THE ENDOMORPHISMS MONOID OF A HOMOGENEOUS VECTOR BUNDLE

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ABSTRACT. In this paper we give some properties of the algebraic and geometric structure of the endomorphisms monoid of a homogeneous vector bundle.

1. INTRODUCTION

Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic and A an abelian variety over $\Bbbk$. For any $a \in A$ denote by $t_a : A \to A$ the translation by a. A vector bundle $\rho : E \to A$ over A is called *homogeneous* if $E \cong t_a^* E$, for any $a \in A$. An indecomposable homogeneous vector bundles over A are vector bundles of the form $L \otimes F$, where $L \in \mathrm{Pic}^0(A)$, and F is an *unipotent* homogeneous vector bundle, i.e. F admits a filtration by sub-bundles such that the associated graded bundle is a trivial bundle.

Let $\xi = (E, \rho, A)$ and $\xi' = (E', \rho', A)$ be two homogeneous vector bundles over A. A *bundle homomorphism* $(f, \alpha) : \xi \to \xi'$ is a morphism of the underlying bundles; that is, $f : E \to E'$ and $\alpha : A \to A$ are maps such that $\alpha = t_a$ for some $a \in A$, $\rho' f = \alpha \rho$ and the restriction $f_b : \rho^{-1}(b) \to \rho'^{-1}(\alpha(b))$ is linear for each $b \in A$. We denote by $\mathrm{Hom}_{hb}(E, E')$ the set of such homomorphisms and by $Hom_A(E, E')$ the subset of bundle homomorphisms of the form (f, Id_A). We will denote $\mathrm{Hom}_{hb}(E, E)$ (respectively $\mathrm{Hom}_A(E, E)$) by $\mathrm{End}_{hb}(E)$ (respectively $\mathrm{End}_A(E)$).

The monoid $\mathrm{End}_A(E)$ has been studied by a number of authors over at least the last 50 years. From Atiyah's results (see [2]) we have that a homogeneous vector bundle E over A is indecomposable if and only if the subset $N_A(E) \subset \mathrm{End}_A(E)$ consisting of all nilpotent endomorphism of E is a vector subspace which is in fact a 2-sided ideal in $\mathrm{End}_A(E)$, and

$$\mathrm{End}_A(E) = \Bbbk \cdot 1_E \oplus N_A(E)$$

as a $\Bbbk$-vector space, where the direct summand $\Bbbk \cdot 1_E$ consists of all scalar endomorphisms. Miyanishi in [16] described the algebraic structure of the *automorphisms group* $\mathrm{Aut}_{hb}(E)$ of an homogeneous vector bundle $E \to A$

and derived some consequences about the structure of E as vector bundle. Later, Mukai in [17] generalized Miyanishi results for the indecomposable homogeneous vector bundles. In [12] Brion and the second author proved that the *endomorphisms monoid* $\operatorname{End}_{hb}(E)$ is a smooth algebraic monoid with unit group $\operatorname{Aut}_{hb}(E)$, *the group of automorphisms* of E. Moreover, they proved (see [12, Theorem 5.3]) that any normal algebraic monoid M can be embedded as a closed submonoid of the endomorphisms monoid $\operatorname{End}_{hb}(E)$ of an indecomposable homogeneous vector bundle E over $A(M)$, where $A(M)$ is Albanese variety of M. Moreover, its Albanese morphism $\alpha : \operatorname{End}_{hb}(E) \to A$ is a morphism of algebraic monoids, with Kernel $\alpha^{-1}(0) = \operatorname{End}_A(E)$ (see [12] and Remark 3.1 below).

The aim of this paper is to describe the geometric and algebraic structure of the endomorphisms monoid $\operatorname{End}_{hb}(E)$, as well as the relationship between this structure and the structure of E as a vector bundle.

If $\rho : E = L \otimes F \to A$ be an indecomposable homogeneous vector bundle of rank n, where L is a homogeneous line bundle and F is an unipotent vector bundle then for the algebraic structure of $\operatorname{End}_{hb}(E)$ and the structure as vector bundle we prove that,

(1) $\operatorname{End}_{hb}(E) \to A$ is a homogeneous vector bundle with fiber isomorphic to $\operatorname{End}_A(E)$. In particular, $\operatorname{End}_A(E)$ is a finite-dimensional $\Bbbk$-algebra with $\dim \operatorname{End}_A(E) \leq 1 + \frac{n(n-1)}{2}$. Moreover, $\operatorname{End}_{hb}(E)$ is obtained by successive extensions of L (see Theorems 3.3 and 4.17).

(2) If L is a line bundle then $\operatorname{End}_{hb}(L) \cong L$ (see Lemma 4.9 and Corollary 4.10).

(3) The Kernel of the algebraic monoid $\operatorname{End}_{hb}(E)$ is the zero section

$$\Theta(E) = \{\theta_a : E \to E : \theta(v_x) = 0_{x+a} \ \forall \, v_x \in E_x\} = \operatorname{Ker}\big(\operatorname{End}_{hb}(E)\big).$$

In particular, $\operatorname{Ker}\big(\operatorname{End}_{hb}(E)\big)$ is an algebraic group, isomorphic to the abelian variety A (see Corollary 4.13).

(4) Let $\mathcal{N}_{hb}(E)$ denote the set of pseudo-nilpotent elements, Then the algebraic monoid $\operatorname{End}_{hb}(E)$ decomposes as a disjoint union

$$\operatorname{End}_{hb}(E) = \operatorname{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E)$$

(see Theorem 4.16). In particular, $\mathcal{N}_{hb}(E)$ is an ideal of $\operatorname{End}_{hb}(E)$. Moreover, $\mathcal{N}_{hb}(E)$ is a homogeneous vector bundle, obtained by successive extensions of L (see Proposition 4.18).

(5) There exists a exact sequence of vector bundles

$$0 \longrightarrow \mathcal{N}_{hb}(E) \longrightarrow \operatorname{End}_{hb}(E) \xrightarrow{\ \rho\ } \operatorname{End}_{hb}(L) \cong L \longrightarrow 0$$

Moreover, the morphisms appearing in the sequence are compatible with the structures of semigroup, and the sequence splits if and only if $E \cong L$ (see Theorem 4.19).

Let $\rho : E = L \otimes F \to A$ be an indecomposable homogeneous vector bundle of rank n. Denote by E_0 the fibre $\rho^{-1}(0) \subset E$ where $0 \in A$ is the

unit element. Recall that *the induced space* $\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0$ is defined as the geometric quotient of $\mathrm{Aut}_{hb}(E) \times E_0$ under the diagonal action of $\mathrm{Aut}_A(E)$ (see Definition 2.1 below). In Theorem 4.1 we prove that

an indecomposable homogeneous vector bundle $\rho : E \to A$ is the induced space from the action of $\mathrm{Aut}_A(E)$ on E_0 to an action of the automorphisms group $\mathrm{Aut}_{hb}(E)$, i.e.

$$E \cong \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0.$$

The above description allow us to describe the structure of $\mathrm{End}_{hb}(E)$ as vector bundle when E is a decomposable vector bundle (see Theorem 4.12). More precisely, if $E = \bigoplus_{i,j} L_i \otimes F_{i,j}$, where L_i is a homogeneous line bundle and $F_{i,j}$ an unipotent bundle, then

$$\mathrm{End}_{hb}(E) \cong \bigoplus_i L_i \otimes \left(\oplus_{j,k} \mathrm{Hom}_{hb}(F_{i,j}, F_{i,k}) \right)$$

where $\mathrm{Hom}_{hb}(F_{i,j}, F_{i,k})$ is the monoid of homomorphisms from $F_{i,j}$ to $F_{i,j}$. Actually, (see Theorems 3.6 and 3.6 and Corollary 3.10) we have that if (E, ρ, A) and (E', ρ', A) are two homogeneous vector bundles over A then $\mathrm{Hom}_{hb}(E, E')$ is a homogeneous vector bundle over A.

The paper is organized as follows: in section 2 we recall the basic results about algebraic monoids and homogeneous vector bundles needed in the subsequents sections. In section 3 we establish the first results on the structure of $\mathrm{Hom}_{hb}(E, E')$. In section 4 we prove the main results of this work, which relate the structure of the homogeneous vector bundle of E and the structure of $\mathrm{End}_{hb}(E)$ as vector bundle and as algebraic monoid. In section 5 we do explicit calculations of $\mathrm{End}_{hb}(E)$ and $\mathrm{End}_A(E)$ when E is a homogeneous vector bundle of small rank.

2. General results

In this paper we fix an algebraically closed field $\Bbbk$ of arbitrary characteristic and an abelian variety A over $\Bbbk$ and recall the basic results on algebraic monoids and homogeneous vector bundles over A that we will need. The main references for the theory of algebraic monoids are [11, 12, 19, 20] and [1, 2, 3, 16, 17] for the theory of homogeneous vector bundles.

2.1. **Algebraic monoids.**

Recall that an *algebraic monoid* is an algebraic variety M together with a morphism $m : M \times M \to M$ such that m is an associative product and there exists a neutral element $1 \in M$. The *unit group* of M is the group of invertible elements

$$G(M) = \left\{ g \in M : \exists\, g^{-1}\, ,\, gg^{-1} = g^{-1}g = 1 \right\}.$$

It is well known that $G(M)$ is an algebraic group, open in M (see [19]). If M is an irreducible algebraic monoid, then its *Kernel*, denoted by $\mathrm{Ker}(M)$, is the minimum closed ideal, that is $\mathrm{Ker}(M)$ is the minimum closed subset $Z \subset M$ such that $MZM = M$. The Kernel of an algebraic monoid always exists: if M is an affine algebraic monoid, then its $\mathrm{Ker}(M)$ is the unique closed $\big(G(M) \times G(M)\big)$-orbit (see [19, 12]).

Let M, N be algebraic monoids. A morphism $\varphi : M \to N$ is a *morphism of algebraic monoids* if $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in M$ and $\varphi(1_M) = 1_N$. We denote such morphism as $\varphi_{am} : M \to N$. If a morphism of algebraic monoids $\varphi_{am} : M \to N$ is an isomorphism, we say that M and N are *isomorphic* as algebraic monoids, and denote as $M \cong_{am} N$.

Definition 2.1. Let $H \subset G$ be algebraic subgroup of an algebraic group G such that H acts on an algebraic variety X. The *induced space* $G *_H X$ is defined as the geometric quotient of $G \times X$ under the H-action $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. Under mild conditions on X (e.g. X is covered by quasi-projective H-stable open subsets), this quotient exists. Clearly, $G *_H X$ is a G-variety, for the action induced by $a \cdot (g, x) = (ag, x)$. We will denote the class of (g, x) in $G *_H X$ by $[g, x]$. We refer the reader to [4], where some basic facts about induced spaces were proved, and to [21] for a survey on this construction.

Remark 2.2. Let G be an algebraic group and let $H \subset G$ be a closed subgroup, with H acting over an algebraic variety X. Then the morphism $\pi : G *_H X \to G/H$ induced by $(g, x) \mapsto [g] = gH$ is a fiber bundle over G/H with fiber isomorphic to X. If moreover X is a H-module, then $G *_H X \to G/H$ is a vector bundle (see [21]).

The Chevalley's structure theorem for an algebraic monoid says that if G is algebraic group and $A(G)$ is the Albanese group then the Albanese morphism $p : G \to A(G)$ fits into an exact sequence of algebraic groups

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \overset{p}{\longrightarrow} A(G) \longrightarrow 0$$

where G_{aff} is a normal connected affine algebraic group (since the group $A(G)$ is commutative, its law will be denoted additively).

