HERMITE NORMAL FORMS WITH A GIVEN δ-VECTOR

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ABSTRACT. Let $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ be the δ-vector of an integral polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d. By means of Hermite normal forms of square matrices, the problem of classifying the possible integral simplices with a given δ-vector $(\delta_0, \delta_1, \ldots, \delta_d)$, where $\sum_{i=0}^{d} \delta_i \leq 4$, will be studied. In consequence, following the previous work of characterizing the δ-vectors with $\sum_{i=0}^{d} \delta_i \leq 3$, the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$ will be classified. And each possible δ-vectors can be obtained by simplices.

1. INTRODUCTION

1.1. **A classification problem.** Let $\mathbb{Z}^{d\times d}$ denote the set of $d \times d$ integral matrices (a_{ij}). Recall that a matrix $A \in \mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$ of dimension d, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *unimodular equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and an integral vector w, such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$.

Given a d dimensional integral polytope $\mathcal{P}$, we can define its δ-*vector* $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ to be the coefficients appearing in the following generating function for its Ehrhart polynomial $i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^d|$ (see next subsection for more about δ-vectors):

$$\sum_{n\geq 0} i(\mathcal{P}, n)t^n = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}}.$$

Clearly, if $\mathcal{P}$ and $\mathcal{Q}$ are unimodular equivalent, then $\delta(\mathcal{P}) = \delta(\mathcal{Q})$. Conversely, given a vector $v \in \mathbb{Z}_{\geq 0}^{d+1}$, it is natural to ask what are all the integral polytopes $\mathcal{P}$ under unimodular equivalence, such that $\delta(\mathcal{P}) = v$.

In this paper, we will focus on this problem for simplices with one vertex at the origin. In addition, we do not allow any shifts in the equivalence, i.e., d dimensional integral polytopes $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if there exists a unimodular matrix U, such that $\mathcal{Q} = f_U(\mathcal{P})$. Let $\mathcal{P}$ be an integral simplex in $\mathbb{R}^d$ of dimension d with the vertices $\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_d$. Define $M(\mathcal{P}) \in \mathbb{Z}^{d\times d}$ to be the matrix with the row vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. Then we have the following connection between the matrix $M(\mathcal{P})$ and the δ-vector of $\mathcal{P}$: $|\det(M(\mathcal{P}))| = \sum_{i\geq 0} \delta_i$. In this setting $\mathcal{P}$ and $\mathcal{P}'$ are equivalent if and only if

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$M(\mathcal{P})$ and $M(\mathcal{P}')$ have the same Hermite normal form. The *Hermite normal form* of an integral square matrix B is the unique nonnegative lower triangular matrix $A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{d \times d}$ such that $A = BU$ for some unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and $0 \leq a_{ij} < a_{ii}$ for all $1 \leq j < i$ (see, for example, [6, Chapter 4]).

In other words, we can pick the Hermite normal form as the representative in each equivalence class and study the following

Problem 1.1. Given a vector $v \in \mathbb{Z}_{\geq 0}^{d+1}$, classify all possible $d \times d$ matrices $A \in \mathbb{Z}^{d \times d}$ which are in Hermite normal form with $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d) = v$, where $\mathcal{P} \subset \mathbb{R}^d$ is the integral simplex whose vertices are the row vectors of A together with the origin in $\mathbb{R}^d$.

We will present a solution of Problem 1.1 when $\sum_{i=0}^{d} \delta_i \leq 4$ in Section 4.

1.2. **Background for δ-vectors.** Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral polytope of dimension d and $\partial\mathcal{P}$ its boundary. Define the numerical functions $i(\mathcal{P}, n)$ and $i^*(\mathcal{P}, n)$ by setting

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad i^*(\mathcal{P}, n) = |n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Here $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X.

The systematic study of $i(\mathcal{P}, n)$ and $i^*(\mathcal{P}, n)$ originated in Ehrhart around 1955, who established the following fundamental properties:

(0.1) $i(\mathcal{P}, n)$ is a polynomial in n of degree d;

(0.2) $i(\mathcal{P}, 0) = 1$;

(0.3) (reciprocity law) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. An introduction to Ehrhart polynomials is discussed in [8, pp. 235–241] and [2, Part II].

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1) \qquad (1 - \lambda)^{d+1} \left[1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n) \lambda^n \right] = \sum_{i=0}^{\infty} \delta_i \lambda^i.$$

In particular, $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d + 1)$. Thus, if $\delta_1 = 0$, then $\mathcal{P}$ is a simplex. The above facts (0.1) and (0.2) together with a well-known result on generating function ([8, Corollary 4.3.1]) guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$$

which appears in Eq. (1) is the *δ-vector* of $\mathcal{P}$ and the polnomial

$$\delta_{\mathcal{P}}(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$$

which also appears in Eq. (1) is the *δ-polynomial* of $\mathcal{P}$.

It follows from the reciprocity law (0.3) that

$$(2) \qquad (1 - \lambda)^{d+1} \left[\sum_{n=1}^{\infty} i^*(\mathcal{P}, n) \lambda^n \right] = \sum_{i=0}^{d} \delta_{d-i} \lambda^{i+1}$$

In particular, $\delta_d = |(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Each δ_i is nonnegative (Stanley [9]). If $\delta_d \neq 0$, then $\delta_1 \leq \delta_i$ for every $1 \leq i < d$ ([3]). Eq. (2) says that

$$\max\{\, i \,:\, \delta_i \neq 0 \,\} + \min\{\, i \,:\, i(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset \,\} = d + 1.$$

Let $s = \max\{\, i \,:\, \delta_i \neq 0 \,\}$. Stanley [10] shows that

$$\text{(3)} \qquad \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq [s/2]$$

by using Cohen–Macaulay rings. The inequalities

$$\text{(4)} \quad \delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leq \delta_2 + \delta_3 + \cdots + \delta_i + \delta_{i+1}, \quad 1 \leq i \leq [(d-1)/2]$$

appear in [3, Remark (1.4)].

1.3. **Two related classification problems.** One of the most fundamental problems of enumerative combinatorics is to find a combinatorial characterization of all vectors that can be realized as the δ-vector of some integral polytope. For example, restrictions like $\delta_0 = 1$, $\delta_i \geq 0$, (3) and (4) are necessary conditions for a vector to be a δ-vector of some integral polytope.

