CLASSIFICATION OF NILPOTENT ASSOCIATIVE ALGEBRAS OF SMALL DIMENSION

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Abstract. We describe a method for classifying nilpotent associative algebras, that is analogous to the Skjelbred-Sund method for classifying nilpotent Lie algebras (cf. [19], [8]). Subsequently we classify nilpotent associative algebras of dimensions ≤ 3 over any field, and 4-dimensional commutative nilpotent associative algebras over finite fields and over $\mathbb{R}$.

1. Introduction

The classification of associative algebras is an old and often recurring problem. The first investigation into it was perhaps done by Peirce ([16]). Many other publications related to the problem have appeared. Without any claim of completeness, we mention work by Hazlett ([9], nilpotent algebras of dimension ≤ 4 over $\mathbb{C}$), Mazzola ([11] - associative unitary algebras of dimension 5 over algebraically closed fields of characteristic not 2, [12] - nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields of characteristic not 2,3), and recently, Poonen ([18] - nilpotent commutative associative algebras of dimension ≤ 5, over algebraically closed fields).

All these publications have in common that the ground field is assumed to be algebraically closed. In this paper we consider the problem over non algebraically closed fields. This is motivated by two applications. The first concerns the classification of Hopf Galois structures on field extensions L/K (see [4], [7]). Here the classification of commutative nilpotent algebras over finite fields comes into play. The second is related to the classification of Novikov algebras ([3]) - a Novikov algebra with an abelian Lie algebra is nothing but a commutative associative algebra.

In Sections 3, 4, we obtain a classification of nilpotent associative algebras of dimensions ≤ 3, over any field. In Section 5 we classify the nilpotent commutative associative algebras of dimension 4, over finite fields and $\mathbb{R}$. We end with some comments on the situation when the ground field is $\mathbb{Q}$. In Section 2 we describe the method that we use for classifying nilpotent associative algebras. It is very similar to a well-known method for classifying nilpotent Lie algebras (cf. [19], [8]). We note that similar ideas also have come up in the classification of p-groups ([13], [14]).

It turns out that over an infinite field there is an infinite number of isomorphism classes of 3-dimensional nilpotent associative algebras. In fact, there is a series depending on one parameter, where different values of the parameter yield non-isomorphic algebras. On the other hand, over $\mathbb{F}_q$ there are $q+6$ (isomorphism classes of) such algebras, if q is odd, and $q+5$ if q is even. The classification of the nilpotent commutative associative algebras of dimension 4 is different for odd and even order finite fields. However, in both cases there are 11 isomorphism classes of such algebras. Over $\mathbb{R}$ there are 12 isomorphism classes.

2. The classification method

The proofs of the main results in this section are simply translations of those for Lie algebras (cf. [19], [8]), and are therefore omitted.

Throughout the ground field of the vector spaces and algebras will be denoted F.

2.1. Central extensions.

Let A be an associative algebra, V a vector space, and $\theta : A \times A \to V$ a bilinear map. Set $A_\theta = A \oplus V$. For $a, b \in A$, $v, w \in V$ we define $(a+v)(b+w) = ab + \theta(a,b)$. Then A_θ is an associative algebra if and only if

$$\theta(ab, c) = \theta(a, bc) \text{ for all } a, b, c \in A.$$

The bilinear θ satisfying this are called cocycles. The set of all cocycles is denoted $Z^2(A,V)$. The algebra A_θ is called a ($\dim V$-dimensional) central extension of A by V (note that $A_\theta V = VA_\theta = 0$).

Let $\nu : A \to V$ be a linear map, and define $\eta(a,b) = \nu(ab)$. Then η is a cocycle, called a coboundary. The set of all coboundaries is denoted $B^2(A,V)$. Let η be a coboundary; then $A_\theta \cong A_{\theta+\eta}$. Therefore we consider the set $H^2(A,V) = Z^2(A,V)/B^2(A,V)$. If we view V as a trivial A-bimodule, then $H^2(A,V)$ is the second Hochschild-cohomology space (cf. [17]).

Now let B be an associative algebra. By $C(B)$ we denote the ideal consisting of all $b \in B$ with $bB = Bb = 0$. We call this the *multiplication kernel* of B. Suppose that $C(B)$ is nonzero, and set $V = C(B)$, and $A = B/C(B)$. Then there is a $\theta \in H^2(A,V)$ such that $B \cong A_\theta$.