Remark 2.3. Brion and Rittatore in [11, 12] generalize Chevalley's decomposition to irreducible, normal, algebraic monoids. They prove that if M is a irreducible, normal, algebraic monoid with unit group G, then M admits a Chevalley's decomposition:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & M_{\text{aff}} = \overline{G_{\text{aff}}} & \longrightarrow & M = \overline{G} & \xrightarrow{\ p\ } & A(G) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & G_{\text{aff}} & \longrightarrow & G & \xrightarrow[p|_G]{} & A(G) & \longrightarrow & 0
\end{array}$$

where $p : M \to A(G) = G/G_{\text{aff}}$ (respectively $p|_G : G \to A(G)$) is the Albanese morphism of M (respectively G). Moreover, if Z^0 denotes the connected center of G, then $A(G) \cong Z^0/(Z^0 \cap G_{\text{aff}})$ and

$$M = G \cdot M_{\text{aff}} = Z^0 \cdot M_{\text{aff}}.$$

Actually they prove that,

$$M \cong G *_{G_{\text{aff}}} M_{\text{aff}} \cong Z^0 *_{Z^0 \cap G_{\text{aff}}} M_{\text{aff}}.$$

From Brion and Rittatore results we have the following corollary.

Corollary 2.4. *Let M be an irreducible algebraic monoid, with unit group G. Then $\text{Ker}(M) = G\,\text{Ker}(M_{\text{aff}})G = G\,\text{Ker}(M_{\text{aff}}) = Z^0\,\text{Ker}(M_{\text{aff}})$ where Z^0 is the connected center of G.*

Proof. Since $M = Z^0 M_{\text{aff}}$, it follows that $\text{Ker}(M_{\text{aff}}) \subset \text{Ker}(M)$, and hence $G\,\text{Ker}(M_{\text{aff}})G \subset \text{Ker}(M)$. Since both terms in the last inclusion are $\big(G(M)\times G(M)\big)$-orbits, the first equality follows.

It is clear that $\big(G\,\text{Ker}(M_{\text{aff}})G\big) \cap M_{\text{aff}} = \text{Ker}(M_{\text{aff}})$ and from the decompositions $M = Z^0 M_{\text{aff}} = M_{\text{aff}} Z^0$ and $G = Z^0 G_{\text{aff}}$, we deduce that

$$G\,\text{Ker}(M_{\text{aff}})G = Z^0\,\text{Ker}(M_{\text{aff}}) = G\,\text{Ker}(M_{\text{aff}}).$$

$\square$

2.2. Homogeneous vector bundles.

Definition 2.5. A vector bundle $E \to A$ is called *homogeneous* if for all $a \in A$, $E \cong t_a^* E$ where t_a is the translation by a.

Note that a line bundle is homogeneous if and only if it is algebraically equivalent to zero (see [15, Sect. 9]). In particular the trivial bundle $\mathcal{O}_A$ is homogeneous.

Let $\xi = (E, \rho, A)$ and $\xi' = (E', \rho', A)$ be two homogeneous vector bundles over A and let $(f, \alpha) : \xi \to \xi'$ be a bundle homomorphism. We have a commutative diagram:

$$\begin{array}{ccc}
E & \xrightarrow{\ f\ } & E' \\
\downarrow & & \downarrow \\
A & \xrightarrow[\alpha]{} & A
\end{array}$$

Since the projections ρ and ρ' are the Albanese morphisms of E and E' respectively, (see for example [15, Cor. 3.9]), it follows that $f : E \to E'$ induces the morphism $\alpha : A \to A$ which is a translation

$$t_a : A \to A, \quad x \mapsto a + x$$

for some $a \in A$.

We denote $\mathrm{Hom}_{hb}(E, E')$ the set of such bundle homomorphisms and $\mathrm{Hom}_A(E, E')$ the subset of bundle homomorphisms of the form (f, Id_A), that is, those that fix the base. If $E^* \otimes E' = \mathrm{Hom}(E, E')$, then $\mathrm{Hom}_A(E, E') = H^0(A, \mathrm{Hom}(E, E'))$; in particular, $\mathrm{End}_A(E) = H^0(A, E^* \otimes E)$.

It is clear that if $(f, t_a) : E \to E'$ and $(g, t_b) : E' \to E''$ are bundle morphisms then $\big((g \circ f), t_{b+a}\big) : E \to E''$ is a bundle morphism. If E is a homogeneous vector bundle, then $\mathrm{Hom}_{hb}(E, E) := \mathrm{End}_{hb}(E)$ is a monoid under composition, called the *endomorphisms monoid*. The group $\mathrm{Aut}_{hb}(E)$ of *automorphisms* of E is the unit group of this monoid. Clearly $(f, t_a) \in \mathrm{Aut}_{hb}(E)$ if and only if $f : E \to E$ is an isomorphism of algebraic varieties.

The map

$$\pi : \mathrm{End}_{hb}(E) \to A, \quad \pi(f, t_a) = a,$$

is a morphism of monoids. In particular, the fiber at the unit element $0 \in A$ is the *algebra of endomorphisms* $\mathrm{End}_A(E)$ of the vector bundle E. It is well known that if E is indecomposbale, then $\mathrm{End}_A(E)$ is a finite-dimensional $\Bbbk$-algebra; in particular, an irreducible, affine, smooth, algebraic monoid, with unit group $\mathrm{Aut}_A(E) := \pi^{-1}(0) \cap \mathrm{Aut}(E)$. We say that E is *simple* if $\mathrm{End}_A(E) = \Bbbk$.

For convenience of notation we will denote sometimes a bundle homomorphism $(f, \alpha) : \xi \to \xi'$ just as $f : E \to E'$ since $\alpha = t_a$ is determined by f and call it just a homomorphism. If (f, Id_A) is an isomorphism, we will write $E \cong_{vb} E'$.

Remark 2.6. Let $\pi : \mathrm{Hom}_{hb}(E, E') \to A$ be defined as $\pi(f, t_a) = a$.

(1) Let $\pi^{-1}(a)$ be the fibre of $\pi : \mathrm{Hom}_{hb}(E, E') \to A$ at $a \in A$. There is a natural bijection between $\pi^{-1}(a)$ and the set $\mathrm{Hom}_A(E, t_a^* E')$. Indeed, given $(f, t_a) \in \mathrm{Hom}_{hb}(E, E')$ there is a homomorphism $f^a : E \to t_a^* E'$ of vector bundles over A, such that the following diagram

is commutative.

(2) $\pi : \mathrm{Hom}_{hb}(E, E') \to A$ is a fibration over A with fiber at $a \in A$ canonically bijective to $\mathrm{Hom}_A(E, t_a^* E')$ which is a finite-dimensional k-vector space. In particular, $\mathrm{Hom}_A(E, E') = \pi^{-1}(0)$.

Definition 2.7. We say that a vector bundle E of rank $r > 1$ is obtained by *successive extensions of a vector bundle* R, of length s, if there exist extensions

$$\rho_1 : \quad 0 \longrightarrow E_0 \xrightarrow{i_1} E_1 \xrightarrow{p_1} R \longrightarrow 0$$
$$\rho_2 : \quad 0 \longrightarrow E_1 \xrightarrow{i_2} E_2 \xrightarrow{p_2} R \longrightarrow 0$$
$$\rho_3 : \quad 0 \longrightarrow E_2 \xrightarrow{i_3} E_3 \xrightarrow{p_3} R \longrightarrow 0$$
$$\vdots \qquad\qquad \vdots$$
$$\rho_s : \quad 0 \longrightarrow E_{s-1} \xrightarrow{i_s} E_s \xrightarrow{p_s} R \longrightarrow 0$$

such that $E_s \cong E$ and $E_0 = R$.

In such case we say that $(\rho_1, \ldots, \rho_s)$ are extensions associated to E. In particular if E_i is homogeneous for all $i = 0, \ldots, s$ we say that E is obtained by *successive extensions of the homogeneous vector bundle* R. If E is obtained as successive extensions of the trivial bundle $\mathcal{O}_A$, we say that E is a *unipotent vector bundle*.

Note that E is obtained by successive extensions of a vector bundle R if and only if there exists a filtration

$$R = E_0 \subset E_1 \subset E_2 \subset E_3 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that $E_i/E_{i-1} \cong R$ for $i = 1, \ldots, s$. The graded bundle i.e. $gr(E) = \oplus E_i/E_{i-1}$, associated to this filtration is isomorphic to $\oplus^s R$. In particular, if E is unipotent then $gr(E) = \oplus \mathcal{O}_A$.

Proposition 2.8. *If E is a vector bundle obtained by successive extensions of a vector bundle R, of length s, then $\dim_{\Bbbk} \mathrm{End}_A(E) \geq 2$.*

Proof. Let

$$\rho_1 : \quad 0 \longrightarrow R \xrightarrow{i_1} \quad E_1 \quad \xrightarrow{p_1} R \longrightarrow 0$$
$$\rho_2 : \quad 0 \longrightarrow E_1 \xrightarrow{i_2} \quad E_2 \quad \xrightarrow{p_2} R \longrightarrow 0$$
$$\rho_3 : \quad 0 \longrightarrow E_2 \xrightarrow{i_3} \quad E_3 \quad \xrightarrow{p_3} R \longrightarrow 0$$
$$\vdots \qquad\qquad \vdots$$
$$\rho_s : \quad 0 \longrightarrow E_{s-1} \xrightarrow{i_s} E_s \cong E \xrightarrow{p_s} R \longrightarrow 0$$

be the extensions associated to E. The composition $\varphi = i_s \circ \cdots \circ i_2 \circ i_1 \circ p_s \neq 0$ defines a non invertible endomorphism of E. Therefore, $\dim_{\Bbbk} \mathrm{End}_A(E) \geq 2$. $\qquad\square$

In the following remark we recall the main results of [16] and [17] that we use in the rest of this paper.

Remark 2.9. Let $E \to A$ be a vector bundle over an abelian variety.

(1) E is an indecomposable homogeneous vector bundle if and only if
$E \cong L \otimes F$ where $L \in \mathrm{Pic}^0(A)$ and F is a unipotent vector bundle.
(2) E is homogeneous if and only if E decomposes as a direct sum $E = \bigoplus L_i \otimes F_i$, where $L_i \in \mathrm{Pic}^0(A)$ and F_i is a unipotent vector bundle.
(3) Unipotent vector bundles are homogeneous.
(4) A vector bundle E that is a successive extensions of a homogeneous line bundle L, is homogeneous.
(5) If E is indecomposable then E is obtained by extensions of a homogeneous line bundle L. Moreover, one can choose the associated filtration $0 \subsetneq E_1 = L \subsetneq \cdots \subsetneq E_i \subsetneq \cdots \subsetneq E_n = E$ in such a way that E_i is stable by $\mathrm{Aut}_{hb}(E)$ for all i.

From the results of Mukai in [17] we have that the category $\mathcal{H}_A$ of homogeneous vector bundles over an abelian variety A, with morphisms $\mathrm{Hom}_A(E, E')$, is equivalent to the category of coherent sheaves over the *dual abelian variety* $\widehat{A}$, with support a finite number of points. In particular the category of homogeneous vector bundles is abelian. Mukai's paper involves a transform which provides the equivalence between the derived categories. This technique has come to be known as the Fourier-Mukai transform.

Under the above equivalence, the full subcategory $\mathcal{H}_{L,A}$ of indecomposable vector bundles obtained by successive extensions of a homogeneous line bundle L, is equivalent to the category of coherent sheaves with support on a point $\widehat{x} \in \widehat{A}$. In particular a homogeneous line bundle L correspond to the sheaf $\mathcal{O}_{\widehat{x}}$. The subcategory $\mathcal{U}_A$ of unipotent vector bundles is equivalent to the category of finite length $\mathcal{O}_{\widehat{A},\widehat{0}}$-modules. In particular, the category of unipotent vector bundles is abelian.

Remark 2.10. Note that for every homogeneous line bundle L, the functor $F \mapsto L \otimes F$ is an equivalence of categories between $\mathcal{U}_A$ and $\mathcal{H}_{L,A}$. $\qquad\square$

We finish this section by recalling the concept of stability introduced by Gieseker in [13], and its relation with homogeneous bundles, as given by Mukai in [17].

