(\emptyset) = 0$, and

2) If $\{E_i\}_{i=0}^{\infty}$ is a countable collection of disjoint Borel sets, then $\mu(\bigcup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} \mu(E_i)$, and the convergence of this sum is absolute: $\sum_{i=0}^{\infty} |\mu(E_i)| < \infty$. In particular, $|\mu(E)| < \infty$ for each Borel set E.

A quaternionic Borel measure μ on $\mathbb{H}^n$ may be written uniquely as $\mu = \mu_1 + \mu_2 i + \mu_3 j + \mu_4 k$, where each $\mu_r : \mathfrak{B}(\mathbb{H}^n) \longrightarrow \mathbb{R}$ is a Borel signed measure. By the Jordan decomposition theorem, (see, e.g., [6]), each μ_r can be expressed uniquely as the difference of two mutually singular positive Borel measures μ_r^+ and μ_r^- on $\mathbb{H}^n$. That is, there exist two disjoint Borel sets A_r and B_r whose union is $\mathbb{H}^n$, such that $\mu_r^+(B_r) = \mu_r^-(A_r) = 0$ and $\mu_r = \mu_r^+ - \mu_r^-$. In particular, condition 2) implies that each $\mu_r^{\pm}$ is finite, as $\mu_r^+(\mathbb{H}^n) = \mu_r^+(A_r) = \mu_r(A_r) < \infty$ and $\mu_r^-(\mathbb{H}^n) = \mu_r^-(B_r) = -\mu_r(B_r) < \infty$.

A Borel set E will be called null with respect to μ if $\mu(E') = 0$ for each Borel set $E' \subseteq E$. By the Jordan decompositions of the μ_r, this is equivalent to E having measure zero with respect to each of the $\mu_r^{\pm}$: If E is null for μ, then $\mu_r^+(E) = \mu_r^+(E \cap A_r) = \mu_r(E \cap A_r) = 0$ for each r, and similarly $\mu_r^-(E) = 0$ for each r as well. Conversely, if E has measure zero for each $\mu_r^{\pm}$ and $E' \subseteq E$ is Borel, then $\mu_r(E') = \mu_r^+(E') - \mu_r^-(E') = 0$.

We will call a subset of $\mathbb{H}^n$ a “real hyperplane” if it is a hyperplane in $\mathbb{R}^{4n}$ under the canonical identification with $\mathbb{H}^n$, and we will call a quaternionic Borel measure μ on $\mathbb{H}^n$ “proper” if each real hyperplane in $\mathbb{H}^n$ is null with respect to μ. For each $g \in G$, $\partial\mathcal{R}_{l,g} = \mathrm{H}_l + \partial R_g \mathbf{a}$, where each H_l is a $(4n-4)$ dimensional affine space in $\mathbb{R}^{4n}$, and as each ∂R_g is contained in a finite union of hyperplanes in $\mathbb{R}^4$, $\partial\mathcal{R}_{l,g}$ is contained in a finite union of hyperplanes in $\mathbb{R}^{4n}$. A subset of a null set is a null set and the union of null sets is also a null set, so each $\partial\mathcal{R}_{l,g}$ is a null set with respect to any proper quaternionic Borel measure on $\mathbb{H}^n$. Finally, we prove Lemma 1.

Proof. We proceed as in [9]. Let μ be a proper, quaternionic Borel measure on $\mathbb{H}^n$. As $\mu = (\mu_1^+ - \mu_1^-) + (\mu_2^+ - \mu_2^-)i + (\mu_3^+ - \mu_3^-)j + (\mu_4^+ - \mu_4^-)k$, continuity of the association $w \mapsto \mu(\mathcal{R}_{l,1}(w))$ is equivalent to continuity of the map $w \mapsto \mu_r^{\pm}(\mathcal{R}_{l,1}(w))$ for each r. Let $\{w_m\}_{m=1}^{\infty}$ be a sequence in $S^{4n+3} - X$ converging to w, and let $h_m = \chi_{\mathcal{R}_{l,1}(w_m)}$ and $h = \chi_{\mathcal{R}_{l,1}(w)}$. We will show that h_m converges to h pointwise outside of a null set, so that h_m converges to h almost everywhere with respect to each $\mu_r^{\pm}$.