We conclude that any algebra with a nontrivial multiplication kernel can be obtained as a central extension of a algebra of smaller dimension. So in particular, all nilpotent algebras can be constructed this way.

2.2. The radical.

When constructing nilpotent algebras as $A_\theta = A \oplus V$, we want to restrict to θ such that $C(A_\theta) = V$. If the multiplication kernel of A_θ is bigger, then it can be constructed as a central extension of a different algebra. (This way we avoid constructing the same algebra as central extension of different algebras.) Now the radical of a $\theta \in Z^2(A,V)$ is

$$\theta^\perp = \{a \in A \mid \theta(a,b) = \theta(b,a) = 0 \text{ for all } b \in A\}.$$

Then $C(A_\theta) = (\theta^\perp \cap C(A)) + V$, and hence

Proposition 2.1. $\theta^\perp \cap C(A) = 0$ *if and only if* $C(A_\theta) = V$.

2.3. Isomorphism.

Let $e_1, \ldots, e_s$ be a basis of V, and $\theta \in Z^2(A,V)$. Then

$$\theta(a,b) = \sum_{i=1}^{s} \theta_i(a,b) e_i,$$

where $\theta_i \in Z^2(A,F)$. Furthermore, θ is a coboundary if and only if all θ_i are.

Let $\phi \in \mathrm{Aut}(A)$. For $\eta \in Z^2(A,V)$ define $\phi\eta(a,b) = \eta(\phi(a), \phi(b))$. Then $\phi\eta \in Z^2(A,V)$. So $\mathrm{Aut}(A)$ acts on $Z^2(A,V)$. Also, $\eta \in B^2(A,V)$ if and only if $\phi\eta \in B^2(A,V)$ so $\mathrm{Aut}(A)$ acts on $H^2(A,V)$.

Proposition 2.2. *Let* $\theta(a,b) = \sum_{i=1}^{s} \theta_i(a,b)e_i$ *and* $\eta(a,b) = \sum_{i=1}^{s} \eta_i(a,b)e_i$. *Suppose that* $\theta^\perp \cap C(A) = \eta^\perp \cap C(A) = 0$. *Then* $A_\theta \cong A_\eta$ *if and only if there is a* $\phi \in \mathrm{Aut}(A)$ *such that the* $\phi\eta_i$ *span the same subspace of* $H^2(A,F)$ *as the* θ_i.

2.4. **Central components.** Let $A = I_1 \oplus I_2$ be the direct sum of two ideals. Suppose that I_2 is contained in the multiplication kernel of A. Then I_2 is called a *central component* of A.

Proposition 2.3. *Let θ be such that $\theta^\perp \cap C(A) = 0$. Then A_θ has no central components if and only if $\theta_1, \dots, \theta_s$ are linearly independent.*

2.5. **The classification procedure.** Based on Propositions 2.1, 2.2, 2.3 we formulate a procedure that takes as input a nilpotent algebra A of dimension $n-s$. It outputs all nilpotent algebras B of dimension n such that $B/C(B) \cong A$, and B has no central components. For this we need some more terminology. Let Ω be an s-dimensional subspace of $H^2(A,F)$ spanned by $\theta_1, \dots, \theta_s$. Let V be an s-dimensional vector space spanned by $e_1, \dots, e_s$. Then we define $\theta \in H^2(A,V)$ by $\theta(a,b) = \sum_i \theta_i(a,b)e_i$. We call θ the cocycle corresponding to Ω (or more precisely, to the chosen basis of Ω). Furthermore, we say that Ω is *useful* if $\theta^\perp \cap C(A) = 0$. Note that $\theta^\perp$ is the intersection of the $\theta_i^\perp$.

Now the procedure runs as follows:

(1) Determine $Z^2(A,F)$, $B^2(A,F)$ and $H^2(A,F)$.
(2) Determine the orbits of $\mathrm{Aut}(A)$ on the set of useful s-dimensional subspaces of $H^2(A,F)$.
(3) For each orbit let θ be the cocycle corresponding to a representative of it, and construct A_θ.