For any torsion-free sheaf F over A let $\chi(F) = \Sigma(-1)^i \dim H^i(A, F)$ be the Euler characteristic of F. Denote by $p(F)$ the rational number $p(F) := \chi(F)/\mathrm{rk}(E)$.

A torsion-free sheaf E over A is Gieseker-stable (respectively Gieseker-semistable) if for all coherent subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ and torsion free quotient we have that

$$p\big(F(k)\big) < p\big(E(k)\big) \qquad \big(\text{respectively } p\big(F(k)\big) \leq p\big(E(k)\big)\big)$$

for all sufficiently large integers $k \in \mathbb{Z}$. Note that stability implies semistability but the converse is not true.

Remark 2.11. Mukai proved in [17, Proposition 6.13] that homogeneous vector bundles are Gieseker-semistable. Moreover, a homogeneous vector

bundle E is Gieseker-stable if and only if it is simple. It follows from Proposition 2.8 that a homogeneous vector bundle E is Gieseker-stable if and only if E is a homogeneous line bundle.

3. The endomorphisms monoid of a homogeneous vector bundle

Let $E \to A$, $E' \to A$ be homogeneous vector bundles. In this section we endow $\mathrm{Hom}_{hb}(E, E')$ with a structure of homogeneous vector bundle induced by the canonical action of either $\mathrm{Aut}_A(E)$ or of $\mathrm{Aut}_A(E')$ on $\mathrm{Hom}_A(E, E')$. In Theorem 3.12 we show that both structures are isomorphic.

We begin this section by recalling the description of the algebraic structure (i.e. as algebraic monoid) of $\mathrm{End}_{hb}(E)$. Recall first (see [2]) that a homogeneous vector bundle E over A is indecomposable if and only if the subset $N_A(E) \subset \mathrm{End}_A(E)$ consisting of all nilpotent endomorphisms of E is a vector subspace which is an ideal in $\mathrm{End}_A(E)$, and

$$\mathrm{End}_A(E) = \Bbbk \cdot 1_E \oplus N_A(E)$$

as a $\Bbbk$-vector space.

If E is an indecomposable homogeneous vector bundle, then $\mathrm{Aut}_A(E) \cong G_m \times U_A(E)$, where $G_m = \Bbbk^*$ and $U_A(E)$ is the unipotent affine subgroup $\mathrm{Id} + \mathcal{N}_A(E)$. Miyanishi in [16, Lem 1.1] proved that $\mathrm{Aut}_{hb}(E)$ as an algebraic group is an extension of A by $\mathrm{Aut}_A(E)$, that is, we have the exact sequence

$$1 \longrightarrow \mathrm{Aut}_A(E) \longrightarrow \mathrm{Aut}_{hb}(E) \longrightarrow A \longrightarrow 0$$

Remark 3.1. Brion and Rittatore in [12] gave the Chevalley's decomposition of $\mathrm{End}_{hb}(E)$ as an algebraic monoid. They prove that $\mathrm{End}_{hb}(E)$ has a structure of a non-singular irreducible algebraic monoid such that its action on E is algebraic. Moreover they show the Albanese morphism of $\mathrm{End}_{hb}(E)$ is $\pi : \mathrm{End}_{hb}(E) \to A$ where $\pi((f, t_a)) = a$, and that $\mathrm{End}(E)_{\mathrm{aff}} = \mathrm{End}_A(E)$ fits in the following exact sequence of monoids

$$1 \longrightarrow \mathrm{End}_A(E) \longrightarrow \mathrm{End}_{hb}(E) \longrightarrow A \longrightarrow 0.$$

$\square$

Remark 3.2. Let $\mathrm{Z}_{hb}^0(E)$ be the connected center of $\mathrm{End}_{hb}(E)$ and $\mathrm{Z}_A^0(E) = \mathrm{Z}_{hb}^0(E) \cap \mathrm{End}_A(E)$. From the results of Brion and Rittatore we have the following isomorphisms of algebraic monoids

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \cdot \mathrm{End}_A(E) = \mathrm{Z}_{hb}^0(E) \cdot \mathrm{End}_A(E)$$
$$\cong_{am} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E)$$
$$\cong_{am} \mathrm{Z}_{hb}^0(E) *_{\mathrm{Z}_A^0(E)} \mathrm{End}_A(E).$$

Theorem 3.3. *Let E be a homogeneous vector bundle of rank r over an abelian variety A. Then $\mathrm{End}_{hb}(E)$ is a homogeneous vector bundle with fiber*

isomorphic to $\mathrm{End}_A(E)$. *Moreover, if E is an indecomposable homogeneous vector bundle, then $\mathrm{End}_{hb}(E)$ is a homogeneous vector bundle over A of rank $\leq 1 + r(r-1)/2$.*

Proof. Since $\mathrm{End}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E)$ as algebraic monoids, it follows from general properties of the induced action that $\mathrm{End}_{hb}(E) \to A$ is a vector bundle with fiber isomorphic to $\mathrm{End}_A(E)$, since $\mathrm{End}_A(E)$ is a finite dimensional algebra (see for example [21]). Moreover, since $\mathrm{Aut}_{hb}(E)$ acts by left multiplication by automorphisms on $\mathrm{End}_{hb}(E)$ we have that $\mathrm{End}_{hb}(E)$ is homogeneous. Indeed, given $a in A$, there exists $(f, t_a) in \in \mathrm{Aut}_{hb}(E)$. If $\ell_f : \mathrm{End}_{hb}(E) \to \mathrm{End}_{hb}(E)$ denotes the isomorphism $\ell_f(h) = f \circ h$, $h \in \mathrm{End}_A(E)$, then $\alpha(\ell_f) = t_a$.

As in [6, Prop. 1.1.9] we have that the dimension of the algebra of endomorphisms of semistable bundles E of rank r is upper bounded by $\dim \mathrm{End}(E) \leq 1 + r(r-1)/2$. Indeed, the fiber E_a has a flag invariant under $e_a(\mathrm{End}_A(E))$ where $e_a : \mathrm{End}_A(E) \to \mathrm{End}_{\Bbbk}(E_a)$, $e_a(f) = f|_{E_a}$ is the restriction to the fiber $a \in A$. Since a homogeneous bundles are semistable, the proof in [6, Prop. 1.1.9] generalizes to abelian varieties. Hence,

$$\dim \mathrm{End}_A(E) \leq 1 + r(r-1)/2$$

and then $\mathrm{End}_{hb}(E)$ is a homogeneous vector bundle over A of rank $\leq 1 + r(r-1)/2$. $\square$

In section 5 we will give an explicit description the homogeneous vector bundle $\mathrm{End}_{hb}(E)$ for small rank.

Remark 3.4. $\mathrm{Aut}_A(E)$ acts in two different ways on $\mathrm{End}_A(E)$, either by post-composing, $f \cdot h = f \circ h$, or by pre-composing, $f \cdot h = h \circ f^{-1}$, with $f \in \mathrm{Aut}_A(E)$ and $h \in \mathrm{End}_A(E)$. This allows to endow $\mathrm{End}_{hb}(E)$ with two structures of vector bundle. However, since

$$\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E) \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{End}_A(E),$$

one can prove that in fact these structures coincide. Instead of proving this in full details, we will prove in theorems 3.6 and 3.7 slightly more general results relating the structures of vector bundle of $\mathrm{Hom}_{hb}(E, E')$.

Proposition 3.5. *Let E and E' be two vector bundles over A. Suppose E' is homogeneous. The inclusion $\mathrm{Z}^0_{hb}(E') \hookrightarrow \mathrm{Aut}_{hb}(E')$ induces an isomorphism of the homogeneous vector bundles*

$$\mathrm{Z}^0_{hb}(E') *_{\mathrm{Z}^0_A(E')} \mathrm{Hom}_A(E, E') \cong_{vb} \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E'),$$

where $\mathrm{Z}^0_A(E')$ and $\mathrm{Aut}_A(E')$ act on $\mathrm{End}_A(E, E')$ by post-composing.

Proof. Recall that the induced space $P = \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E')$ is a vector bundle over $\mathrm{Aut}_{hb}(E')/\mathrm{Aut}_A(E') = A$.

It is clear that the canonical action of $\mathrm{Aut}_{hb}(E')$ over P induces a morphism of (abstract) groups $\varphi : \mathrm{Aut}_{hb}(E') \to \mathrm{Aut}(P)$,

$$\varphi(f, t_a) = (\widetilde{f}, t_a),$$

where $\widetilde{f}\big([(h, t_b), h']\big) = \big[(f \circ h, t_{b+a}), h'\big]$. Hence, since the canonical projection $\mathrm{Aut}_{hb}(E') \to A$, $(f, t_a) \mapsto a$, is surjective, it follows that the canonical projection $\mathrm{Aut}_{hb}(P) \to A$ is also surjective. In other words, the vector bundle P is homogeneous. In an analogous way one can prove that the vector bundle $Q = \mathrm{Z}_{hb}^0(E') *_{\mathrm{Z}_A^0(E')} \mathrm{Hom}_A(E, E')$ is homogeneous. In order to finish the proof just observe that the inclusion $\mathrm{Z}_{hb}^0(E') \hookrightarrow \mathrm{Aut}_{hb}(E')$ induces a morphism of homogeneous vector bundles $Q \to P$ which is bijective, hence $Q \cong_{vb} P$. $\qquad\square$

Theorem 3.6. *Let E, E' be vector bundles over the Abelian variety A, with E' homogeneous. Then $\mathrm{Hom}_{hb}(E, E')$ can be endowed with an structure of homogeneous vector bundle, via an*

$$\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}_{hb}(E, E')$$

isomorphism of vector bundles. Moreover,

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \mathrm{Z}_{hb}^0(E') *_{\mathrm{Z}_A^0(E')} \mathrm{Hom}_A(E, E').$$

Proof. We shall prove that there exists a bijection

$$\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \longleftrightarrow \mathrm{Hom}_{hb}(E, E')$$

such that the following diagram

(3.1) $\quad \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \xrightarrow{\quad \phi \quad} \mathrm{Hom}_{hb}(E, E')$

with π' and π mapping to A.

is commutative, where $\pi : \mathrm{Hom}(E, E') \to A$ is the projection $\pi(f, t_a) = a$, and $\pi' : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to A$ is the canonical projection $\pi'(g, h) = [g] \in \mathrm{Aut}_{hb}(E')/\mathrm{Aut}_A(E') \cong A$.

Let $(g, t_a) \in \mathrm{Aut}_{hb}(E')$ and $h \in \mathrm{Hom}_A(E, E')$. Then the following diagram

$$E \xrightarrow{\quad h \quad} E' \cong t_a^* E' \xrightarrow{\quad g \quad} E'$$

is commutative.

Let $\varphi : \mathrm{Aut}_{hb}(E') \times \mathrm{Hom}_A(E, E') \to \mathrm{Hom}_{hb}(E, E')$ be given by

$$\varphi\big((g, t_a), h\big) = (g \circ h, t_a).$$

Clearly φ is constant at the $\mathrm{Aut}_A(E')$-orbits, and hence induces a homomorphism

$$\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}(E, E').$$

By construction ϕ makes the diagram (3.1) commutative.