Each h_m is dominated by $\chi_{\mathbb{H}^n}$, which is in $L^1(\mu_r^{\pm})$ because $\mu_r^{\pm}(\mathbb{H}^n) < \infty$. By the Dominated Convergence Theorem,

$$\lim_{m\to\infty} \mu_r^{\pm}(\mathcal{R}_{l,1}(w_m)) = \lim_{m\to\infty} \int h_m \mathrm{d}\mu_r^{\pm} = \int h \mathrm{d}\mu_r^{\pm} = \mu_r^{\pm}(\mathcal{R}_{l,1}(w)),$$

so the function $w \mapsto \mu_r^{\pm}(\mathcal{R}_{l,1}(w))$ is continuous.

Now we show the convergence of the h_m. Recalling that $w \notin X$, define $\partial\mathcal{R}_{l,1}(w) = \{u \mid < u, (w_1, \ldots, w_n) >_l = -\bar{w}_0 + v; v \in \partial R_1\}$. If $w \notin S^3 \times 0$, then $\partial\mathcal{R}_{l,1}(w)$ is the boundary of the G-region $\mathcal{R}_{l,1}(w)$ and hence is a null set, while if $w = (w_0, 0)$, then $\partial\mathcal{R}_{l,1}(w) = \{u \mid w_0 \in \partial\bar{C}_1'\} = \emptyset$. Thus $\partial\mathcal{R}_{l,1}(w)$ is always a null set, and we will show that the h_m converge to h outside of $\partial\mathcal{R}_{l,1}(w)$.

For each $u \notin \partial\mathcal{R}_{l,1}(w)$, define $\psi_u : S^{4n+3} - X \longrightarrow \mathbb{H}$ by $\psi_u(z) = < u, (z_1, \ldots, z_n) >_l + \bar{z}_0$. For $u \notin \partial\mathcal{R}_{l,1}(z)$, u is in either $Int(\mathcal{R}_{l,1}(z)) := \{u \mid < (u, z_1, \ldots, z_n) >_l = -\bar{z}_0 + v; v \in Int(R_1)\}$ or $Ext(\mathcal{R}_{l,1}(z)) := \{u \mid < u, (z_1, \ldots, z_n) >_l = -\bar{z}_0 + v; v \in Int(\mathbb{H} - R_1)\}$. Moreover, $u \in Int(\mathcal{R}_{l,1}(z))$ iff $z \in \psi_u^{-1}(Int(R_1))$, while $u \in Ext(\mathcal{R}_{l,1}(z))$ iff $z \in \psi_u^{-1}(Int(\mathbb{H} - R_1))$.

Suppose first that $u \in Int(\mathcal{R}_{l,1}(w))$, so that $w \in \psi_u^{-1}(Int(R_1))$. $\psi_u^{-1}(Int(R_1))$ is open because ψ_u is continuous and $Int(R_1)$ is open, and since $w_m \to w$, $w_m \in \psi_u^{-1}(Int(R_1))$ for all sufficiently large m. Hence $u \in Int(\mathcal{R}_{l,1}(w_m))$ for all large enough m, and by the same reasoning $u \in Ext(\mathcal{R}_{l,1}(w_m))$ for all large enough m if $u \in Ext(\mathcal{R}_{l,1}(w))$. Thus for $u \notin \partial\mathcal{R}_{l,1}(w)$, $h_m(w) = h(w)$ for all sufficiently large m and $h_m \to h$ outside of $\partial\mathcal{R}_{l,1}(w)$. $\square$

7. Applications of Theorem 1

7.1. $G = \mathbb{Z}_m$**.** From here on, we will omit the subscripts differentiating left and right quaternionic hyperplanes and regions.

Identifying $\mathbb{H}$ with $\mathbb{C}^2$, any quaternionic Borel measure on $\mathbb{H}^n$ is a pair of complex Borel measures on $\mathbb{C}^{2n}$, so Theorem 1 applies to $2n$ complex Borel measures $\mu_1, \mu_2, \ldots, \mu_{2n}$ on $\mathbb{C}^{2n}$. When $G = \mathbb{Z}_m$, it follows from section 3.1 that the $\mathcal{R}_{\zeta_m^k}$, $0 \leq k < m$, are regular $\frac{1}{m}$ sectors $\mathcal{S}_k$ corresponding to a complex hyperplane, and Theorem 1 gives that $\sum_{k=0}^{m-1} \zeta_m^{-k}\mu_\ell(\mathcal{S}_k) = 0$ for each μ_ℓ, thereby recovering the $\mathbb{Z}_m$ ham sandwich theorem for $\mathbb{C}^{2n}$.