Comments:

- Of course the hard part is Step 2. Note that $\mathrm{Aut}(A)$ is an algebraic group. This means that whether two useful subspaces lie in the same $\mathrm{Aut}(A)$-orbit is equivalent to the existence of a solution over F of a set of polynomial equations. On some occasions we cannot decide solvability by hand. Then we use the technique of Gröbner bases (cf. [5]). This is an algorithmic procedure to compute an equivalent set of polynomial equations that is sometimes easier to solve. On all occasions where we use this the equations have coefficients in $\mathbb{Z}$. For the Gröbner basis calculation we take the ground field to be $\mathbb{Q}$. A priori this yields results that are only valid over fields of characteristic 0. However, the MAGMA computational algebra system ([2]) has the facility to compute the coefficients of an element of the Gröbner basis relative to the input basis. We use this in order to derive conclusions valid in all characteristics. This will be illustrated in more detail in Section 4.
- The procedure only gives those algebras without central components. So we have to add the algebras obtained by taking the direct sum of a smaller-dimensional algebra with a nil-algebra (that has trivial multiplication).
- In dimension 4 we only classify the commutative nilpotent algebras. For that we only have to consider commutative smaller-dimensional algebras A, and cocycles $\theta \in H^2(A,F)$ with $\theta(a,b) = \theta(b,a)$ for all $a, b \in A$.

2.6. **Notation.** Let A be an associative algebra with basis elements $a_1, \dots, a_n$. Then by Δ_{a_i,a_j} we denote the bilinear map $A \times A \to F$ with $\Delta_{a_i,a_j}(a_k,a_l) = 1$ if $i = k$ and $j = l$, and otherwise it takes the value 0. Moreover, by Σ_{a_i,a_j} we denote the symmetric bilinear map with $\Sigma_{a_i,a_j}(a_k,a_l) = 1$ if $i = k$ and $j = l$, or $i = l$ and $j = k$, and it takes the value 0 otherwise.

Throughout the basis elements of the algebras will be denoted by the letters $a, b, \dots$. The multiplication of an algebra is specified by giving only the nonzero products among the basis elements.

3. Dimensions 1 and 2

There is only one nilpotent algebra of dimension 1: it is spanned by a, and $a^2 = 0$. We denote it by $A_{1,1}$.

Now $H^2(A_{1,1}, F)$ is spanned by $\Delta_{a,a}$. So we get two nilpotent algebras of dimension 2, corresponding to $\theta = 0$ and $\theta = \Delta_{a,a}$ respectively. They are $A_{2,1}$, which is spanned by a, b, and all products are zero, and

$$A_{2,2} : a^2 = b.$$

4. Dimension 3

In this section we classify nilpotent associative algebras of dimension 3, over any field.

First we get the algebras that are the direct sum of an algebra of dimension 2 and a 1-dimensional ideal, isomorphic to $A_{1,1}$, spanned by c. We denote them $A_{3,1}$ (all products zero), and

$$A_{3,2} : a^2 = b.$$

There are no 2-dimensional central extensions of $A_{1,1}$. So we consider 1-dimensional central extensions of $A_{2,1}$. Here $H^2(A_{2,1}, F)$ consists of $\theta = \alpha\Delta_{a,a} + \beta\Delta_{a,b} + \gamma\Delta_{b,a} + \delta\Delta_{b,b}$. The automorphism group consists of all

$$\phi = \begin{pmatrix} u & x \\ v & y \end{pmatrix}, \text{ with } uy - vx \neq 0.$$

Write $\phi\theta = \alpha'\Delta_{a,a} + \cdots + \delta'\Delta_{b,b}$. Then

$$\begin{aligned}
\alpha' &= u^2\alpha + uv\beta + uv\gamma + v^2\delta \\
\beta' &= ux\alpha + uy\beta + vx\gamma + vy\delta \\
\gamma' &= ux\alpha + vx\beta + uy\gamma + vy\delta \\
\delta' &= x^2\alpha + xy\beta + xy\gamma + y^2\delta.
\end{aligned}$$

We distinguish a few cases.

Case 1. First suppose that $\delta \neq 0$, then we can divide to get $\delta = 1$. Choose $u = y = 1$, $x = 0$, $v = -\gamma$ to get $\gamma' = 0$ and $\delta' = 1$. So we may assume that $\delta = 1$ and $\gamma = 0$. Choose $x = v = 0$, $y = 1$; this leads to $\delta' = 1$, $\gamma' = 0$, and $\beta' = u\beta$. We can still freely choose $u \neq 0$. So we are left with two cases: $\beta = 0, 1$.