In order to prove the surjectivity of ϕ consider $(f, t_a) \in \mathrm{Hom}(E, E')$, so we have the following commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ f\ } & E' \\
\downarrow & & \downarrow \\
A & \xrightarrow{\ t_a\ } & A.
\end{array}
$$

Since E' is homogeneous, for $a \in A$ there exists $(g, t_{-a}) \in \mathrm{Aut}_{hb}(E')$ such that we have the following commutative diagram

$$
\begin{array}{ccccc}
E & \xrightarrow{\ f\ } & E' \cong t_a^* E' & \xrightarrow{\ g\ } & E' \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\ t_a\ } & A & \xrightarrow{\ t_{-a}\ } & A.
\end{array}
$$
$$t_0 = \mathrm{Id}$$

Hence, the composition $(g, t_{-a}) \circ (f, t_a) = (g \circ f, t_0)$ defines a homomorphism $g \circ f : E \to E'$, as vector bundles over A. Moreover,

$$\phi\big([(g^{-1}, t_a), g \circ f]\big) = (g^{-1} \circ g \circ f, t_a) = (f, t_a),$$

and thus ϕ is surjective.

We claim that ϕ is injective. Indeed, if

$$\big[(g_1, t_{a_1}), h_1\big], \big[(g_2, t_{a_2}), h_2\big] \in \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E')$$

are such that $\phi\big([(g_1, t_{a_1}), h_1)]\big) = \phi\big([(g_2, t_{a_2}), h_2]\big)$ then, by definition of ϕ, we have that $g_1 \circ h_1 = g_2 \circ h_2$ and $t_{a_1} = t_{a_2}$. It follows that $a_1 = a_2$, and hence $g_2^{-1} \circ g_1 \in \mathrm{Aut}_A(E)$.

Since ϕ is linear when restricted to a fiber, we can endow $\mathrm{Hom}(E, E')$ with a structure of vector bundle in such a way that

$$
\begin{aligned}
\big[(g_1, t_{a_1}), h_1\big] &= \big[(g_1 \circ (g_1^{-1} \circ g_2), t_{a_1}), (g_2^{-1} \circ g_1) \circ h_1\big] \\
&= \big[(g_2, t_{a_1}), g_2^{-1} \circ (g_1 \circ h_1)\big] \\
&= \big[(g_2, t_{a_2}), h_2\big].
\end{aligned}
$$

It follows that $\phi : \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}_{hb}(E, E')$ is also injective and hence a bijection. Therefore

$$
\begin{aligned}
\mathrm{Hom}_{hb}(E, E') &\cong_{vb} \mathrm{Z}_{hb}^0(E') *_{\mathrm{Z}_A^0(E')} \mathrm{Hom}_A(E, E') \\
&\cong_{vb} \mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E').
\end{aligned}
$$

The last part follows from Proposition 3.5. $\qquad\qquad\qquad\qquad\square$

If E is homogeneous we have the corresponding theorem.

Theorem 3.7. *Let E, E' be vector bundles over A. If E is homogeneous, then $\mathrm{Hom}_{hb}(E, E')$ can be endowed with an structure of homogeneous vector bundle via*

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \longrightarrow \mathrm{Hom}(E, E'),$$

which becomes an isomorphism of vector bundles. Moreover,

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{Hom}_A(E, E')$$

Theorem 3.7 also follows from the corresponding proposition.

Proposition 3.8. *Let E and E' be two vector bundles over A. Suppose E is homogeneous. The inclusion $\mathrm{Z}^0_{hb}(E) \hookrightarrow \mathrm{Aut}_{hb}(E)$ induces an isomorphism of the vector bundles*

$$\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} \mathrm{Hom}_A(E, E') \cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E'),$$

where $\mathrm{Z}^0_A(E)$ and $\mathrm{Aut}_A(E)$ act on $\mathrm{End}_A(E, E')$ by pre-composing: $g \cdot f = f \circ g^{-1}$, $f \in \mathrm{End}_A(E)$, $g \in \mathrm{Aut}_A(E)$.

The proofs of Theorem 3.7 and Proposition 3.8 are analogous to the proofs of Theorem 3.6 and Proposition 3.5. In this case there exists a bijection

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \longleftrightarrow \mathrm{Hom}(E, E')$$

such that the following diagram

$$
\begin{array}{ccc}
\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') & \xrightarrow{\ \psi\ } & \mathrm{Hom}(E, E') \\
{\scriptstyle \pi''}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\
A & \xrightarrow[\ -\,\mathrm{Id}\ ]{} & A
\end{array}
$$

is commutative where $\pi'' : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \to A$ is the canonical projection $\pi''(g, h) = [g] \in \mathrm{Aut}_{hb}(E)/\mathrm{Aut}_A(E) \cong A$. The homomorphism ψ is given as follows: the canonical action of $\mathrm{Aut}_A(E)$ over $\mathrm{Aut}_{hb}(E) \times \mathrm{Hom}_A(E, E')$ is given by

$$g \cdot \big((f, t_a), h\big) = \big((f \circ g^{-1}, t_a), h \circ g^{-1}\big),$$

where $g \in \mathrm{Aut}_A(E)$, $(f, t_a) \in \mathrm{Aut}(E)$, $h \in \mathrm{Aut}(E)$. Hence, the map

$$\xi : \mathrm{Aut}_{hb}(E) \times \mathrm{Hom}_A(E, E') \to \mathrm{Hom}(E, E') \quad,$$

defined as $\big((f, t_a), h\big) \mapsto (h \circ f^{-1}, t_{-a})$ is constant on the $\mathrm{Aut}_A(E)$-orbits, and hence induces a the homomorphisms

$$\psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \to \mathrm{Hom}(E, E').$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If E, E' are both homogeneous vector bundles, it is not clear *a priori* that the structures of vector bundles on $\mathrm{Hom}_{hb}(E, E')$ given in Theorem 3.6 and Theorem 3.7 are the same. In Theorem 3.12 we will prove that the structure of homogeneous vector bundle on $\mathrm{Hom}_{hb}(E, E')$ is unique, up to isomorphism. First we have the following propositions. Denote by P the vector bundle $Z_{hb}^0(E) *_{Z_A^0(E)} \mathrm{End}_A(E)$; recall that P is the quotient of the action of $Z_A^0(E)$ over $Z_{hb}^0(E) \times \mathrm{End}_A(E)$ given by

$$(3.2) \qquad z \cdot \big((f, t_a), h\big) = \big((f \circ z^{-1}, t_a), z \circ h\big).$$

Let Q be the vector bundle $(Z_{hb}^0(E) *_{Z_A^0(E)} \mathrm{End}_A(E))$, that is the quotient of the action of $Z_A^0(E)$ over $Z_{hb}^0(E) \times \mathrm{End}_A(E)$ defined by

$$z \cdot \big((f, t_a), h\big) = \big((f \circ z^{-1}, t_a), h \circ z^{-1}\big).$$

Proposition 3.9. *If $E \to A$ is a homogeneous vector bundle then $P \cong_{vb}$ $(\text{-Id})^*Q$.*

Proof. Consider the morphism $\xi : Z_{hb}^0(E) \times \mathrm{End}_A(E) \to Q$ given by $\xi\big((f, t_a), h\big) = \big[(f^{-1}, t_{-a}), h\big]$. If we consider the action of $Z_{hb}^0(E)$ over $Z_{hb}^0(E) \times \mathrm{End}_A(E)$ given by (3.2), then for $z \in Z_{hb}^0(E)$ we have that

$$\xi\big(z \cdot \big((f, t_a), h\big)\big) = \xi\big((f \circ z^{-1}, t_a), z \circ h\big) = \big[(z \circ f^{-1}, t_{-a}), z \circ h\big]$$
$$= \big[(f^{-1} \circ z, t_{-a}), h \circ z\big],$$

where for the last equality we use the fact that z commutes with any endomorphism of the homogeneous vector bundle E. Taking into account the definition of Q, it follows that

$$\xi\big(z \cdot \big((f, t_a), h\big)\big) = \big[(f^{-1} \circ z, t_{-a}), h \circ z\big] = \big[(f^{-1}, t_{-a}), h\big].$$

In other words, ξ is constant on the $\mathrm{Aut}_A(E)$-orbits (of the action given by (3.2)), and hence induces a homomorphism of homogeneous vector bundles $\widetilde{\xi} : P \to Q$. An easy calculation shows that $\widetilde{\xi}$ is an isomorphism such that the diagram

commutes. Hence, $\widetilde{\xi}$ induces the required isomorphism $\bar{\xi} : P \to_{vb} (\text{-Id})^*Q$. $\square$

Corollary 3.10. *If E is a homogeneous vector bundle then the structures of homogeneous vector bundle defined on $\mathrm{End}_{hb}(E)$ by ϕ in Theorem 3.6 and ψ in Theorem 3.7 are isomorphic.* $\qquad\square$

Proposition 3.11. *If $E = \bigoplus_i E_i$ and $E' = \bigoplus_j E'_j$ are two homogeneous vector bundles. Consider the structure of homogeneous vector bundle in $\mathrm{Hom}_{hb}(E, E')$ and in $\mathrm{Hom}_{hb}(E_i, E'_j)$ given in Theorem 3.6. Then*

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j).$$

In particular,

$$\mathrm{End}_{hb}(E) \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E_j).$$

Proof. We have that $\mathrm{Hom}_A(E, E') \cong \bigoplus_{i,j} \mathrm{Hom}_A(E_i, E'_j)$. First, we prove that

$$\mathrm{Aut}_{hb}(E') *_{\mathrm{Aut}_A(E')} \mathrm{Hom}_A(E, E') \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j).$$

Let

$$\iota_{i,j} : \mathrm{Hom}_A(E_i, E'_j) \hookrightarrow \mathrm{Hom}_A(E, E'),$$

be the canonical inclusion, that is, $\iota_{i,j}(f_{i,j})(v) = f_{i,j}(v_i)$, where $f_{i,j} \in \mathrm{Hom}_A(E_i, E'_j)$, $v \in E$ and if $v \in E_a$ then $v = \sum_i v_i$, with $v_i \in (E_i)_a$.

For any j, the automorphisms group $\mathrm{Aut}_{hb}(E'_j)$ can be immersed as a sub-semigroup of $\mathrm{End}_{hb}(E')$ *via* the canonical inclusion $\iota_j(f_j, t_a) = (\iota_j(f_j), t_a)$, where if $v' \in E'$ belongs to the fiber E'_b and decomposes as $v' = \sum v'_j$, $v'_j \in E'_j$, then $\iota_j(f_j)(v') = f_j(v'_j)$.

Consider the morphism

$$\varphi_{i,j} : \mathrm{Hom}_{hb}(E_i, E'_j) \cong_{vb} \mathrm{Aut}_{hb}(E'_j) *_{\mathrm{Aut}_A(E'_j)} \mathrm{Hom}_A(E_i, E'_j) \to \mathrm{Hom}_{hb}(E, E')$$

induced by

$$\begin{aligned} \psi_{i,j} : \mathrm{Aut}_{hb}(E'_j) \times \mathrm{Hom}_A(E_i, E'_j) &\to \mathrm{Hom}_{hb}(E, E') \\ \psi_{i,j}\big((f_j, t_a), f_{i,j}\big) &= \big(\iota_j(f_j) \circ \iota_{i,j}(f_{i,j}), t_a\big). \end{aligned}$$

Then the direct sum

$$\varphi = \sum_{i,j} \varphi_{i,j} : \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j) \to \mathrm{Hom}_{hb}(E, E')$$

is clearly a homomorphism of homogeneous vector bundles of the same rank (since $\mathrm{Hom}_A(E, E') = \bigoplus_{i,j} \mathrm{Hom}_A(E_i, E'_j)$). In order to prove that φ is an isomorphism, it suffices to prove that φ is injective. Let $\sum_{i,j}\big[(h_{i,j}, t_a), f_{i,j}\big]$ and $\sum_{i,j}\big[(h'_{i,j}, t_{a'}), f'_{i,j}\big]$ in $\bigoplus_{i,j} \mathrm{Aut}_{hb}(E'_j) *_{\mathrm{Aut}_A(E'_j)} \mathrm{Hom}_A(E_i, E'_j)$ be such that

$$\varphi\big(\sum_{i,j}\big[(h_{i,j}, t_a), f_{i,j}\big]\big) = \varphi\big(\sum_{i,j}\big[(h'_{i,j}, t_{a'}), f'_{i,j}\big]\big).$$