The union $\cup_{k=0}^{m-1}\partial\mathcal{S}_k$ is a regular m-fan centered at a codimension 2 affine subspace. Letting each complex Borel measure be a proper, Borel measure on $\mathbb{R}^{4n}$, one concludes in particular that given $2n$ proper, Borel measures on $\mathbb{R}^{4n}$, there is a regular 3-fan trisecting each measure, a pair of orthogonal hyperplanes which bisect each measure, and a regular 6-fan whose corresponding opposite regular $\frac{1}{3}$-sectors always have equal measure (see [9]).

If $\mu_1, \mu_2, \mu_3, \mu_4$ are proper, Borel measures on $\mathbb{R}^{4n}$, then $\mu := \mu_1 + \mu_2 i + \mu_3 j + \mu_4 k$ is a proper quaternionic Borel measure on $\mathbb{H}^n$. The $\mathbb{Z}_2$ regions $\mathcal{R}_1$ and $\mathcal{R}_{-1}$ of Theorem 1 are the half-spaces S^+ and S^- of a hyperplane in $\mathbb{R}^{4n}$ (see section 3.1), and as each measure is proper, we recover the original ham sandwich theorem on $\mathbb{R}^{4n}$: $\mu_\ell(S^+) = \mu_\ell(S^-) = \frac{1}{2}\mu_\ell(\mathbb{R}^{4n})$ for each μ_ℓ.

7.2. $G = D_m^*$**.** Recall from section 3.2.1 that D_m^* triangulates S^3 as the union of the two solid tori, where each torus is a ring of regular $2m$ prisms C_g, and $\mathbb{R}^4$ is the union of the two rings of cones R_g on these prisms. Realizing each $(4n-4)$-dimensional affine space H as a quaternionic hyperplane H_ℓ, $\mathbb{R}^{4n}$ is the union of two rings of $2m$ "prism wedges" $\mathcal{R}_g = \mathrm{H} + R_g\mathbf{a}$. Similarly, the complement $\mathrm{H}^\perp$ of each codimension 4 affine subspace H of

$\mathbb{R}^{4n+k}$, $1 \leq k \leq 3$, can be partitioned into copies of the D_m^*-regions R_g, thereby partitioning $\mathbb{R}^{4n+k}$ into prism wedges which we again denote by $\mathcal{R}_g$, and $\mathbb{R}^{4n+k}$ is the union of two rings of $2m$ wedges each.

Let $\mu_1, \ldots, \mu_n$ be proper, finite Borel measures on $\mathbb{R}^{4n}$. Each μ_ℓ can be seen as a proper quaternionic Borel measure taking values in $[0, \infty)$, so by Theorem 1 there is a codimension 4 affine space H and $4m$ corresponding D_m^* regions such that for each μ_ℓ,

$$\sum_{p=0}^{2m-1} \zeta_{2m}^{-p} \mu_\ell(\mathcal{R}_{\zeta_{2m}^p}) = 0 \tag{9}$$

$$\sum_{p=0}^{2m-1} \zeta_{2m}^{-p} \mu_\ell(\mathcal{R}_{\zeta_{2m}^p j}) = 0 \tag{10}$$

Thus (9) and (10) show that for each of the two rings forming $\mathbb{R}^{4n}$, the $\mathbb{Z}_{2m}$ average of the measures of the ring's wedges is zero for each μ_ℓ.