Case 1a. If $\beta = 0$, then we get the cocycles $\theta^1_\alpha = \alpha\Delta_{a,a} + \Delta_{b,b}$. If we choose $x = v = 0$ then $\delta' = 1$, $\gamma' = \beta' = 0$ and $\alpha' = u^2\alpha$. So we see that θ^1_α and $\theta^1_{u^2\alpha}$ for any $u \neq 0$, lie in the same $\mathrm{Aut}(A_{2,1})$-orbit. In order to show the converse let ϕ be as above. Then $\phi\theta^1_\alpha = \lambda\theta^1_\beta$ (for some nonzero $\lambda \in F$) amounts to the following polynomial equations

$$\begin{aligned}
f_1 &:= u^2\alpha + v^2 - \lambda\beta = 0, \\
f_2 &:= ux\alpha + vy = 0, \\
f_3 &:= x^2\alpha + y^2 - \lambda = 0.
\end{aligned}$$

To these we add

$$f_4 := D(uy - vx) - 1 = 0,$$

which ensures that $\det\phi \neq 0$.

Now a reduced Gröbner basis of the ideal generated by $f_1, \ldots, f_4$ contains the polynomials $u^2\alpha - y^2\beta$, and $v^2 - x^2\alpha\beta$. Using Magma it is not only possible to compute this Gröbner basis, but also to write its elements in terms of the f_i. In this case we have

$$u^2\alpha - y^2\beta = (Dvx+1)f_1 + (Dxy\beta - Duv)f_2 - (Dvx\beta+\beta)f_3 + (v^2 - x^2\alpha\beta)f_4$$
$$v^2 - x^2\alpha\beta = -Dvxf_1 + (Duv - Dxy\beta)f_2 + Dvx\beta f_3 + (x^2\alpha\beta - v^2)f_4.$$

We see that the coefficients that appear all lie in $\mathbb{Z}$; so these equations are valid over any field F. Hence if there is a $\phi \in \mathrm{Aut}(A_{2,1})$ with $\phi\theta^1_\alpha = \lambda\theta^1_\beta$, then there are $u, v, x, y \in F$ with $u^2\alpha - y^2\beta = v^2 - x^2\alpha\beta = 0$ and $uy - vx \neq 0$. Now this implies that there exists $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$. The conclusion is that the spaces spanned by θ^1_α and θ^1_β lie in the same $\mathrm{Aut}(A_{2,1})$-orbit if and only if there is a $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$.

Case 1b. If $\beta = 1$, then we get the cocycles $\theta^2_\alpha = \alpha\Delta_{a,a} + \Delta_{a,b} + \Delta_{b,b}$. Note that these cannot be $\mathrm{Aut}(A_{2,1})$-conjugate to a θ^1_α as the latter is symmetric. Also here we use a Gröbner basis calculation, of which we do not give all the details. In this case when we write the polynomial equations that are equivalent to $\phi\theta^2_\alpha = \lambda\theta^2_\beta$ and compute a Gröbner basis, then we find that it contains $\alpha - \beta$. Also, writing $\alpha - \beta$ in terms of the initial polynomials (as above) we conclude that this is valid over all fields. So, in this case the spaces spanned by θ^2_α and θ^2_β lie in the same $\mathrm{Aut}(A_{2,1})$-orbit if and only if $\alpha = \beta$.

Case 2. If $\delta = 0$ but $\alpha \neq 0$, then we can choose $u = y = 0$, and $x = v = 1$, and get $\delta \neq 0$, and we are back in Case 1.

Case 3. If $\alpha = \delta = 0$, and $\beta \neq -\gamma$, then we can choose x and y such that $xy \neq 0$; then $\delta' \neq 0$, and again we are back in Case 1.

Case 4. The remaining case is $\alpha = \delta = 0$ and $\beta = -\gamma \neq 0$. Then after dividing we get $\theta^3 = \Delta_{a,b} - \Delta_{b,a}$. We have that $\phi\theta^3$ is a multiple of θ^3. Hence it is not conjugate to any of the previous cocycles.

So we get the nonzero cocycles θ^1_α, θ^2_α, and θ^3. For the first we need $\alpha \neq 0$, otherwise a lies in the radical. This leads to the algebras:

$$A^\alpha_{3,3} : a^2 = \alpha c,\ b^2 = c,\ \alpha \neq 0$$
$$A^\alpha_{3,4} : a^2 = \alpha c,\ b^2 = c,\ ab = c,$$
$$A_{3,5} : ab = c,\ ba = -c.$$

From the above discussion it follows that $A^\alpha_{3,3}$ is isomorphic to $A^\beta_{3,3}$ if and only if there is an $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$.