In [4], it is shown that when $\sum_{i=0}^{d} \delta_i \leq 3$, (3) and (4) is both necessary and sufficient. Therefore, they give a combinatorial characterization of the possible δ-vectors. However, this is not true for $\sum_{i=0}^{d} \delta_i = 4$. In section 5, we show that (3) and (4) with an additional condition will be both necessary and sufficient to give all possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$. And in this case, all δ-vectors can be obtained by simplices (Theorem 5.1).

Another question related with Problem 1.1 is to consider for a given vector, whether or not there is a unique Hermite normal form. A more general version is the following:

Problem 1.2. [1, Open Problem 3.41] *Given integral polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$ of dimension d with $\delta(\mathcal{P}) = \delta(\mathcal{Q})$, what conditions guarantee the existence of a vector $\mathbf{w} \in \mathbb{R}^d$ together with a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ such that $\mathcal{Q} = f_U(\mathcal{P}) + \mathbf{w}$?*

Since [1, Open Problem 3.41] allows a shift in the equivalence class, Problem 1.1 is different from the simplex version of [1, Open Problem 3.41]. See the following

Example 1.3. Let $d = 2$ and $\mathcal{P}$ the triangle with the vertices $(0,0), (1,0)$ and $(0,2)$ and $\mathcal{P}' = \mathcal{P} + (-1,0)$. Thus $\mathcal{P}'$ is the triangle with the vertices $(-1,0), (0,0)$ and $(-1,2)$. Let A be the 2×2 matrix with the row vectors $(1,0)$ and $(0,2)$, and A' the 2×2 matrix with the row vectors $(-1,0)$ and $(-1,2)$. Then the Hermite normal form of A is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and that of A' is $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.

1.4. **Structure of this paper.** The way we approach Problem 1.1 is to develop an algorithm for any Hermite normal form A to compute its δ-vector (See Theorem 2.1 in Section 2). This actually gives a new way to compute the δ-vector for any integral simplex via its Hermite normal form. This algorithm can be very efficient for simplices with small volumes and prime volumes.

Based on this algorithm, as a by-product, we can derive some conditions for Hermite normal forms to have "shifted symmetric" δ-vector, namely, $\delta_i = \delta_{d+1-i}$. We will discuss these conditions for two classes of Hermite normal forms in Section 3.

In Section 4, we apply Theorem 2.1 and get a solution to Problem 1.1 when $\sum_{i=0}^d \delta_i \leq 4$. In section 5, we show that (3) and (4) with an additional condition will give all possible δ-vectors with $\sum_{i=0}^d \delta_i = 4$. And in this case, all δ-vectors can be obtained by simplices (Theorem 5.1).

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2. An algorithm for the computation of the δ-vector of a simplex

In this section, we introduce an algorithm for the δ-vector of integral simplices arising from Hermite normal forms.

Let $M \in \mathbb{Z}^{d\times d}$. We write $\mathcal{P}(M)$ for the integral simplex whose vertices are the row vectors of M together with the origin in $\mathbb{R}^d$. We will present an algorithm to compute the δ-vector of $\mathcal{P}(M)$. To make the notation clear, we assume $d = 3$. The general case is completely analogous. Let A be the Hermite normal form of M. We have that $\{\mathcal{P}(M) \cap \mathbb{Z}^d\}$ is in bijection with $\{\mathcal{P}(A) \cap \mathbb{Z}^d\}$. By definition,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where each a_{ij} is a nonnegative integer.

For a vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, consider

$$b(\lambda) = (\lambda_1, \lambda_2, \lambda_3)A = (a_{11}\lambda_1 + a_{21}\lambda_2 + a_{31}\lambda_3, a_{22}\lambda_2 + a_{32}\lambda_3, a_{33}\lambda_3).$$

Then it is clear that the set of interior points inside $\mathcal{P}(A)$ ($\{(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3\}$) is in bijection with the set

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0,\ \lambda_1 + \lambda_2 + \lambda_3 < 1 \text{ and } b(\lambda) \in \mathbb{Z}^3\}.$$

An observation is that $\{n(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3\}$, for any $n \in \mathbb{N}$, is in bijection with

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0,\ \lambda_1 + \lambda_2 + \lambda_3 < n \text{ and } b(\lambda) \in \mathbb{Z}^3\}.$$

We first consider all positive vectors λ satisfying $b(\lambda) \in \mathbb{Z}^3$. By the lower triangularity of the Hermite normal form, we can start from the last term of $b(\lambda)$ and move forward. It is not hard to see that each vector λ should have the following form: ($\{r\}$ is the fractional part of a rational number r.)

$$\lambda_3^{k,k_3} = \frac{k}{a_{33}} + k_3,$$

$$\lambda_2^{jk,k_2} = \frac{j - \{a_{32}\lambda_3^k\}}{a_{22}} + k_2,$$

and

$$\lambda_1^{ijk,k_1} = \frac{i - \{a_{21}\lambda_2^{jk} + a_{31}\lambda_3^k\}}{a_{11}} + k_1,$$

for some nonnegative integers k_3, k_2, k_1. In the above formula, $k \in \{1, 2, \dots, a_{33}\}$, $j \in \{1, 2, \dots, a_{22}\}$, $i \in \{1, 2, \dots, a_{11}\}$ and $\lambda_1^{ijk} = \lambda_1^{ijk,0}, \lambda_2^{jk} = \lambda_2^{jk,0}, \lambda_3^k = \lambda_3^{k,0}$. We call all the vectors λ with the same index (i, j, k) *the congruence class of* (i, j, k).

Now we go to the condition $\lambda_1 + \lambda_2 + \lambda_3 < n$ in the above bijection. As n increases, we ask when is the first time that a congruence class (i, j, k) starts to produce interior points inside $n\mathcal{P}(A)$. In other words, fix (i, j, k). We want the smallest n such that $\lambda_1 + \lambda_2 + \lambda_3 < n$. Then it is clear that this happens when $k_1 = k_2 = k_3 = 0$ and

$$n = s_{ijk} = \lfloor \lambda_1^{ijk} + \lambda_2^{jk} + \lambda_3^k \rfloor + 1,$$

where $\lfloor r \rfloor$ for a rational number is the biggest integer not larger than r.