Then

$$\sum \varphi_{i,j}\big([(h_{i,j}, t_a), f_{i,j}]\big) = \sum \varphi_{i,j}\big([(h'_{i,j}, t_{a'}), f'_{i,j}]\big),$$

and hence $\big(\sum_{i,j} \iota_j(h_{i,j}) \circ \iota_{i,j}(f_{i,j}), t_a\big) = \big(\sum_{i,j} \iota_{j'}(h'_{i,j}) \circ \iota_{i',j'}(f'_{i,j}), t_{a'}\big)$. It follows that $a = a'$ and that

$$(3.3) \qquad \Phi = \sum_{i,j} \iota_j(h_{i,j}) \circ \iota_{i,j}(f_{i,j}) = \sum_{i,j} \iota_{j'}(h'_{i,j}) \circ \iota_{i',j'}(f'_{i,j}) = \Phi'.$$

Since equation (3.3) holds if and only if $\Phi|_{E_i} = \Phi'|_{E_i}$ for all i, it follows that (3.3) holds if and only if for all i,

$$\sum_j \iota_j(h_{i,j}) \circ \iota_{i,j}(f_{i,j})|_{E_i} = \Phi|_{E_i} = \Phi'|_{E_i} = \sum_j \iota_j(h'_{i,j}) \circ \iota_{i',j}(f'_{i,j})|_{E_i} : E_i \to E'$$

Observe that $\iota_j(h_{i,j})|_{E'_j} = h_{i,j} : E'_j \to E'_j$ and $\iota_j(h'_{i,j})|_{E'_j} = h'_{i,j} : E'_j \to E'_j$ are automorphisms such that $\iota_{i,j}(f_{i,j})|_{E_i} = f_{i,j} : E_i \to E'_j$, $\iota_{i,j}(f'_{i,j})|_{E_i} = f'_{i,j} : E_i \to E'_j$. It follows that $\Phi = \Phi'$ if and only if

$$\iota_j(h_{i,j}) \circ \iota_{i,j}(f_{i,j}) = \sum_j \iota_j(h'_{i,j}) \circ \iota_{i',j}(f'_{i,j}) \quad \forall\, i, j.$$

Hence, $h_{i,j} \circ f_{i,j} = h'_{i,j} \circ f'_{i,j}$ for all i, j, and $[(h_{i,j}, t_a), f_{i,j}] = [(h'_{i,j}, t_a), f'_{i,j}]$. Indeed, $h_{i,j}^{-1} \circ h'_{i,j} \in \mathrm{Aut}_A(E'_j)$, and thus for all i, j we have that

$$\begin{aligned}
[(h_{i,j}, t_a), f_{i,j}] &= \big[(h_{i,j} \circ h_{i,j}^{-1} \circ h'_{i,j}, t_a), h'_{i,j}{}^{-1} \circ h_{i,j} \circ f_{i,j}\big] \\
&= \big[h'_{i,j}, t_a), h'_{i,j}{}^{-1} \circ h'_{i,j} \circ f'_{i,j}\big] \\
&= \big[h'_{i,j}, t_a), f'_{i,j}\big].
\end{aligned}$$

Therefore,

$$\mathrm{Hom}_{hb}(E, E') \cong_{vb} \bigoplus_{i,j} \mathrm{Hom}_{hb}(E_i, E'_j).$$

$\square$

In an analogous way, one can prove a similar decomposition when considering the structures of vector bundles given by the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{hb}(E, E') &\cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{Hom}_A(E, E') \\
\mathrm{Hom}_{hb}(E_i, E'_j) &\cong_{vb} \mathrm{Aut}_{hb}(E_j) *_{\mathrm{Aut}_A(E_j)} \mathrm{Hom}_A(E_i, E'_j)
\end{aligned}$$

Theorem 3.12. *Let E and E' be homogeneous vector bundles. The structures of vector bundle on $\mathrm{Hom}_{hb}(E, E')$ given in Theorem 3.6 and in Theorem 3.7 are isomorphic.*

Proof. Consider the vector bundle $E \oplus E'$. Then, we have isomorphisms

$$\mathrm{End}_{hb}(E \oplus E') \cong_{vb} \mathrm{Hom}_{hb}(E, E') \oplus \mathrm{Hom}_{hb}(E', E) \oplus \mathrm{End}_{hb}(E) \oplus \mathrm{End}_{hb}(E')$$

From Corollary 3.10 the structure of vector bundle given by theorems 3.6 and 3.7 are isomorphic, the isomorphism being induced by $\bar{\xi} : P \to Q$

as in the proof of Proposition 3.9. It is clear that by construction this isomorphism must then induce and isomorphism between the structures of $\mathrm{Hom}_{hb}(E, E')$ given in Theorem 3.6 and in Theorem 3.7, and the same for $\mathrm{Hom}_{hb}(E', E)$. □

4. Relationship between the structure of a homogeneous bundle and its endomorphisms monoid

We begin this section by showing that a homogeneous vector bundle $E \to A$ is obtained as an extension of its fiber E_0 at 0 by the principal bundle $\mathrm{Aut}_{hb}(E) \to A$.

Theorem 4.1. *Let $\rho : E \to A$ be a homogeneous vector bundle. Then as vector bundles over A*

$$E \cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \cong_{vb} \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} E_0.$$

Proof. Recall that $A \cong \mathrm{Aut}_{hb}(E)/\mathrm{Aut}_A(E)$. We first define

$$\phi : \mathrm{Aut}_{hb}(E) \times E_0 \to E$$

as $\big((f, t_a), v\big) \mapsto f(v) \in E_a$, where $(f, t_a) \in \mathrm{Aut}_{hb}(E)$ and $v \in E_0$. Clearly, ϕ is constant on the $\mathrm{Aut}_A(E)$-orbits, and hence induces a homomorphism $\varphi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \to E$.

It is easy to see that φ is in fact a isomorphism of vector bundles. Indeed, given $a \in A$, consider $(f, t_a) \in \mathrm{Aut}_{hb}(E)$. Then

$$\big(\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0\big)_a = \big\{\big((f, t_a), v\big) : v \in E_0\big\},$$

and $f|_{E_0} : E_0 \to E_a$, is a linear isomorphism. Hence, the restriction $\varphi_a : \big(\mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0\big)_a \to E_A$, $\varphi_a\big((f, t_a), v\big) = f(v)$, is a linear isomorphism, and hence φ is a isomorphism of vector bundles.

Since $\mathrm{Z}^0_{hb}(E) \to A$ is surjective, it is clear that we can apply the same argument to prove that $E \cong \mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} E_0$. □

Corollary 4.2. *Let $E \to A$ be an indecomposable homogeneous vector bundle. Then:*

(i) *E_0 is an indecomposable $\mathrm{End}_A(E)$-module;*
(ii) *E_0 is an indecomposable $\mathrm{Z}^0_A(E)$-module.*

Proof. In order to prove (i), suppose that $E_0 \cong V_1 \oplus V_2$ as $\mathrm{Z}^0_A(E)$-modules. Then

$$\big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} V_1\big) \oplus \big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} V_2\big) \cong_{vb} \big(\mathrm{Z}^0_{hb}(E) *_{\mathrm{Z}^0_A(E)} (V_1 \oplus V_2)\big) \cong_{vb} E$$

as vector bundles over A, where the last isomorphism is given by Theorem 4.1; hence, E is decomposable. It is clear that (i) implies (ii). □

The converse of Corollary 4.2 is false, as the following example shows.

Example 4.3. Let L be a homogeneous line bundle over A and consider $E = L \oplus L$. Then $\mathrm{Aut}_A(E) \cong \mathrm{GL}_2(\Bbbk)$ and E_0 is a indecomposable $\mathrm{Aut}_A(E)$-module.

As a consequence of Theorem 4.1, we have the following corollary which characterizes the trivial homogeneous vector bundles of rank r. We denote by $I_r = \bigoplus_{i=1}^r \mathcal{O}_A = A \times \Bbbk^r$, the trivial bundle of rank r.

Corollary 4.4. *Let $E \to A$ be a homogeneous vector bundle or rank r. Then E is isomorphic to the trivial bundle of rank r if and only if $\mathrm{End}_{hb}(E) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$.*

Proof. It is clear that the trivial bundle I_r, of rank r, has endomorphisms monoid $\mathrm{End}_{hb}(I_r) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$.

Let $E \to A$ be such that $\mathrm{End}_{hb}(E) \cong_{am} A \times \mathrm{End}(\Bbbk^n)$. Then, since $E \cong_{vb} \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0$, it follows that

$$E \cong_{vb} \big(A \times \mathrm{GL}_n(\Bbbk)\big) *_{\mathrm{GL}_n(\Bbbk)} \Bbbk^n \cong_{vb} A \times \Bbbk^n.$$

$\square$

In [16, Lem 1.4] Miyanishi gives a characterization of homogeneous vector bundles $E \to X$ (over complete homogeneous spaces) in terms of the existence of schematic sections for certain fibrations. For the abelian case, Theorem 4.1 gives a simple proof of such result. Recall that a schematic section of a fibration $\pi : \mathrm{Aut}_{hb}(E) \to A$ is a morphism $\sigma : A \to \mathrm{Aut}_{hb}(E)$ such that $\pi \circ \sigma = \mathrm{Id}_A$.

Corollary 4.5. *Let $E \to A$ be a homogeneous vector bundle of rank r. If $\pi : \mathrm{Aut}_{hb}(E) \to A$ has a schematic section, then $E \cong_{vb} I_r$.*

Proof. Let $\sigma : A \to \mathrm{Aut}_{hb}(E)$, $\sigma(a) = \big(\sigma_1(a), t_a\big)$, be a schematic section, and let $\varphi : A \times E_0 \to \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 \cong_{vb} E$, be the morphism given by $\varphi(a, v) = \big[(\sigma_1(a), t_a), v\big]$. Clearly, φ is a homomorphism of homogeneous vector bundles. We claim that φ is injective. If this is the case, then φ is an isomorphism, since both vector bundles have the same rank.

Let $(a, v), (a', v') \in A \times E_0$ be such that $\varphi(a, v) = \varphi(a, v')$. Then

$$\big[(\sigma_1(a), t_a), v\big] = \big[(\sigma_1(a'), t_{a'}), v'\big],$$

and it follows that $a = a'$, and hence $v = v'$. $\square$

Corollary 4.6. *Let $\rho : E \to A$ and $\rho' : E' \to A$ be homogeneous vector bundles. Then the following statements are equivalent:*

(i) $E \cong_{vb} E'$;

(ii) $\mathrm{Aut}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E')$, and $E_0 \cong E'_0$ as rational $\mathrm{Aut}_A(E') \cong \mathrm{Aut}_A(E)$-modules;

(iii) $\mathrm{Aut}_{hb}(E) \cong_{am} \mathrm{Aut}_{hb}(E')$, and $E_0 \cong E'_0$ as rational $\mathrm{Z}_A^0(E') \cong \mathrm{Z}_A^0(E)$-modules.

Proof. The implications (i) $\implies$ (ii) $\implies$ (iii) are clear.