Next we consider 1-dimensional central extensions of $A_{2,2}$. Here we get that $Z^2(A_{2,2}, F)$ is spanned by $\Delta_{a,a}$ and $\Delta_{a,b} + \Delta_{b,a}$. Moreover, $B^2(A_{2,2}, F)$ is spanned by $\Delta_{a,a}$. So we get only one cocycle $\theta = \Delta_{a,b} + \Delta_{b,a}$, yielding the algebra

$$A_{3,6} : a^2 = b, ab = ba = c.$$

Concluding, we have the following nilpotent 3-dimensional algebras: $A_{3,1}$, $A_{3,2}$, $A^\alpha_{3,3}$, where $\alpha \in F^*/(F^*)^2$, $A^\alpha_{3,4}$, where $\alpha \in F$, $A_{3,5}$, $A_{3,6}$. So over an infinite field there is an infinite number of them, wheras over $\mathbb{F}_q$ there are $q + 6$ for q odd, and $q + 5$ for q even.

5. Commutative nilpotent algebras of dimension 4

In this section we consider the classification of the 4-dimensional commutative nilpotent associative algebras. First we assume that the ground field is of characteristic different from 2. We obtain a slightly rough classification, meaning that there will be a series of algebras $A^{\alpha,\beta}_{4,6}$, depending on two parameters α, β, among which we cannot describe the isomorphisms in general. In subsequent subsections we then complete the classification over finite fields of odd order, finite fields of even order and $\mathbb{R}$. Finally we comment on what hapens over $\mathbb{Q}$.

From the previous section we get that the 3-dimensional commutative nilpotent algebras (over fields of characterisitc not 2) are: $A_{3,1}$, $A_{3,2}$, $A_{3,3}^{\alpha}$, $A_{3,6}$.

First we get the algebras that are the direct sum of a 3-dimensional algebra, and $A_{1,1}$. This way we get $A_{4,1} = A_{3,1} \oplus A_{1,1}$, $A_{4,2} = A_{3,2} \oplus A_{1,1}$, $A_{4,3}^{\alpha} = A_{3,3}^{\alpha} \oplus A_{1,1}$, $A_{4,4} = A_{3,6} \oplus A_{1,1}$.

Next we consider 2-dimensional extensions of $A_{2,1}$. As we want to construct commutative algebras we consider the space V consisting of $\alpha\Sigma_{a,a} + \beta\Sigma_{a,b} + \gamma\Sigma_{b,b}$. We need to classify the $\mathrm{GL}_2(F)$-orbits of 2-dimensional subspaces of V. This is the same as classifying the $\mathrm{GL}_2(F)$-orbits on the lines in the space $V \wedge V$. Write the matrix of an element $\phi \in \mathrm{GL}_2(F)$ as in the previous section. Abbreviate $e_1 = \Sigma_{a,a}$, $e_2 = \Sigma_{a,b}$, $e_3 = \Sigma_{b,b}$. Write $\delta = \det(\sigma) = uy - vx$. Then $\phi(\alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3) = \alpha' e_1 \wedge e_2 + \beta' e_1 \wedge e_3 + \gamma' e_2 \wedge e_3$, where

$$\begin{aligned}
\alpha' &= \delta(u^2\alpha + uv\beta + v^2\gamma)\\
\beta' &= \delta(2ux\alpha + (uy + vx)\beta + 2vy\gamma)\\
\gamma' &= \delta(x^2\alpha + xy\beta + y^2\gamma).
\end{aligned}$$

If $\gamma = 0$ then we can choose x, y such that $\gamma' \neq 0$. So we may assume $\gamma \neq 0$, and after dividing, $\gamma = 1$. Then setting $x = 0$, $u = y = 1$, $v = -\frac{1}{2}\beta$ we get $\beta' = 0$. So we may assume that $\beta = 0$. So we get the vectors $w_\alpha = \alpha e_1 \wedge e_2 + e_2 \wedge e_3$. Now the polynomial equations that are equivalent to $\phi w_\alpha = \lambda w_\beta$ are the same as those considered in Section 4, Case 1a. Hence if the lines through w_α and w_β lie in the same $\mathrm{Aut}(A_{2,1})$-orbit then there is $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$.