Finally, when n grows larger than s_{ijk}, we want to consider how many interior points this fixed congruence class produces. Let $n = s_{ijk} + \ell$, so each interior point corresponds to a choice of k_1, k_2, k_3 in the formula of λ_1^{ijk,k_1}, λ_2^{ij,k_2} and λ_3^{i,k_3} such that $k_1 + k_2 + k_3 \le \ell$. There are $\left(\binom{d+1}{\ell}\right)$ choices in total.

To sum up, we have the following two observations for each congruence class (i, j, k), $k \in \{1, 2, \dots, a_{33}\}$, $j \in \{1, 2, \dots, a_{22}\}$, $i \in \{1, 2, \dots, a_{11}\}$:

(1) s_{ijk} is the smallest n such that this congruence class contributes interior points in the n-th dilation of $\mathcal{P}(A)$;
(2) In the $(s_{ijk}+\ell)$-th dilation of $\mathcal{P}(A)$, this congruence class contributes $\left(\binom{d+1}{\ell}\right)$ interior points.

Therefore, the following Theorem holds. We state it for a general dimension d, and the proof is analogous to the case $d = 3$.

Theorem 2.1. *Let $\mathcal{P}(A)$ be a d dimensional simplex corresponding to a $d \times d$ matrix $A = (a_{ij})$. Then the generating function for the interior points of $n\mathcal{P}(A)$, $i^*(\mathcal{P}(A), n) = |n(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^d|$ is*

$$\sum_{n \ge 1} i^*(\mathcal{P}(A), n) t^n = (1 - t)^{-(d+1)} \sum_{\substack{(i_1, \dots, i_d) \\ 1 \le i_j \le a_{ij}}} t^{s_{i_1 \dots i_d}},$$

where

$$s_{i_1 \dots i_d} = \left\lfloor \sum_{k=1}^{d} \lambda_k^{i_k, i_{k+1}, \dots i_d} \right\rfloor + 1,$$

with

$$\lambda_d^{i_d} = \frac{i_d}{a_{dd}},$$

and for each $1 \leq k < d$*,*

$$\lambda_k^{i_k, i_{k+1}, \ldots i_d} = a_{kk}^{-1} \left(i_k - \left\{ \sum_{h=k+1}^{d} a_{hk} \lambda_h^{i_h i_{h+1} \ldots i_d} \right\} \right).$$

By the reciprocity law (0.3), we have

$$\delta_{\mathcal{P}(A)}(t) = \sum_{\substack{(i_1, \ldots, i_d) \\ 1 \leq i_j \leq a_{ij}}} t^{d+1-s_{i_1 \ldots i_d}}.$$

Example 2.2. Let A be the 4×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Consider

$$b(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)A = (\lambda_1 + \lambda_3 + \lambda_4, \lambda_2 + \lambda_3, 2\lambda_3 + \lambda_4, 3\lambda_4).$$

Denote

$$\lambda_4^j = \frac{j}{3}, \text{ for } j = 1, 2, 3, \quad \lambda_3^{ij} = \frac{i - \{\lambda_4^j\}}{2}, \text{ for } i = 1, 2,$$

$$\lambda_2^{ij} = 1 - \{\lambda_3^{ij}\}, \quad \lambda_1^{ij} = 1 - \{\lambda_3^{ij} + \lambda_4^j\}$$

and

$$s_{ij} = 1 + \lfloor \lambda_1^{ij} + \lambda_2^{ij} + \lambda_3^{ij} + \lambda_4^j \rfloor.$$

Then we have

$$s_{11} = 2, s_{21} = 3, s_{12} = 2, s_{22} = 3, s_{13} = 3, s_{23} = 5,$$

$$\delta_{\mathcal{P}(A)}(t) = \sum_{i=1}^{3} \sum_{j=1}^{2} t^{d+1-s_{ij}} = 1 + 3t^2 + 2t^3,$$

and thus

$$\delta(\mathcal{P}(A)) = (1, 0, 3, 2, 0).$$

3. Shifted symmetric δ-vectors

In this section, we define shifted symmetric δ-vectors and study its conditions for some special Hermite forms. Results in this section are direct applications of the algorithm developed in the previous section (Theorem 2.1). In [5], the second author studied shifted symmetric δ-vectors without using the algorithm.

We call that a δ-vector is shifted symmetric, if $\delta_i = \delta_{d+1-i}$, $1 \le i \le d$. For example, (1, 1, 2, 2, 1, 2, 2, 1) is shifted symmetric.

We want this definition because it simply arises from the algorithm for the "one row" Hermite normal forms as discussed in the first subsection. In the second subsection, we will consider a special "one row" Hermite normal form, which allow us to have better results.

3.1. **"One row" Hermite normal forms.** Consider all $d \times d$ matrices with determinant D and the following Hermite normal forms for some $k \in \{1, 2, \dots, d\}$.

$$A_D = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ a_1 & \cdots & a_{k-1} & D & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{5}$$

where $a_1, \dots, a_{k-1}$ are nonnegative integers smaller than D and all other terms are zero. Let d_j denote the number of j's among these a_ℓ's, for $j = 1, \dots, D-1$. Then we can simplify Theorem 2.1 for these "one row" Hermite normal forms.