This result can be extended to n finite, proper Borel measures μ_ℓ, $1 \leq \ell \leq n$, on $\mathbb{R}^{4n+k}$, $1 \leq k \leq 3$. The projection $\pi : \mathbb{R}^{4n+k} \longrightarrow \mathbb{R}^{4n}$ onto the last $4n$ coordinates of $\mathbb{R}^{4n+k}$ induces n finite Borel measures $\pi(\mu_\ell)$ on $\mathbb{R}^{4n}$ by setting $\pi(\mu_\ell)(E) = \mu_\ell(\pi^{-1}(E))$ for each Borel set $E \subseteq \mathbb{R}^{4n}$. The pull-back of an affine subspace of $\mathbb{R}^{4n}$ is an affine subspace of $\mathbb{R}^{4n+k}$ of the same codimension, so in particular each $\pi(\mu_\ell)$ is proper, and since a prism wedge $\mathcal{R}_g$ in $\mathbb{R}^{4n}$ pulls back to a prism wedge in $\mathbb{R}^{4n+k}$, applying Theorem 1 to the $\pi(\mu_\ell)$ shows that there is some codimension 4 affine subspace H in $\mathbb{R}^{4n+k}$ and corresponding prism wedges $\mathcal{R}_g$ which satisfy (9) and (10).

We conclude with the cases $m = 2$ and $m = 3$. When $m = 2$, the C_g are cubes (section 3.2.1) and the $\mathcal{R}_g$ are cubical wedges. Examining real and imaginary parts in (9) and (10) yields

$$\begin{array}{ll} \mu_\ell(\mathcal{R}_1) = \mu_\ell(\mathcal{R}_{-1}) & \mu_\ell(\mathcal{R}_j) = \mu_\ell(\mathcal{R}_{-j}) \\ \mu_\ell(\mathcal{R}_i) = \mu_\ell(\mathcal{R}_{-i}) & \mu_\ell(\mathcal{R}_k) = \mu_\ell(\mathcal{R}_{-k}) \end{array} \tag{11}$$

$R_{-g} = -R_g$ for each g, so $\mathcal{R}_{-g}$ and $\mathcal{R}_g$ are opposite regions of $\mathbb{R}^{4n+k}$ with respect to the affine space H, and (11) says that for each μ_ℓ, opposite wedges have equal measure.

Corollary 1: Given $\lfloor \frac{n}{4} \rfloor$ finite, proper Borel measures $\mu_1, \ldots, \mu_{\lfloor \frac{n}{4} \rfloor}$ on $\mathbb{R}^n$, there exists a codimension 4 affine space and a corresponding decomposition of $\mathbb{R}^n$ into 2 rings of 4 cubical wedges $\mathcal{R}_g$, $g \in Q_8$, such that for each μ_ℓ, opposite wedges have equal measure: $\mu_\ell(\mathcal{R}_g) = \mu_\ell(\mathcal{R}_{-g})$ for each $g \in Q_8$.

When $m = 3$, the $\mathcal{R}_g$ are hexagonal prism wedges, six in each of the two rings comprising $\mathbb{R}^n$, and within each ring there are six "$\frac{1}{3}$ rings" $\mathcal{T}_g$ formed by taking the union of two adjacent wedges (e.g., $\mathcal{T}_{\zeta_6^p} := \mathcal{R}_{\zeta_6^p} \cup \mathcal{R}_{\zeta_6^{p+1}}$ in the first ring), and the $\frac{1}{3}$ ring opposite to $\mathcal{T}_g$ with respect to H is $\mathcal{T}_{-g}$. A calculation using (9) and (10) yields that opposite $\frac{1}{3}$ rings always have equal measure:

$$\mu_\ell(\mathcal{T}_{\zeta_6^p}) = \mu_\ell(\mathcal{T}_{-\zeta_6^p}), 0 \leq p \leq 2 \tag{12}$$

$$\mu_\ell(\mathcal{T}_{\zeta_6^p j}) = \mu_\ell(\mathcal{T}_{-\zeta_6^p j}), 0 \leq p \leq 2 \tag{13}$$

for each μ_ℓ.

Corollary 2: Given $\lfloor \frac{n}{4} \rfloor$ finite, proper Borel measures on $\mathbb{R}^n$ $\mu_1, \ldots, \mu_{\lfloor \frac{n}{4} \rfloor}$, there exists a codimension 4 affine space and a corresponding decomposition of $\mathbb{R}^n$ into 2 rings of 6 hexagonal prism wedges $\mathcal{R}_g$, $g \in D_3^*$, such that for each μ_ℓ, opposite $\frac{1}{3}$ rings have equal measure: $\mu_\ell(\mathcal{T}_g) = \mu_\ell(\mathcal{T}_{-g})$ for each $g \in D_3^*$.

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