Now $w_\alpha = e_2 \wedge (-\alpha e_1 + e_3)$. So the subspace of V corresponding to it is spanned by $-\alpha\Sigma_{a,a} + \Sigma_{b,b}$ and $\Sigma_{a,b}$. We get the algebras

$$A_{4,5}^{\alpha} : a^2 = -\alpha c,\ ab = ba = d,\ b^2 = c.$$

From the above discussion it follows that $A_{4,5}^{\alpha} \cong A_{4,5}^{\beta}$ implies that there is a $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$. Conversely, if there is such an ε, then setting $a' = \varepsilon^{-1}a$, $b' = b$, $c' = c$, $d' = \varepsilon^{-1}d$ we get a basis of $A_{4,5}^{\alpha}$, but with the multiplication table of $A_{4,5}^{\beta}$. The conclusion is that $A_{4,5}^{\alpha} \cong A_{4,5}^{\beta}$ if and only if there is a $\varepsilon \in F^*$ with $\alpha = \varepsilon^2\beta$.

$A_{2,2}$ does not have 2-dimensional central extensions, so we are left with determining the 1-dimensional extensions of $A_{3,i}$.

We start with $A_{3,1}$. The usable 1-dimensional subspaces of the space of symmetric cocycles are spanned by nondegenerate symmetric bilinear forms on $A_{3,1}$. It is well-known (cf. [10]) that such a form is equivalent to a diagonal one. In other words, after dividing to get the first coefficient equal to 1, we are left with the cocycles $\theta_{\alpha,\beta} = \Sigma_{a,a} + \alpha\Sigma_{b,b} + \beta\Sigma_{c,c}$, with $\alpha, \beta \neq 0$. Now the problem is to describe exactly when there exists a $\phi \in \mathrm{Aut}(A_{3,1}) = \mathrm{GL}_3(F)$ with $\phi\theta_{\alpha,\beta} = \lambda\theta_{\gamma,\delta}$. The answer to this question depends strongly on the ground field. For algebraically closed fields it is easy: there is only one orbit, with representative $\theta_{1,1}$. In the next subsections we treat finite fields and $\mathbb{R}$. Over $\mathbb{Q}$ the question appears to be very difficult (cf. [1]). We comment on it in the final subsection.

The $\theta_{\alpha,\beta}$ correspond to the algebras

$$A_{4,6}^{\alpha,\beta} : a^2 = d,\ b^2 = \alpha d,\ c^2 = \beta d.$$

Now we consider $A_{3,2}$. We have that $H^2(A_{3,2}, F)$ consists of $\theta = \alpha\Sigma_{a,b} + \beta\Sigma_{a,c} + \gamma\Sigma_{c,c}$. Furthermore the automorphism group consists of

$$\phi = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11}^2 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

Writing $\phi\theta = \alpha'\Sigma_{a,b} + \beta'\Sigma_{a,c} + \gamma'\Sigma_{c,c}$ we have

$$\begin{aligned}\alpha' &= a_{11}^3\alpha\\ \beta' &= a_{11}a_{23}\alpha + a_{11}a_{33}\beta + a_{31}a_{33}\gamma\\ \gamma' &= a_{33}^2\gamma.\end{aligned}$$

We need $\alpha \neq 0$, and one of β, γ nonzero, in oder to have that b or c do not lie in the radical of θ. So after dividing we may asume $\alpha = 1$. Choose $a_{31} = 0$ and $a_{11} = a_{33} = 1$ and $a_{23} = -\beta$. Then $\beta' = 0$. So we may assume $\beta = 0$. This then implies $\gamma \neq 0$. So we get the algebra

$$A^\gamma : a^2 = b,\ ab = ba = d,\ c^2 = \gamma d.$$

Now putting $a' = \gamma a$, $b' = \gamma^2 b$, $c' = \gamma c$, $d' = \gamma^3 d$ we get the same multiplication table, but the parameter has changed to 1. So we may assume $\gamma = 1$, and we get only one algebra

$$A_{4,7} = A^1.$$

Now consider $A_{3,3}^\alpha$. For a $\theta \in Z^2(A_{3,3}^\alpha, F)$ we have that $\theta(u, c) = 0$ for all $u \in A_{3,3}^\alpha$. It follows that c always lies in the radical. So we do not need to consider central extensions of this algebra.

Finally we consider $A_{3,6}$. Here $H^2(A_{3,6}, F)$ is spanned by $\theta = \Sigma_{a,c} + \Sigma_{b,b}$. So we get the algebra

$$A_{4,8} : a^2 = b,\ ab = ba = c,\ ac = ca = d,\ b^2 = d.$$

5.1. **Odd order finite ground fields.** Here we assume the ground field F to be finite and of odd characteristic. In order to complete the classification in this case we need to describe the isomorphisms among the algebras $A_{4,6}^{\alpha,\beta}$, or, equivalently, the $\mathrm{GL}_3(F)$-orbits of usable 1-dimensional subspaces of the space of symmetric cocycles of $A_{3,1}$.