Corollary 3.1. *Let $M \in \mathbb{Z}^{d\times d}$ with $\det(M) = D$ and $\mathcal{P}(M)$ be the corresponding integral simplex. If its Hermite normal form looks like matrix (5), then we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^{D} t^{d+1-s_i},$$

where

$$s_i = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d. \tag{6}$$

Proof. Consider

$$b(\lambda) = (\lambda_1, \dots, \lambda_k, \dots, \lambda_d)A_D = (\lambda_1 + a_1\lambda_k, \dots, \lambda_{k-1} + a_{k-1}\lambda_k, D\lambda_k, \lambda_{k+1}, \dots, \lambda_d).$$

Using notation from the proof of Theorem 2.1, we have, for $i = 1, 2, \dots, D$,

$$\lambda_k^i = \frac{i}{D},\ \lambda_\ell^i = 1 - \left\{ a_\ell \frac{i}{D} \right\},\ \text{for } \ell = 1, \dots, k-1$$

and

$$\lambda_{k+1}^i = \cdots = \lambda_d^i = 1.$$

Therefore, $s_i = 1 + \lfloor \lambda_1^i + \cdots + \lambda_d^i \rfloor = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d.$ $\square$

Now we are going to deduce a symmetry property of the δ-vectors by the following fact: if D does not divide x, we have

$$\left\{ \frac{-x}{D} \right\} = 1 - \left\{ \frac{x}{D} \right\} \text{ and } \left\lfloor \frac{-x}{D} \right\rfloor = - \left\lfloor \frac{x}{D} \right\rfloor - 1. \tag{7}$$

Consider s_{D-i} for $i = 1, \ldots, D-1$. By formula (6), we have

$$s_{D-i} = \left\lfloor -\frac{i}{D} - \sum_{j=1}^{D-1} \left\{ -\frac{ij}{D} \right\} d_j \right\rfloor + d + 1.$$

By (7), if D does not divide ij for all $i, j = 1, \ldots, D-1$, we have

$$s_{D-i} = \left\lfloor -\frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d + 1 - \sum_{j=1}^{D-1} d_j.$$

Now compare s_{D-i} with formula (6) for s_i, and by (7) again, we have the following symmetry property.

Proposition 3.2 (Shifted symmetry for "one row")**.** *For a matrix $M \in \mathbb{Z}^{d \times d}$ with Hermite normal form (5), we have $s_i + s_{D-i} = d + 1$, for $i = 1, \ldots, D-1$, which implies $\delta_i = \delta_{d+1-i}$ by reciprocity, if the following three conditions hold:*

1. *D does not divide ij for all $i, j = 1, \ldots, D-1$. This is true, for example, if D is prime;*
2. *$(-\frac{i}{D} + \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j) \notin \mathbb{Z}$;*
3. *$\sum_{j=1}^{D-1} d_j = d - 1$.*

These conditions are not very easy to check, so we consider a special case of Hermite normal forms (5).

3.2. **"All $D-1$ one row" Hermite normal forms.** Assume in addition $d_{D-1} = D-1$ in Corollary 3.1, i.e., the Hermite normal form looks like

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ D-1 & D-1 & \cdots & D-1 & D \end{pmatrix}. \tag{8}$$

Then we have

Corollary 3.3 (All $D-1$)**.** *For a matrix $M \in \mathbb{Z}^{d\times d}$ with Hermite normal form (8), we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^{D-1} t^{d+1-s_i}, \text{ where } s_i = \left\lfloor \frac{id}{D} \right\rfloor + 1.$$

By this formula, the conditions for shifted symmetry in Proposition 3.2 can be simplified.

Proposition 3.4 (Shifted symmetry for "all $D-1$ one row")**.** *Let $M \in \mathbb{Z}^{d\times d}$ with Hermite normal form (8). Then*

(1) *$\delta_i = \delta_{d+1-i}$ if D and d are coprime.*
(2) *When $D = id$, for $i \in \mathbb{N}$, the δ-vector is*

$$(1, \underbrace{i, \dots, i}_{d-1}, i-1),$$

which is not shifted symmetric. But for $i = 2$, we have $\delta_i = \delta_{d-i}$ (Gorenstein).

It is also interesting to see the changes of δ-vectors if we fix d and let D increase. For example, for $d = 7$, we have the following table:

D	δ-vector	D	δ-vector	D	δ-vector
9	1 1 1 1 **2** 1 1 1	16	1 2 2 2 **3** 2 2 2	23	1 3 3 3 **4** 3 3 3
10	1 1 1 **2** 1 **2** 1 1	17	1 2 2 **3** 2 **3** 2 2	24	1 3 3 **4** 3 **4** 3 3
11	1 1 **2** 1 **2** 1 **2** 1	18	1 2 **3** 2 **3** 2 **3** 2	25	1 3 **4** 3 **4** 3 **4** 3
12	1 1 **2 2** 1 **2 2** 1	19	1 2 **3 3** 2 **3 3** 2	26	1 3 **4 4** 3 **4 4** 3
13	1 1 **2 2 2 2 2** 1	20	1 2 **3 3 3 3 3** 2	27	1 3 **4 4 4 4 4** 3
14	1 2 2 2 2 2 2 1	**21**	1 3 3 3 3 3 3 2	**28**	1 4 4 4 4 4 4 3
15	1 **2 2 2 2 2 2 2**	22	1 **3 3 3 3 3 3 3**		$\cdots$

4. Classification of Hermite normal forms with a given δ-vector

In this section, we will give another application of the algorithm Theorem 2.1. Consider Problem 1.1 first with the assumption that matrix $A \in \mathbb{Z}^{d\times d}$ has prime determinant, i.e., A is of the form (5), with only one general row. By Corollary 3.1, in order to classify all possible Hermite normal forms (5) with a given δ-vector $(\delta_0, \delta_1, \dots, \delta_d)$, we need to find all nonnegative integer solutions $(d_1, d_2, \dots, d_{D-1})$ with $d_1 + d_2 + \cdots + d_{D-1} \le D-1$ such that

$$\#\{i : d+1-s_i = j, \text{ for } i = 1, \dots, D\} = \delta_j, \text{ for } j = 0, \dots, d.$$

By Corollary 3.1, we can build equations with "floor" expressions for $(d_1, d_2, \dots, d_{D-1})$. Remove the "floor" expressions, we obtain D linear equations of $(d_1, d_2, \dots, d_{D-1})$ with different constant terms but the same $D \times D$ coefficient matrix M with ij entry

$\{(ij) \bmod D\}$, which is a number in $\{0, 1, \ldots, D-1\}$. Then we first get all integer solutions $(d_1, d_2, \ldots, d_{D-1})$, and then test every candidates by the restrictions of nonnegativity and $d_1 + d_2 + \cdots + d_{D-1} \leq D - 1$.

For $D = 2$ and 3, the coefficient matrix M is nonsingular, so we can write down the complete solutions, as presented in the first two subsections. For larger primes, the coefficient matrix becomes singular, so there are free varibles in the integer solutions $(d_1, d_2, \ldots, d_{D-1})$, which make it very hard to simplify the final solutions after the test.