Assume that (iii) holds. Let $\psi : \text{Aut}_{hb}(E) \to \text{Aut}_{hb}(E')$ be an isomorphism of algebraic groups and let $\Phi : E_0 \to E'_0$ be a morphism of $\text{Z}_A^0(E)$-modules, that is $\Phi(g \cdot v) = \psi(g) \cdot \Phi(v)$. Then the morphism $\varphi : \text{Z}^0(E) \times E_0 \to E'$, defined as $\varphi(g, v) = \psi(g)\big(\Phi(v)\big)$ induces the required isomorphism $E \to E'$. $\qquad\qquad\square$

Remark 4.7. It is well known that $\text{Aut}_A(E) \cong_{am} \text{Aut}_A(E')$ (or $\text{End}_A(E) \cong_{am}$ $\text{End}_A(E')$) does not imply that $E \cong E'$. However, Corollary 4.4 shows the trivial bundle is characterized by its endomorphisms monoid. One can see that in the case of the trivial bundle, $\text{Z}_A^0(I_r) = \Bbbk^* \text{Id}$ acts by homotheties in the fiber. In the general case, the group $\text{Z}_A^0(E)$ could be larger and there could exist two different irreducible representations of the same dimension. This is the main obstruction to generalize Corollary 4.4, and raises the following

Question 4.8. *Let $E, E' \to A$ be two indecomposable homogeneous vector bundles. Does the existence of an isomorphism $\text{Aut}_{hb}(E) \cong_{am} \text{Aut}_{hb}(E')$ (or $\text{End}_{hb}(E) \cong_{am} \text{End}_{hb}(E')$) imply that $E \cong E'$?*

The following Lemma is a straightforward generalization of [14, Lem. 4.3]. We omit the proof, since it is an easy adaptation of the cited result (see also [20, Thm. 2]).

Lemma 4.9. *Let $\rho : L \to A$ be an homogeneous line bundle. Then there exists an structure of algebraic monoid $L \times L \to L$ such that ρ is a morphism of algebraic monoids. The fiber $\rho^{-1}(0) = L_0 \cong \Bbbk$ is central in L. In particular, L is a commutative algebraic monoid. Moreover, the unit group for this monoid is $G(L) = L \setminus \Theta(L))$, where $\Theta(L)$ is the image of the zero section of L.*

Corollary 4.10. *Let $\rho : L \to A$ be an homogeneous line bundle. Then $\text{End}_{hb}(L) \cong_{vb} L$.*

Proof. By Lemma 4.9 L is an algebraic monoid. For any $x \in L$ let $l_x : L \to L$ be the endomorphisms defined as $l_x(y) = xy$ (the product on the algebraic monoid L). Hence, L is a sub-bundle of $\text{End}_{hb}(L)$. But $\text{End}_A(L) \cong \Bbbk$; hence $\text{End}_{hb}(L)$ is a line bundle, and $L = \text{End}_{hb}(L)$. $\qquad\square$

4.1. **Structure of the endomorphisms monoid of a homogeneous vector bundle.**

Let $E \to A$ be a homogeneous vector bundle over an abelian variety. We have proved in Theorem 3.3 that the Chevalley's decomposition of $\text{End}_{hb}(E)$ induces on $\text{End}_{hb}(E)$ a structure of vector bundle over A, of fiber $\text{End}_A(E)$. We establish now the relationship between the decomposition given by Miyanishi (see Remark 2.9) and the structure of the endomorphisms monoid, generalizing in this way Corollary 4.10:

Recall from Remark 2.9 that an indecomposable homogeneous vector bundle $E \to A$ is of the form $E \cong L \otimes F$, where L is an homogeneous line bundle and F is unipotent homogeneous vector bundle. Hence, E is a successive extensions of the line bundle L.

Proposition 4.11. *For $i = 1, 2$, let $E_i \to A$ be an indecomposable homogeneous vector bundle of rank $n_i = \mathrm{rk}(E_i)$, and $E_i \cong L_i \otimes F_i$, where L_i is an homogeneous line bundle and F_i is unipotent homogeneous vector bundle. Then*

(1) If $L_1 \not\cong_{hb} L_2$, then

$$\mathrm{Hom}_{hb}(E, E') = \{\theta_a : E \to E' : a \in A\} \cong_{vb} A \times \{0\},$$

where if $v \in (E)_x$, then $\theta_a(v) = 0_{a+x} \in E'$.

(2) If $L_1 \cong_{vb} L_2 \cong L$, then

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong_{vb} L \otimes \mathrm{Hom}_{hb}(F_1, F_2).$$

In particular, $\mathrm{End}_{hb}(E) \cong_{vb} L \otimes \mathrm{End}_{hb}(F)$.

Proof. By Proposition 3.8,

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong \mathrm{Aut}_{hb}(E_1) *_{\mathrm{Aut}_A(E_1)} \mathrm{Hom}_A(E_1, E_2).$$

We claim that if $L_1 \not\cong L_2$, then $\mathrm{Hom}_A(E_1, E_2) = 0$. Thus,

$$\mathrm{Hom}_{hb}(E_1, E_2) \cong \mathrm{Aut}_{hb}(E_1) *_{\mathrm{Aut}_A(E_1)} \{0\} \cong A \times \{0\}.$$

Indeed, let

$$L = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_{n_1-1} \subset H_{n_1} = E_1$$

be the filtration associated to such E_1 and $0 \neq \varphi \in \mathrm{Hom}_A(E_1, E_2)$. Let $i \in \{0, \ldots, n_1 - 1\}$ be such that $H_i \subset \mathrm{Ker}(\varphi)$ but $H_{i+1} \not\subset \mathrm{Ker}(\varphi)$. Let $j \in \{0, \ldots, n' - 1\}$ be such that $\mathrm{Im}(\varphi) \subset K'_{j+1}$, $\mathrm{Im}(\varphi) \not\subset K'_j$ where

$$0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n_2-1} \subset K_{n_2} = E_2$$

be the filtration associated to such E_2.

Then φ induces a non zero morphism $\widetilde{\varphi} : L_1 \cong H_{i+1}/H_i \to K_{j+1}/K_j \cong L_2$. Since both are algebraically equivalent to zero, $\widetilde{\varphi}$ is an isomorphism, and hence $L_1 \cong L_2$.

Suppose now that $L_1 \cong_{vb} L_2 = L$. Then

$$\mathrm{Hom}_A(E_1, E_2) \cong (L \otimes F_1)^\vee \otimes (L \otimes F_2))$$
$$\cong F_1^\vee \otimes F_2$$
$$\cong \mathrm{Hom}_A(F_1, F_2).$$

It follows that $\mathrm{Hom}_{hb}(E, E')$ and $L \otimes \mathrm{Hom}_{hb}(F, F')$ are homogeneous vector bundles of the same rank. Consider the homomorphism $\varphi : L \otimes \mathrm{Hom}_{hb}(F, F') \to$

$\text{Hom}_{hb}(E, E')$ given by $\varphi\big((l, t_a) \otimes (h, t_a)\big) = (l \otimes h, t_a)$, where we are identifying $L \cong_{vb} \text{End}_{hb}(L)$, and $l \otimes h(v \otimes w) = l(v) \otimes h(w)$, for $v \otimes w \in L \otimes F$. φ is an injection, and hence an isomorphism of vector bundles. $\qquad \square$

Let $E \to A$ be a homogeneous vector bundle. As a consequence of Theorem 3.12 and Proposition 4.11, we have the following explicit description of $\text{End}_{hb}(E)$.

Theorem 4.12. *Let $E = \bigoplus_{i,j} L_i \otimes F_{i,j}$ and $E' = \bigoplus_{i,j} L_i \otimes F'_{i,j}$ be a homogeneous vector bundles, where L_i are homogeneous line bundles, F_{ij} and F'_{ij} unipotent homogeneous vector bundles and $L_i \not\cong L_j$ if $i \neq j$. Then*

$$\text{Hom}_{hb}(E, E') \cong_{vb} \bigoplus_i L_i \otimes \big(\oplus_{j,k} \text{Hom}_{hb}(F_{i,j}, F'_{i,k})\big).$$

In particular,

$$\text{End}_{hb}(E) \cong_{vb} \bigoplus_i L_i \otimes \big(\oplus_{j,k} \text{Hom}_{hb}(F_{i,j}, F_{i,k})\big).$$

$\qquad \square$

In view of Theorem 4.12, the description of the endomorphisms monoid of a homogeneous vector bundle follows from the description of $\text{Hom}_{hb}(F, F')$ for two unipotent indecomposable homogeneous vector bundles. In Section 5 we describe $\text{Hom}_{hb}(F, F')$ for vector bundles of small rank.

By construction, the map $\sigma : A \to \text{End}_{hb}(E)$ given by $a \mapsto \theta_a$ is the zero section of the vector bundle $\text{End}_{hb}(E) \to A$. As an easy application of Corollary 2.4 we have the following result.

Corollary 4.13. *Let $\rho : E \to A$ be a homogeneous vector bundle over an abelian variety. Then the algebraic monoid $\text{End}_{hb}(E)$ has Kernel*

$$\text{Ker}\big(\text{End}_{hb}(E)\big) = \Theta(E) = \big\{\theta_a : E \to E : \theta_a(v) = 0_{\rho(v)+a}\big\}.$$

In particular, $\text{Ker}\big(\text{End}_{hb}(E)\big)$ is an algebraic group, isomorphic to the abelian variety A. $\qquad \square$

We shall give now a decomposition of $\text{End}_{hb}(E)$ using pseudo-nilpotent endomorphisms.

Definition 4.14. Let $E \to A$ be an homogeneous vector bundle over an abelian variety. An endomorphism $f \in \text{End}_{hb}(E)$ is *pseudo-nilpotent of index n* if $f^n \in \Theta(E) = \text{Ker}\big(\text{End}_{hb}(E)\big)$ whereas $f^{n-1} \notin \Theta(E)$. Denote by $\mathcal{N}_{hb}(E)$ the set of pseudo-nilpotent endomorphisms.

Example 4.15. Let L be a homogeneous line bundle. The algebraic monoid $\text{End}_{hb}(L) = L$ decomposes in a disjoint union of its unit group and its $\text{Kernel}(L) = L = \text{Aut}_{hb}(L) \sqcup \Theta(L)$. In particular, $\mathcal{N}_{hb}(L) = \Theta(L)$.

From Atiyah's results (see [2]) we have that for indecomposable vector bundles $\mathrm{End}_A(E) = \Bbbk \cdot \mathrm{Id}_E \oplus N_A(E)$ – recall that $N_A(E) \subset \mathrm{End}_A(E)$ is the subset of idempotents elements. For indecomposable homogeneous vector bundle E over an abelian variety we have the following decomposition of $\mathrm{End}_{hb}(E)$.

Theorem 4.16. *Let $E \to A$ be a indecomposable homogeneous vector bundle over an abelian variety. Then:*

(1) The algebraic monoid $\mathrm{End}_{hb}(E)$ decomposes as the disjoint union of $\mathrm{Aut}_{hb}(E)$ and $\mathcal{N}_{hb}(E)$, that is,

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E).$$

In particular, $\mathcal{N}_{hb}(E)$ is an ideal of $\mathrm{End}_{hb}(E)$.

(2) The set $\mathcal{N}_{hb}(E)$ of pseudo-nilpotent elements is a homogeneous vector bundle over A of $\mathrm{rk}\,\mathcal{N}_{hb}(E) = \mathrm{rk}\,\mathrm{End}_{hb}(E) - 1$. Moreover,

$$\mathcal{N}_{hb}(E) = \mathrm{Z}_{hb}^0(E) \cdot \mathrm{N}_A(E) \cong \mathrm{Z}_{hb}^0(E) *_{\mathrm{Z}_A^0(E)} \mathrm{N}_A(E).$$

where $\cdot$ denotes the composition (the product in $\mathrm{End}_{hb}(E)$). In particular, the fiber of $\pi : \mathcal{N}_{hb}(E) \to A$ is isomorphic to $\mathrm{N}_A(E)$, and π is a morphism of algebraic semigroups.