Let V be a 3-dimensional vector space over F, with basis a, b, c. Up to equivalence there are two nondegenerate symmetric bilinear forms, $\Sigma_{a,a} + \Sigma_{b,b} + \Sigma_{c,c}$ and $\Sigma_{a,a} + \Sigma_{b,b} + \gamma\Sigma_{c,c}$, where γ is a non-square in F (see [15]). Denote the first form by θ. By [15], 62:1 quadratic forms are universal in this situation, so there exist $a_{31}, a_{32} \in F$ with $a_{31}^2 + a_{32}^2 = \gamma$. Now set $a' = a_{32}b + a_{31}c$, $b' = -a_{31}b + a_{32}c$, $c' = \gamma a$. Note that they are linearly independent as $a_{31}^2 + a_{32}^2 = \gamma \neq 0$. Then we get that $\theta = \gamma\Sigma_{a',a'} + \gamma\Sigma_{b',b'} + \gamma^2\Sigma_{c',c'}$. We see that after dividing by γ we get the second form. Hence of the algebras $A_{4,6}^{\alpha,\beta}$ there remains only one: $A_{4,6}^{1,1}$.

Concluding, we have the following algebras over a finite field of odd characteristic: $A_{4,1}$, $A_{4,2}$, $A_{4,3}^\alpha$ ($\alpha = 1, \gamma$), $A_{4,4}$, $A_{4,5}^\alpha$ ($\alpha = 0, 1, \gamma$), $A_{4,6}^{1,1}$, $A_{4,7}$, $A_{4,8}$. Here γ is a fixed non-square in F. So we get 11 algebras.

5.2. **Even order finite ground fields.** Now we suppose that the field is finite of characteristic 2. Here we also need to consider 1-dimensional central extensions of $A_{3,5}$. However, a short calculation shows that any cocycle of $A_{3,5}$ will have c in the radical. So we can dispense with this case.

In characteristic 2, the 2-dimensional extensions of $A_{2,1}$ yield extra problems, due to the coefficients 2 that appear in the action of $\mathrm{GL}_2(F)$. Here we cannot get $\beta' = 0$.

First of all, assuming $\beta = 0$ we get the algebras $A_{4,5}^\alpha$. However, also the isomorphism condition changes. Writing the equations we see directly that $A_{4,5}^\alpha \cong A_{4,5}^\beta$ if and only if there are $u, v, x, y \in F$ with $uy + vx \neq 0$ and

$$\alpha = \frac{y^2\beta + v^2}{x^2\beta + u^2}.$$

This defines a $\mathrm{GL}_2(F)$-action on F. In the case of a finite field, everything is a square, and we see that $A^{\alpha}_{4,5} \cong A^0_{4,5}$.

Now assume $\beta \neq 0$. Then choose $x = 0$, $y = 1$, $u = \beta^{-1}$. Then (after dividing by the determinant) we get $\gamma' = \beta' = 1$. This yields the vectors $w_\alpha = \alpha e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3 = (e_1 + e_2) \wedge (\alpha e_2 + e_3)$. So the subspace corresponding to w_α is spanned by $\Sigma_{a,a} + \Sigma_{a,b}$ and $\alpha\Sigma_{a,b} + \Sigma_{b,b}$, yielding the algebras

$$B^{\alpha}_{4,1} : \; a^2 = c, \; ab = ba = c + \alpha d, \; b^2 = d.$$

Write $\phi \in \mathrm{Aut}(A_{2,1})$ as in Section 4. Then the polynomial equations equivalent to $\phi w_\alpha = \lambda w_\beta$ amount to

$$\begin{aligned} &u^2\alpha + uv + v^2 + (uy + vx)\beta = 0 \\ &x^2\alpha + xy + y^2 + uy + vx = 0 \\ &uy + vx \neq 0. \end{aligned}$$