The idea is similar for Hermite normal forms with non prime determinant. Instead of using Corollary 3.1, we need to use the formulas in Theorem 2.1. In the third subsection, we will present the complete solution for $D = 4$ as an example.

4.1. **A solution of Problem 1.1 when $\sum_{i=0}^d \delta_i = 2$.** The goal of this subsection is to give a solution of Problem 1.1 when $\sum_{i=0}^d \delta_i = 2$, i.e., given a δ-vector $\delta(\mathcal{P})$ with $\sum_{i=0}^d \delta_i = 2$, we classify all the integral simplices with $\delta(\mathcal{P})$ arising from Hermite normal forms with determinant 2.

We consider all Hermite normal forms (5) with $D = 2$, namely,

$$A_2 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 2 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{9}$$

where there are d_1 1's among the $*$'s. Notice that the position of the row with a 2 does not affect the δ-vector, so the only variable is d_1. By Corollary 3.1, we have a formula for the δ-vector of this integral simplex $\mathcal{P}(A_2)$. Denote

$$k = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor.$$

Then one has $\delta_0 = \delta_k = 1$.

By this formula, we can characterize all Hermite normal forms with a given δ-vector. Let $\delta_0 = \delta_i = 1$. Then by solving the equation $i = 1 - \left\lfloor \frac{1-d_1}{2} \right\rfloor$, we obtain $d_1 = 2i - 2$ and $d_1 = 2i - 1$, both cases will give us the desired δ-vector.

Notice that there is a constraint on d_1 to be $0 \leq d_1 \leq d - 1$. Not all δ-vectors are obtained by from simplices. But we can easily get the restriction of i and the corresponding d_1 as follows (by $d_1 \geq 0$, we have $i \geq 1$):

(1) If $i \leq d/2$, $d_1 = 2i - 2$ and $d_1 = 2i - 1$ both work, and these give all the matrices with this δ-vector.
(2) If $i = (d + 1)/2$, only $d_1 = 2i - 2 = d - 1$ works.

(3) If $i > (d+1)/2$, there is no solution.

4.2. **A solution of Problem 1.1 when $\sum_{i=0}^{d} \delta_i = 3$.** We consider all Hermite normal forms (5) with $D = 3$, namely,

$$A_3 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 3 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{10}$$

where there are d_1 1's and d_2 2's among the $*$'s. Since the position of the row with a 3 does not affect the δ-vector, so the only variables are d_1 and d_2. Also, by Corollary 3.1, we have $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^{k_1} + t^{k_2}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - 2d_2 - d_1}{3} \right\rfloor \text{ and } k_2 = 2 - \left\lfloor \frac{2 - d_2 - 2d_1}{3} \right\rfloor.$$

Then by the formula, similar to the case of $\sum_{i=0}^{d} \delta_i = 2$, though a little more complicated, we can characterize all Hermite normal forms with a given δ-vector. Let $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^i + t^j$. Set

$$\begin{cases} i = 1 - \left\lfloor \frac{1-2d_2-d_1}{3} \right\rfloor \\ j = 2 - \left\lfloor \frac{2-d_2-2d_1}{3} \right\rfloor \end{cases}.$$

(Later reverse the role of i and j if $i \neq j$, in both equations and solutions.) After computations, the solutions for (d_1, d_2) are

$$d^{(1)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 1, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2j - i - 1 \\ d_2 = 2i - j - 1 \end{cases} \quad \text{and} \quad d^{(3)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_2) that $d_1, d_2 \geq 0$ and $d_1 + d_2 \leq d - 1$, we have the following characterizations:

TABLE 1. Characterizations for matrices of the form (10)

$2j$	$2i$	$i+j$	solutions
$\geq i$	$\geq j+1$	$\leq d$	$d^{(1)}$
$\geq i+1$	$\geq j+1$	$\leq d+1$	$d^{(2)}$
$\geq i$	$\geq j+2$	$\leq d+1$	$d^{(3)}$

(1) If $2j \geq i, 2i \geq j+1$ and $i + j \leq d$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

(2) If $2j \geq i+1, 2i \geq j+1$ and $i+j \leq d+1$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

(3) If $2j \geq i, 2i \geq j+2$ and $i+j \leq d+1$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

(4) If $\{i,j\}$ in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its δ-vector.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = d-1 \\ d_2 = 0 \end{cases}$$

works when $i = (d+2)/3$ and $j = (2d+1)/3$. This happens when $d \equiv 1 \pmod 3$ and there is only one matrix with $d_1 = d-1$ and $d_2 = 0$. Similary, only the solution

$$d^{(3)} = \begin{cases} d_1 = 0 \\ d_2 = d-1 \end{cases}$$

works when $i = (2d+2)/3$ and $j = (d+1)/3$. This happens when $d \equiv 2 \pmod 3$ and again, there is only one matrix with $d_1 = 0$ and $d_2 = d-1$.

4.3. **A solution of Problem 1.1 when $\sum_{i=0}^{d} \delta_i = 4$.** When the determinant is 4, there are two cases of Hermite normal forms. One is the Hermite normal forms (5) with $D = 4$, namely,

$$A_4 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 4 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{11}$$

where there are d_1 1's, d_2 2's and d_3 3's among $*$'s. The other one looks like

$$A_4' = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ * & \cdots & * & 2 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ \dot{*} & \cdots & \dot{*} & \bar{*} & \dot{*} & \cdots & \dot{*} & 2 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \end{pmatrix}, \tag{12}$$

where there are d_1 1's (resp. d_1' 1's) among $*$'s (resp. $\dot{*}$'s), there are e_1 1's (resp. e_1' 1's) among $*$'s (resp. $\dot{*}$'s) of which the $\dot{*}$ (resp. $*$) in the same column is not 1. Also, set $d_1'' = e_1 + e_1'$. (For example, a 6×6 Hermite normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 2 \end{pmatrix}$$

is a matrix (12) with $d_1 = d_1' = 2, e_1 = e_1' = 1, d_1'' = 2$ and $\bar{*} = 1$.)