Proof. Recall from Remark 3.2 that

$$\mathrm{End}_{hb}(E) = \mathrm{Z}_{hb}^0(E) \cdot \mathrm{End}_A(E) \cong \mathrm{Z}_{hb}^0(E) *_{\mathrm{Z}_A^0(E)} \mathrm{End}_A(E).$$

Let $f \in \mathrm{End}_A(E)$ and $z \in \mathrm{Z}_{hb}^0(E)$. Since $\mathrm{End}_A(E) = \Bbbk \,\mathrm{Id} \oplus \mathrm{N}_A(E)$ it follows that either $f \in \mathrm{Aut}_A(E)$ or $f \in \mathrm{N}_A(E)$. If $f \in \mathrm{Aut}_A(E)$, then $z \cdot f \in \mathrm{Aut}_{hb}(E)$. If $f \in \mathrm{N}_A(E)$, with $f^n = 0$, then clearly $(z \cdot f)^n = z^n \cdot f^n = \theta_{\pi(z^n)}$, where $\pi : \mathrm{Z}_{hb}^0(E) \to A$ is the canonical projection. Therefore,

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E).$$

Note that in particular we have proved that

$$\mathcal{N}_{hb}(E) = \mathrm{Z}_{hb}^0(E) \cdot \mathrm{N}_A(E) \cong \mathrm{Z}_{hb}^0(E) *_{\mathrm{Z}_A^0(E)} \mathrm{N}_A(E).$$

Since $\mathcal{N}_{hb}(E) = \mathrm{End}_{hb}(E) \setminus \mathrm{Aut}_{hb}(E)$, it follows that $\mathcal{N}_{hb}(E)$ is an ideal. In particular, $\mathcal{N}_{hb}(E)$ is $\mathrm{Aut}_{hb}(E)$-stable, and hence a homogeneous vector bundle, since composing with an automorphism gives an automorphism of $\mathrm{End}_{hb}(E)$.

Finally, the equality $\mathrm{rk}\,\mathcal{N}_{hb}(E) = \mathrm{rk}\,\mathrm{End}_{hb}(E) - 1$ follows again from the fact that $\mathrm{End}_A(E) = \Bbbk \,\mathrm{Id} \oplus N_A(E)$ and $N_A(E) \neq 0$. $\qquad\square$

Theorem 4.17. *Let $E \to A$ be a indecomposable vector bundle obtained by successive extensions of the homogeneous line bundle L. Then $\mathrm{End}_{hb}(E)$ is obtained by successive extensions of the homogeneous line bundle L.*

Proof. Indeed, let $L' \subset \text{End}_{hb}(E)$ be a homogeneous line sub-bundle and $\rho : \text{End}_{hb}(E) \to_{vb} L$ as in Theorem 4.19. Let $f \in L' \cap \text{End}_A(E)$ be a non zero nilpotent element. In other words, $f \in L'_0 \setminus \{\theta_0\}$. Let $e \in E_0$ be such that $f(e) \neq 0$. Since $\text{End}_{hb}(E) \cong_{vb} \text{Aut}_{hb}(E) *_{\text{Aut}_A(E)} \text{End}_A(E)$, for every $a \in A$, there exists $(h_a, t_a) \in \text{Aut}_{hb}(E)$ such that $L'_a = \Bbbk(h_a \circ f)$. Hence, $\varphi L' \to E$, $\varphi(l) = l(e)$ is an injective morphism of homogeneous vector bundles, and since E is obtained by successive extensions of L, it follows that $L' \cong L$. Thus, $\text{End}_{hb}(E)$ is also obtained by successive extensions of L. $\qquad\square$

Proposition 4.18. *Let $E \to A$ be an indecomposable homogeneous vector bundle of rank $r \geq 2$, obtained by successive extension of the homogeneous line bundle L. Then there exists an injective morphism of vector bundles $\psi : L \hookrightarrow_{vb} \mathcal{N}_{hb}(E)$. In particular, $\text{rk}\,\mathcal{N}_{hb}(E) \geq 1$, and if $\text{rk}\,\mathcal{N}_{hb}(E) \geq 2$ then $\mathcal{N}_{hb}(E)$ is obtained by successive extensions of the line bundle L.*

Proof. By theorems 4.16 and 4.17, $\mathcal{N}_{hb}(E)$ is a homogeneous vector bundle, obtained by succesive extensions of L (sinceit is a sub-bundle of $\text{End}_{hb}(E)$). Hence, we only need to prove that $(rk)\,\mathcal{N}_{hb}(E) \geq 1$.

Let $0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$ be a filtration such that $E_i/E_{i-1} \cong L$. By proposition 2.8, there exists $0 \neq \varphi \in \text{End}_A(E)$ such that $\varphi^2 = 0$, and thus $(rk)\,\mathcal{N}_{hb}(E) \neq 0$. $\qquad\square$

We now to state and prove a generalization of Miyanishi structure theorem (see Remark 2.9).

Theorem 4.19. *Let $E \cong L \otimes F \to A$ be an indecomposable homogeneous vector bundle, where L is a homogeneous line bundle and F is an unipotent homogeneous vector bundle. Then there exists an exact sequence of vector bundles over A, compatible with the structures of algebraic semigroups (i.e. the morphisms are compatible with the composition)*

$$0 \longrightarrow \mathcal{N}_{hb}(E) \longrightarrow \text{End}_{hb}(E) \overset{\rho}{\longrightarrow} \text{End}_{hb}(L) \cong L \longrightarrow 0$$

Moreover, the exact sequence splits if and only if $E \cong L$.

Proof. Since $L \subset E$ is $\text{Aut}_{hb}(E)$-stable (Remark 2.9) is also $\text{End}_{hb}(E)$-stable. The restriction $\rho : \text{End}_{hb}(E) \to \text{End}_{hb}(L)$ is a morphism of algebraic monoids, in particular it is compatible with the structure of vector bundles.

By Theorem 4.16, $\text{End}_{hb}(E) = \text{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E)$. It is clear that if $g \in \text{Aut}_{hb}(E)$, then $\rho(g) \in \text{Aut}_{hb}(L) = L \setminus \Theta(L)$. If $(f, t_a) \in \mathcal{N}_{hb}(E)$, then there exists $n \in \mathbb{N}$ such that $f^n = \theta_{na}$. It follows that the restriction $f|_L \in \mathcal{N}_{hb}(L) = \Theta(L)$. Hence, $\mathcal{N}_{hb}(E) = \text{Ker}(\rho)$.

Assume now that the exact sequence splits, then there exists an immersion of homogeneous vector bundles $\iota : L \hookrightarrow \text{End}_{hb}(E)$, such that $\rho \circ \iota = \text{Id}_L$. In particular, $\iota(L \setminus \Theta(L)) \subset \text{Aut}_{hb}(E)$. Let E_0 be the fiber of E over $0 \in A$

and consider the morphism of vector bundles

$$\varphi : L \otimes E_0 \cong \mathrm{End}_{hb}(L) \otimes E_0 \to E \quad , \quad \varphi(f \otimes v) = f(v).$$

Let $e \in E$ be such that $\pi(e) = a$. Let $f \in L \setminus \Theta(L)$ be such that $\alpha(f) = a$. Then $\varphi\big(\iota(f) \otimes \iota(f)^{-1}(e)\big) = e$, and it follows that φ is a surjective morphism of homegeneous vector bundles of the same rank. Thus, φ is an isomorphism. But $L \otimes E_0$ is decomposable unless $\dim E_0 = 1$. It follows that $E \cong_{vb} L$. $\quad\square$

Remark 4.20. We can resume the results proved until now about the structure of $\mathrm{End}_{hb}(E)$ as follows: Let $E = L \otimes F \to A$ be an indecomposable homogeneous vector bundle, where L is a homogeneous line bundle and F is an unipotent vector bundle. Then,

(1) $\mathrm{End}_{hb}(E)$ is a non-singular algebraic monoid. Its Albanese morphism $\alpha : \mathrm{End}_{hb}(E) \to A$ is a morphism of algebraic monoids, with Kernel $\alpha^{-1}(0) = \mathrm{End}_A(E)$ (see [12] and Remark 3.1).

(2) $\mathrm{End}_{hb}(E) \to A$ is a homogeneous vector bundle, obtained by successive extensions of L (see theorems 3.3 and 4.17).

(3) The Kernel of the algebraic monoid $\mathrm{End}_{hb}(E)$ is the zero section

$$\Theta(E) = \{\theta_a : E \to E : \theta(v_x) = 0_{x+a} \ \forall \, v_x \in E_x\} = \mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big).$$

In particular, $\mathrm{Ker}\big(\mathrm{End}_{hb}(E)\big)$ is an algebraic group, isomorphic to the abelian variety A (see Corollary 4.13).

(4) Let $\mathcal{N}_{hb}(E)$ denote the set of pseudo-nilpotent elements, Then the algebraic monoid $\mathrm{End}_{hb}(E)$ decomposes as a disjoint union

$$\mathrm{End}_{hb}(E) = \mathrm{Aut}_{hb}(E) \sqcup \mathcal{N}_{hb}(E)$$

(see Theorem 4.16). In particular, $\mathcal{N}_{hb}(E)$ is an ideal of $\mathrm{End}_{hb}(E)$. Moreover, $\mathcal{N}_{hb}(E)$ is a homogeneous vector bundle, obtained by successive extensions of L (see Proposition 4.18).

(5) There exists a exact sequence of vector bundles

$$0 \longrightarrow \mathcal{N}_{hb}(E) \longrightarrow \mathrm{End}_{hb}(E) \overset{\rho}{\longrightarrow} \mathrm{End}_{hb}(L) \cong L \longrightarrow 0$$

Moreover, the morphisms appearing in the sequence are compatible with the structures of semigroup, and the sequence splits if and only if $E \cong L$ (see Theorem 4.19).

5. Explicit calculations in small rank

In this section we describe explicitly $\mathrm{End}_{hb}(E)$ and $\mathrm{Hom}_{hb}(E, E')$ when the homogeneous vector bundles have small rank.

5.1. Homomorphisms between a homogeneous line bundle and an homogeneous vector bundle.

From Theorem 4.12 without loss of generality we work with indecomposable unipotent homogeneous vector bundles. As we saw in Section 4, any line bundle is an algebraic monoid, and is isomorphic to its endomorphism monoid. We give a description of $\mathrm{Hom}_{hb}(E, E')$ when one of the homogeneous vector bundles is a line bundle and the other is indecomposable and unipotent homogeneous vector bundle.

Proposition 5.1. *Let F be a indecomposable unipotent homogeneous vector bundle, $\mathrm{rk}\, F = n \geq 2$ and L a homogeneous line bundle. Then,*

(1) *if $L = \mathcal{O}_A$ is the trivial bundle then*
  - $\mathrm{Hom}_{hb}(\mathcal{O}_A, F)$ *is trivial, with fiber isomorphic to* $\mathrm{Hom}_A(\mathcal{O}_A, F) = H^0(A, F)$, *i.e.*

$$\mathrm{Hom}_{hb}(\mathcal{O}_A, F) \cong_{vb} A \times H^0(A, F).$$

  - $\mathrm{Hom}_{hb}(F, \mathcal{O}_A)$ *is trivial, with fiber isomorphic to* $\mathrm{Hom}_A(F, \mathcal{O}_A) = H^0(A, F^\vee)$, *i.e.*

$$\mathrm{Hom}_{hb}(F, \mathcal{O}_A) \cong_{vb} A \times H^0(A, F^\vee).$$

(2) *If $L \neq \mathcal{O}_A$ then $\mathrm{Hom}_{hb}(F, L) = \mathrm{Hom}_{hb}(L, F) = A \times \{0\}$.*

Proof. The proposition follows from Theorem 3.12, since

$$\mathrm{Hom}_{hb}(\mathcal{O}_A, F) \cong_{vb} \mathrm{Aut}_{hb}(\mathcal{O}_A) *_{\mathrm{Aut}_A(\mathcal{O}_A)} \mathrm{Hom}_A(\mathcal{O}_A, F)$$
$$=_{vb} (A \times \Bbbk^*) *_{\Bbbk^* \mathrm{Id}} \mathrm{Hom}_A(\mathcal{O}_A, F)$$
$$\cong_{vb} A \times \mathrm{Hom}_A(\mathcal{O}_A, F).$$

and

$$\mathrm{Hom}_{hb}(F, \mathcal{O}_A) \cong_{vb} \mathrm{Aut}_{hb}(\mathcal{O}_A) *_{\mathrm{Aut}_A(\mathcal{O}_A)} \mathrm{Hom}_A(F, \mathcal{O}_A)$$
$$=_{vb} (A \times \Bbbk^*) *_{\Bbbk^* \mathrm{Id}} \mathrm{Hom}_A(F, \mathcal{O}_A)$$
$$\cong_{vb} A \times \mathrm{Hom}_A(F, \mathcal{O}_A).$$