First, we consider the Hermite normal forms (11). Then, by Corollary 3.1, we have $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_2 - 3d_3}{4} \right\rfloor, k_2 = 1 - \left\lfloor \frac{1 - d_1 - d_3}{2} \right\rfloor \text{ and } k_3 = 1 - \left\lfloor \frac{3 - 3d_1 - 2d_2 - d_3}{4} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations, according to the order of k_1, k_2 and k_3:

$$\begin{cases} i = 1 - \lfloor \frac{1-d_1-2d_2-3d_3}{4} \rfloor \\ j = 1 - \lfloor \frac{1-d_1-d_3}{2} \rfloor \\ k = 1 - \lfloor \frac{3-3d_1-2d_2-d_3}{4} \rfloor \end{cases}$$

(Later replace the roles of i, j and k if any of the three are distinct.) After computations, the solutions for (d_1, d_2, d_3) are

$$d^{(1)} = \begin{cases} d_1 = -i + j + k - 1 \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1, \end{cases} \qquad d^{(2)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 2, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1 \end{cases} \quad \text{and} \quad d^{(4)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k - 1 \\ d_3 = i + j - k - 1. \end{cases}$$

In addition, by the restriction on (d_1, d_2, d_3) that $d_1, d_2, d_3 \geq 0$ and $d_1 + d_2 + d_3 \leq d - 1$, we have the following characterization:

(1) If $j + k \geq i + 1, 2j \leq i + k \leq d + 1$ and $i + j \geq k + 1$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.
(2) If $j + k \geq i, 2j \leq i + k \leq d + 1$ and $i + j \geq k + 2$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.
(3) If $j + k \geq i, 2j \leq i + k \leq d$ and $i + j \geq k + 1$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

Table 2. Characterization for matrices of the form (11)

$j+k$	$2j$	$i+j$	solutions
$\geq i+1$	$\leq i+k \leq d+1$	$\geq k+1$	$d^{(1)}$
$\geq i$	$\leq i+k \leq d+1$	$\geq k+2$	$d^{(2)}$
$\geq i$	$\leq i+k \leq d$	$\geq k+1$	$d^{(3)}$
$\geq i$	$\leq i+k-1 \leq d$	$\geq k+1$	$d^{(4)}$

(4) If $j+k \geq i, 2j+1 \leq i+k \leq d+1$ and $i+j \geq k+1$, then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.

(5) If $\{i,j,k\}$ in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its δ-vector.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = 0 \\ d_2 = 0 \\ d_3 = d-1 \end{cases}$$

works when $i = (3d+3)/4, j = (d+1)/2$ and $k = (d+1)/4$. This happens when $d \equiv 3 \pmod 4$ and there is only one matrix with $d_3 = d-1$. Similarly, only the solution

$$d^{(1)} = \begin{cases} d_1 = d-1 \\ d_2 = 0 \\ d_3 = 0 \end{cases}$$

works when $i = (d+3)/4, j = (d+1)/2$ and $k = (3d+1)/4$. This happens when $d \equiv 1 \pmod 4$ and again, there is only one matrix with $d_1 = d-1$.

Next, we consider the Hermite normal forms (12). However, we need to consider two cases, which are the cases when $\bar{*} = 0$ and $\bar{*} = 1$.

First, we consider the case with $\bar{*} = 0$. Notice that the variables are d_1, d_1' and d_1''. Obviously we cannot use Corollary 3.1, but we apply Theorem 2.1 directly. Thus we have $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = \left\lfloor \frac{d_1+2}{2} \right\rfloor, \ k_2 = \left\lfloor \frac{d_1'+2}{2} \right\rfloor \text{ and } k_3 = \left\lfloor \frac{d_1''+2}{2} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations, according to the order of k_1, k_2 and k_3:

$$\begin{cases} i = \lfloor \frac{d_1+2}{2} \rfloor \\ j = \left\lfloor \frac{d_1'+2}{2} \right\rfloor \\ k = \left\lfloor \frac{d_1''+2}{2} \right\rfloor \end{cases}$$

or replace the role of i, j and k if i, j and k are distinct, in all equations and solutions. After computations, since $d_1 + d_1' + d_1''$ is even, the solutions of (d_1, d_1', d_1'') are

$$d^{(1)} = \begin{cases} d_1 = 2i - 2 \\ d_1' = 2j - 1 \\ d_1'' = 2k - 1, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2i - 1 \\ d_1' = 2j - 2 \\ d_1'' = 2k - 1, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = 2i - 1 \\ d_1' = 2j - 1 \\ d_1'' = 2k - 2 \end{cases} \quad \text{and} \quad d^{(4)} = \begin{cases} d_1 = 2i - 2 \\ d_1' = 2j - 2 \\ d_1'' = 2k - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_1', d_1'') that $0 \le d_1 \le d-2$, $0 \le d_1' \le d-2$, $0 \le d_1'' \le d-2$, $d_1 + d_1' + d_1'' \le 2(d-2)$, $d_1'' \le d_1 + d_1'$, $d_1' \le d_1 + d_1''$ and $d_1 \le d_1' + d_1''$, we have the following characterization:

Table 3. Characterization for matrices of the form (12) with $\bar{*} = 0$

i	j	k	$i+j$	$i+k$	$j+k$	$i+j+k$	solutions
$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	$\ge k+1$	$\ge j+1$	$\ge i$	$\le d$	$d^{(1)}$
$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	$\ge k+1$	$\ge j$	$\ge i+1$	$\le d$	$d^{(2)}$
$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\ge k$	$\ge j+1$	$\ge i+1$	$\le d$	$d^{(3)}$
$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\ge k+1$	$\ge j+1$	$\ge i+1$	$\le d+1$	$d^{(4)}$

(1) If $i \le \lfloor d/2 \rfloor,\ j, k \le \lfloor (d-1)/2 \rfloor$, $i+j+k \le d, k+1 \le i+j, j+1 \le i+k$ and $i \le j+k$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.
(2) If $j \le \lfloor d/2 \rfloor,\ i, k \le \lfloor (d-1)/2 \rfloor$, $i+j+k \le d, k+1 \le i+j, j \le i+k$ and $i+1 \le j+k$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.
(3) If $k \le \lfloor d/2 \rfloor,\ i, j \le \lfloor (d-1)/2 \rfloor$, $i+j+k \le d, k \le i+j, j+1 \le i+k$ and $i+1 \le j+k$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.
(4) If $i, j, k \le \lfloor d/2 \rfloor$, $i+j+k \le d+1, k+1 \le i+j, j+1 \le i+k$ and $i+1 \le j+k$, then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.
(5) If $\{i, j, k\}$ in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its δ-vector.