Part (2) follows from Proposition 4.11. $\qquad\square$

Corollary 5.2. *Let $E = L \otimes F$ be a indecomposable homogeneous vector bundle of $\mathrm{rk}\, E = n \geq 2$ with $L \neq \mathcal{O}_A$ and L' a homogeneous line bundle. Then,*

(1) *if $L = L'$ then*
  - $\mathrm{Hom}_{hb}(L, E) \cong_{hb} \oplus^r L$ *where $r = \dim H^0(A, F)$.*
  - $\mathrm{Hom}_{hb}(E, L) \cong_{hb} \oplus^s L$ *where $s = \dim H^0(A, F^\vee)$.*

(2) *If $L \neq L'$ then $\mathrm{Hom}_{hb}(E, L') \cong_{hb} \mathrm{Hom}_{hb}(L', E) \cong_{hb} A \times \{0\}$.*

Proof. The corollary follows from Proposition 5.1 since from Proposition 4.11 $\mathrm{Hom}_{hb}(L, E) \cong_{hb} L \otimes \mathrm{Hom}_{hb}(\mathcal{O}_A, F)$ and $\mathrm{Hom}_{hb}(E, L) \cong_{hb} L \otimes \mathrm{Hom}_{hb}(F, \mathcal{O}_A)$ if $L = L'$ and $\mathrm{Hom}_{hb}(L, E) \cong_{hb} \mathrm{Hom}_{hb}(E, L) \cong_{hb} A \times \{0\}$ if $L \neq L'$. $\qquad\square$

5.2. **Homomorphisms between indecomposable homogeneous vector bundles of rank** 2.

Let E and E' be two non-isomorphic indecomposable homogeneous vector bundles of rank 2. Let

$$\rho_E : 0 \to L \xrightarrow{j} E \xrightarrow{\pi} L \to 0$$

and

$$\rho_{E'} : 0 \to L' \xrightarrow{i_1} E' \xrightarrow{p_1} L' \to 0$$

be the extensions associated to E and E' respectively. By Proposition 4.11, if $L \neq L'$ then $\operatorname{Hom}_A(E, E') = 0$.

If $L \cong_{vb} L'$, let $0 \neq \phi = i_1 \circ \pi \in \operatorname{Hom}_A(E, E')$. We shall prove that $\operatorname{Hom}_A(E, E') = \Bbbk\phi$.

Proposition 5.3. *Let E and E' be two non-isomorphic indecomposable homogeneous vector bundles of rank* 2*, obtained by successive extensions of a line bundle L. Then* $\operatorname{Hom}_{hb}(E, E') \cong_{vb} L$.

Proof. By Theorem 4.17, we only need to show that $\operatorname{Hom}_A(E, E') = \Bbbk\phi$, where ϕ is as in the introduction of this paragraph.

Let $0 \neq \varphi \in \operatorname{Hom}_A(E, E')$. Since E and E' are non-isomorphic, the image $\varphi(E)$ is a line sub-bundle L_0 of E'. Moreover, $\psi : p_1 \circ \varphi : E \to L$ is a non zero homomorphism of homogeneous vector bundles. If $\varphi(E) = L_0 \neq L$ we get a contradiction since from Corollary 5.2 we have that $\operatorname{Hom}_{hb}(E, L) \cong_{hb} A \times \{0\}$. Thus, $L_0 \cong_{hb} L$.

If $\varphi \neq \lambda\phi$, $0 \neq \varphi \circ j : L \to L_0$ is an isomorphism, and then $\varphi \circ (\varphi \circ j)^{-1} : E \to L$ is a spitting for j, which is a contradiction. Hence, $\varphi = \lambda\phi$, with $\lambda \in \Bbbk$. $\square$

5.3. **Endomorphisms monoid of indecomposable homogeneous vector bundles of small rank.**

In Proposition 4.11 we prove that if $E = L \otimes F$ is an indecomposable homogeneous vector bundle then $\operatorname{End}_{hb}(E) = L \otimes \operatorname{End}_{hb}(F)$. We are interested now in describing $\operatorname{End}_{hb}(F)$. Recall that F is a successive extensions of the trivial bundle $\mathcal{O}_A$ and hence E is a successive extensions of L. The algebra of endomorphisms of vector bundles of small rank over a curve that are successive extensions of line bundles has been studied by Brambila-Paz in [6, 7, 8, 9, 10]. Similar results apply for vector bundles over abelian varieties. We shall use these results in order to give an explicit description of the endomorphisms monoid of indecomposable homogeneous vector bundles of rank 2 and 3.

Let $E = L \otimes F \to A$ be a indecomposable homogeneous vector bundle of rank 2 over A. Hence E fits in the following exact sequence of vector bundles

$$0 \to L \xrightarrow{i} E \xrightarrow{p} L \to 0$$

From Theorem 3.3 we have that $\dim \operatorname{End}_A(E) \leq 2$ and from the above exact sequence we have that $0 \neq \varphi = i \circ p : E \to E$ satisfied $\varphi^2 = 0$, hence $\operatorname{rk} \mathcal{N}_{hb}(E) \geq 1$ (see Proposition 4.18). Therefore $\operatorname{End}_{hb}(E)$ is a homogeneous vector bundle of rank 2. Moreover, by a result of Atiyah (see [2]) it follows that $\operatorname{End}_A(E) \cong \Bbbk \operatorname{Id} \oplus \Bbbk \varphi$. Hence, we have the following Proposition:

Proposition 5.4. *Let $E \to A$ be an indecomposable homogeneous vector bundle of rank 2. Then $\operatorname{End}_{hb}(E)$ is a commutative algebraic monoid, and $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^2)$. Moreover, $\operatorname{End}_{hb}(E) \cong_{vb} E$.*

Proof. The only assertion that remains to prove is the last one. For this, observe that
$$\operatorname{End}_A(E) \cong_{am} \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) : a, b \in \Bbbk \right\},$$
with action over the fiber E_0 given as follows: consider an isomorphism $E_0 \cong \Bbbk^2$, in such a way that $(1,0) \in \operatorname{Ker}(p)_0$. In other words, $(1,0)$ belongs to the fiber over $0 \in A$, of the $\operatorname{Aut}(E)$-stable line bundle $L \subset E$. Under this identification, the action $\operatorname{End}_A(E) \times E_0 \to E_0$ is given by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) \cdot (x,y) = (ax + by, ay)$.

On the other hand, the action of $\operatorname{Aut}_A(E)$ on $\operatorname{End}_A(E)$ is given by $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) \cdot \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix} \right) = \left(\begin{smallmatrix} ax & ay+bx \\ 0 & ax \end{smallmatrix} \right)$. Thus, there exists an isomorphism of $\operatorname{Aut}_A(E)$-modules $\varphi : E_0 \to \operatorname{End}_A(E)$. It follows that the morphism
$$\begin{aligned} \psi : \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} E_0 &\to \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} \operatorname{End}_A(E) \\ \psi\big([(f, t_a), e_0] \big) &= \big[(f, t_a), \varphi(e_0) \big], \end{aligned}$$
is an isomorphism of vector bundles. Thus, $E \cong_{vb} \operatorname{End}_{hb}(E)$. $\qquad \square$

Remark 5.5. Actually Proposition 5.6 could be done first for indecomposable unipotent vector bundles F to obtain $\operatorname{End}_{hb}(F) = F$. Then if $E = L \otimes F$ we have $\operatorname{End}_{hb}(E) = L \otimes \operatorname{End}_{hb}(F) = L \otimes F = E$.

Let $E \to A$ be an indecomposable homogeneous vector bundle of rank $r \geq 2$, such that $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^r)$. As in the rank 2 case, there exists an isomorphism $\varphi : E_0 \to \operatorname{End}_A(E)$ of $\operatorname{Aut}_A(E)$-modules which induce a isomorphism
$$\begin{aligned} \psi : \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} E_0 &\to \operatorname{Aut}_{hb}(E) *_{\operatorname{Aut}_A(E)} \operatorname{End}_A(E) \\ \psi\big([(f, t_a), e_0] \big) &= \big[(f, t_a), \varphi(e_0) \big], \end{aligned}$$
of vector bundles and hence, $E \cong_{vb} \operatorname{End}_{hb}(E)$. We shall prove in detail the above affirmation for rank 3.

Proposition 5.6. *Let $E \to A$ be an indecomposable homogeneous vector bundle of rank 3 with $\operatorname{End}_A(E) \cong \Bbbk[t]/(t^3)$. Then $\operatorname{End}_{hb}(E)$ is a commutative algebraic monoid, and $\operatorname{End}_{hb}(E) \cong_{vb} E$.*

Proof. It suffices to prove that the representations $\operatorname{Aut}_A(E) \times \operatorname{End}_A(E) \to \operatorname{End}_A(E)$ and $\operatorname{Aut}_A(E) \times E_0 \to E_0$ are isomorphic. In this case,
$$\operatorname{End}_A(E) \cong_{am} \left\{ \left(\begin{smallmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{smallmatrix} \right) a, b \in \Bbbk \right\},$$

and the action over the fiber E_0 given as follows: consider an isomorphism $E_0 \cong \Bbbk^3$, such that $(1,0,0) \in (E_1)_0$, where $L = E_1 \subset E_2 \subset E$ is a $\mathrm{Aut}(E)$-stable filtration. Under this identification the action $\mathrm{End}_A(E) \times E_0 \to E_0$ is given by $\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \cdot (x,y,z) = (ax + by + cz, ay + bz, az)$.

On the other hand, the action of $\mathrm{Aut}_A(E)$ on $\mathrm{End}_A(E)$ is given by

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \cdot \begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} ax & bx+ay & ax+by+cz \\ 0 & ax & bz+ay \\ 0 & 0 & ax \end{pmatrix}.$$

Therefore, there exists an isomorphism $\varphi : E_0 \to \mathrm{End}_A(E)$ of $\mathrm{Aut}_A(E)$-modules and hence the homomorphism

$$\begin{aligned} \psi : \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} E_0 &\to \mathrm{Aut}_{hb}(E) *_{\mathrm{Aut}_A(E)} \mathrm{End}_A(E) \\ \psi\big([(f,t_a), e_0]\big) &= [(f,t_a), \varphi(e_0)], \end{aligned}$$

is an isomorphism of vector bundles. Thus, $E \cong_{vb} \mathrm{End}_{hb}(E)$. $\qquad \square$

Remark 5.7. For indecomposable homogeneous vector bundles of rank 3 we have, from Theorem 3.3, that $\dim \mathrm{End}_A(E) \leq 4$. As in [6, 7] we have that $\mathrm{End}_A(E)$ is a commutative algebra of dimension $2 \leq \dim \mathrm{End}_A(E) \leq 3$ and all the possible algebras for $\mathrm{End}_A(E)$ are $\Bbbk[t]/(t^2)$, $\Bbbk[t]/(t^3)$ or $\Bbbk[r,s]/(r,s)^2$. The structure of the algebra of endomorphisms of $\mathrm{End}_A(E)$, and hence of $\mathrm{End}_{hb}(E)$, will depend on the extensions associated to E and their relations. For higher rank there will be more possibilities for $\mathrm{End}_A(E)$. We expect that the equivalence on categories given by Mukai could be used to describe $\mathrm{End}_A(E)$ for higher rank.

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