Next, we consider the case with $\bar{*} = 1$. By Theorem 2.1, we have $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - (d_1 + 2d_1'')}{4} \right\rfloor, \ k_2 = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor \text{ and } k_3 = 2 - \left\lfloor \frac{3 - (d_1 + 2d_1')}{2} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations, according to the order of k_1, k_2 and k_3:

$$\begin{cases} i = 1 - \left\lfloor \frac{1-(d_1+2d_1'')}{4} \right\rfloor \\ j = 1 - \left\lfloor \frac{1-d_1}{2} \right\rfloor \\ k = 2 - \left\lfloor \frac{3-(d_1+2d_1')}{2} \right\rfloor \end{cases}$$

or replace the roles of i, j and k if i, j and k are distinct. After computations, considering $d_1 + d_1' + d_1''$ is even, the solutions of (d_1, d_1', d_1'') are

$$d^{(1)} = \begin{cases} d_1 = 2j - 1 \\ d_1' = 2k - j - 3 \\ d_1'' = 2i - j - 2, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2j - 1 \\ d_1' = 2k - j - 2 \\ d_1'' = 2i - j - 1, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = 2j - 2 \\ d_1' = 2k - j - 3 \\ d_1'' = 2i - j - 1 \end{cases} \text{ and } d^{(4)} = \begin{cases} d_1 = 2j - 2 \\ d_1' = 2k - j - 2 \\ d_1'' = 2i - j - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_1', d_1'') that $0 \le d_1 \le d-2$, $0 \le d_1' \le d-2$, $0 \le d_1'' \le d-2$, $d_1 + d_1' + d_1'' \le 2(d-2)$, $d_1'' \le d_1 + d_1'$, $d_1' \le d_1 + d_1''$ and $d_1 \le d_1' + d_1''$, we have the following characterization:

Table 4. Characterization for matrices of the form (12) with $\bar{*} = 1$

$2k$	$2i$	$2j$	$i+j$	$i+k$	$j+k$	solutions
$\ge j+3$, $\le d+j+1$	$\ge j+2$, $\le d+j$	$\le d-1$	$\ge k$	$\ge 2j+2$, $\le d+1$	$\ge i+1$	$d^{(1)}$
$\ge j+2$, $\le d+j$	$\ge j+1$, $\le d+j-1$	$\le d-1$	$\ge k$	$\ge 2j+1$, $\le d$	$\ge i+1$	$d^{(2)}$
$\ge j+3$, $\le d+j+1$	$\ge j+1$, $\le d+j-1$	$\le d$	$\ge k$	$\ge 2j+1$, $\le d+1$	$\ge i+2$	$d^{(3)}$
$\ge j+2$, $\le d+j$	$\ge j+2$, $\le d+j$	$\le d$	$\ge k+1$	$\ge 2j+1$, $\le d+1$	$\ge i+1$	$d^{(4)}$

(1) If $j+3 \le 2k \le d+j+1, j+2 \le 2i \le d+j, 2j \le d-1$, $2j+2 \le i+k \le d+1$, $i+1 \le j+k$ and $k \le i+j$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

(2) If $j+2 \le 2k \le d+j, j+1 \le 2i \le d+j-1, 2j \le d-1$, $2j+1 \le i+k \le d$, $i+1 \le j+k$ and $k \le i+j$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

(3) If $j+3 \leq 2k \leq d+j+1, j+1 \leq 2i \leq d+j-1, 2j \leq d$, $2j+1 \leq i+k \leq d+1$, $i+2 \leq j+k$ and $k \leq i+j$ then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

(4) If $j+2 \leq 2k \leq d+j, j+2 \leq 2i \leq d+j, 2j \leq d$, $2j+1 \leq i+k \leq d+1$, $i+1 \leq j+k$ and $k+1 \leq i+j$ then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.

(5) If $\{i,j,k\}$ in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its δ-vector.

Notice that only the solution

$$d^{(3)} = \begin{cases} d_1 = d-2 \\ d_1' = d-2 \\ d_1'' = 0 \end{cases}$$

works when $i = (d+2)/4, j = d/2$ and $k = (3d+2)/4$. This happens when $d \equiv 2 \pmod 4$ and there is only one matrix with $d_1 = d_1' = d-2$. Similarly, only the solution

$$d^{(4)} = \begin{cases} d_1 = d-2 \\ d_1' = 0 \\ d_1'' = d-2 \end{cases}$$

works when $i = 3d/4, j = d/2$ and $k = d/4+1$. This happens when $d \equiv 0 \pmod 4$ and again, there is only one matrix with $d_1 = d_1'' = d-2$.

5. The classification of possible δ-vectors with $\sum_{i=0}^d \delta_i = 4$

In [4], it is shown that when $\sum_{i=0}^d \delta_i \leq 3$, (3) and (4) is both necessary and sufficient. In this section, we classify the possible δ-vectors with $\sum_{i=0}^d \delta_i = 4$ by results from Section 4.3.

Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be a sequence of nonnegative integers with $\sum_{i=0}^d \delta_i = 4$ satisfying the inequalities (3) and (4), where $\delta_0 = 1$ and $\delta_1 \geq \delta_d$, which are the necessary conditions to be a possible δ-vector. Now it is led into the following inequalities from (3) and (4) that (i_1, i_2, i_3) satisfies

$$i_3 \leq i_1 + i_2, \; i_1 + i_3 \leq d+1 \; \text{ and } \; i_2 \leq \lfloor (d+1)/2 \rfloor. \tag{13}$$

Finally, the classification of possible δ-vectors of integral polytopes with $\sum_{i=0}^d \delta_i = 4$ is given by the following

Theorem 5.1. *Let $1 + t^{i_1} + t^{i_2} + t^{i_3}$ be a polynomial with $1 \leq i_1 \leq i_2 \leq i_3 \leq d$. Then there exists an integral polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-polynomial equals to $1 + t^{i_1} + t^{i_2} + t^{i_3}$ if and only if (i_1, i_2, i_3) satisfies (13) and an additional* *condition*

$$2i_2 \leq i_1 + i_3 \text{ or } i_2 + i_3 \leq d + 1. \tag{14}$$

Moreover, all these polytopes are actually simplices.

Proof. There are four cases: (1) $i_1 = i_2 = i_3$, (2) $i_1 < i_2 = i_3$, (3) $i_1 = i_2 < i_3$, (4) $i_1 < i_2 < i_3$. We will show that in each case (13) together with (14) are the necessary and sufficient conditions for $1 + t^{i_1} + t^{i_2} + t^{i_3}$ to be the δ-vector of some integral polytope.

(1) Assume $i_1 = i_2 = i_3 = \ell$. By the inequalities (13), we have $1 \leq \ell \leq \lfloor (d+1)/2 \rfloor$. Set $i = j = k = \ell$. We have

$$j + k \geq i + 1, \ 2j \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1. \tag{15}$$

Thus, by our result on the classification of the case of a matrix (11) (Table 2, the solution $d^{(1)}$), there exists an integral polytope (it is actually a simplex) whose δ-vector is $(1, 0, \ldots, 0, 3, 0, \ldots, 0)$.

On the other hand, if there exists an integral polytope with this δ-vector, then (13) holds since it is a necessary condition. In this case, it follows that both inequalities in (14) hold.

(2) Assume $\ell = i_1 < i_2 = i_3 = \ell'$. By (13), we have $1 \leq \ell < \ell' \leq \lfloor (d+1)/2 \rfloor$. Let $j = \ell$ and $i = k = \ell'$. Then inequalities (15) hold. Thus there exists an integral simplex whose δ-vector is $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0)$.

On the other hand, we have (13) since it is a necessary condition. Then, $i_2 + i_3 \leq d + 1$ follows from $i_2 \leq \lfloor (d+1)/2 \rfloor$.

(3) Assume $\ell = i_1 = i_2 < i_3 = \ell'$. Set $i = \ell'$ and $j = k = \ell$. Then it follows from (13) that

$$j + k \leq i, 2j + 1 \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1.$$

Thus, by our result (Table 2, the solution $d^{(4)}$), there exists an integral simplex whose δ-vector is $(1, 0, \ldots, 0, 2, 0, \ldots, 0, 1, 0, \ldots, 0)$.

On the other hand, if there exists an integral polytope with this δ-vector, then (13) holds since it is a necessary condition. In this case, it follows that both inequalities in (14) hold.

(4) Assume $1 \leq i_1 < i_2 < i_3 \leq d$. Suppose $2i_2 \leq i_1 + i_3$ holds. Set $i = i_3, j = i_2$ and $k = i_1$. Then we have $j + k = i_1 + i_2 \geq i_3 = i$, $2j = 2i_2 \leq i_1 + i_3 = i + k \leq d + 1$ and $i + j = i_2 + i_3 \geq 2i_2 + 1 \geq 2i_1 + 3 > i_1 + 2 = k + 2$. Thus, by our result (Table 2, the solution $d^{(2)}$), there exists an integral simplex whose δ-vector is $(1, 0 \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$.

Suppose $i_2 + i_3 \leq d + 1$ holds. Set $i = i_3, j = i_1$ and $k = i_2$. Then we have $j + k = i_1 + i_2 \geq i_3 = i$, $2j = 2i_1 < i_2 + i_3 = i + k \leq d + 1$ and $i + j = i_1 + i_3 \geq i_1 + i_2 + 1 \geq i_2 + 2 = k + 2$. Thus, by our result (Table 2, the solution $d^{(2)}$), there exists an integral simplex whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$.

On the other hand, assume the contrary of (14): both $2i_2 > i_1+i_3$ and $i_2+i_3 > d+1$ hold. We claim that there exists no integral polytope $\mathcal{P}$ with this δ-vector. First we want to show that if there exists such a polytope, it must be a simplex. Note that the δ-vector satisfies (13). Suppose $i_1 = 1$. It then follows from (13) and $i_2 + i_3 > d + 1$ that $i_2 = (d+1)/2$ and $i_3 = (d+3)/2$. However, this contradicts (4). Therefore $i_1 > 1$, and thus $\delta_1 = 0$. By an explanation after equation (1), $\mathcal{P}$ must be a simplex if $\delta_1 = 0$. Now we can apply our characteristic results for simplices.

If we set $j = i_3$, then $2j = 2i_3 > i_1 + i_2 = i + k$. If we set $j = i_2$, then $2j = 2i_2 > i_1 + i_3 = i + k$. If we set $j = i_1$, then $i + k = i_2 + i_3 > d + 1$. In any case there does not exist an Hermite normal form (11) whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$.

Moreover, since $i + j + k = i_1 + i_2 + i_3 > i_2 + i_3 > d + 1$, there does not exsit an Hermite normal form (12) with $\divideontimes = 0$ whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$.

In addition, if we set $j = i_3$, then $2j = 2i_3 > i_1 + i_2 = i + k$. If we set $j = i_2$, then $2j = 2i_2 > i_1 + i_3 = i + k$. If we set $j = i_1$, then $i + k = i_2 + i_3 > d + 1$. Thus there dose not exist an Hermite normal form (12) with $\divideontimes = 1$ whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$. □

Remark 5.2. From the above proof, we can see that when $\sum_{i=0}^{d} \delta_i = 4$, all the possible δ-vectors can be obtained by simplices. This is also true for all δ-vectors with $\sum_{i=0}^{d} \delta_i \leq 3$, from the proof of Theorem 0.1 in [4]. However, when $\sum_{i=0}^{d} \delta_i = 5$, the δ-vector $(1, 3, 1)$ cannot be obtained from any simplex, while it is a possible δ-vector of a 2-dimensional integral polygon. In fact, suppose that $(1, 3, 1)$ can be obtained from a simplex. Since $\min\{i : \delta_i \neq 0, i > 0\} = 1, \max\{i : \delta_i \neq 0\} = 2$ and $\min\{i : \delta_i \neq 0, i > 0\} = 3 - \max\{i : \delta_i \neq 0\}$, the assumption of [5, Theorem 2.3] is satisfied. Thus the δ-vector must be shifted symmetric, a contradiction.

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