Let $F = \mathbb{F}_{2^m}$ be the ground field, and let $f : F \to F$ be defined $f(T) = T^2 + T$. We claim that the above equations have a solution over F if and only if $\alpha + \beta$ lies in the image of f. Indeed, if v is such that $v^2 + v + \alpha + \beta = 0$ then we set $x = 0$ and $u = y = 1$ and obtain a solution. Conversely, the (reduced) Gröbner basis of the ideal generated by the above polynomials (where we replace the last inequality by the polynomial $D(uy + vx) + 1$) contains the polynomials

$$\begin{aligned} &u^2 + ux + x^2(\alpha + \beta) + xy + y^2 \\ &v^2 + vy + x^2\alpha^2 + xy\alpha + y^2(\alpha + \beta). \end{aligned}$$

So if a solution exists then those polynomials have to vanish as well. If the solution has $x \neq 0$ then we divide the first polynomial by x^2 and get that $\alpha + \beta$ lies in the image of f. On the other hand, if $x = 0$ then we get that conclusion from the second polynomial (note that then $y \neq 0$ so we can divide by y^2).

Now f is a linear map with kernel $\mathbb{F}_2$. It follows that the image of f is a subspace W of order 2^{m-1}. Hence $F = W \cup \gamma_0 + W$, where γ_0 is a fixed element of $F \setminus W$. So here we get two orbits of 1-dimensional subspaces, with representatives w_0 and w_{γ_0}. Hence we also get two algebras: $B^0_{4,1}$ and $B^{\gamma_0}_{4,1}$.

What remains is to describe the 1-dimensional central extensions of $A_{3,1}$. Here we use some theory from [10].

Let θ be a bilinear form on a vector space V of characteristic 2. Then θ is called *alternate* if $\theta(v, v) = 0$ for all $v \in V$. The matrix of such a form can be taken to be block diagonal with blocks

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

or zero. So if $\dim V$ is odd, then such a form is degenerate.

On the other hand, if θ is not alternate, then there exists a basis of V with respect to which the matrix of θ is diagonal. Also, in the case of a finite field, all elements are squares. It follows that we obtain one 1-dimensional central extension of $A_{3,1}$, namely $A^{1,1}_{4,6}$.

Hence we get the following algebras over a finite field of characteristic 2: $A_{4,1}$, $A_{4,2}$, $A^1_{4,3}$, $A_{4,4}$, $A^{\alpha}_{4,5}$ $(\alpha = 0, 1)$, $B^{\alpha}_{4,1}$ $(\alpha = 0, \gamma_0)$, $A^{1,1}_{4,6}$, $A_{4,7}$, $A_{4,8}$.

So, funnily, here we also get 11 algebras.

5.3. **Ground field $\mathbb{R}$.** By [10], Chaper V, Section 9, it follows that there are two $\mathrm{GL}_3(\mathbb{R})$-orbits of usable 1-dimensional subspaces of the space of symmetric cocycles of $A_{3,1}$. They have representatives $\Sigma_{a,a} + \Sigma_{b,b} + \Sigma_{c,c}$ and $\Sigma_{a,a} + \Sigma_{b,b} - \Sigma_{c,c}$. So over $\mathbb{R}$ we get the following algebras: $A_{4,1}$, $A_{4,2}$, $A_{4,3}^{\alpha}$ $(\alpha = -1, 1)$, $A_{4,4}$, $A_{4,5}^{\alpha}$ $(\alpha = -1, 0, 1)$, $A_{4,6}^{1,1}$, $A_{4,6}^{1,-1}$, $A_{4,7}$, $A_{4,8}$. The total number is 12.

5.4. **Ground field $\mathbb{Q}$.** Again the problem is to classify the $\mathrm{GL}(A_{3,1})$-orbits of usable 1-dimensional subspaces of the space of symmetric cocycles of $A_{3,1}$. Let θ, η be two such cocycles. They are called equivalent if they lie in the same $\mathrm{GL}(A_{3,1})$-orbit. Here Hasse's local-global principle holds: θ, η are equivalent if and only if they are equivalent over all p-adic fields, and over $\mathbb{R}$ ([1], §7.5, Theorem 2). This, however, does not yield an explicit classification of the symmetric cocycles up to equivalence. Furthermore, we are interested in a slightly different relation. Write $\theta \sim \eta$ if there is a $\phi \in \mathrm{GL}(A_{3,1})$, and $\lambda \in \mathbb{Q}$ with $\phi\theta = \lambda\eta$. Also here an explicit classification is not in sight.

But, using ideas from [6] we can connect a 4-dimensional central simple associative algebra to the problem.

Let A be the matrix of $\theta_{\alpha,\beta}$, that is A is diagonal with $1, \alpha, \beta$ on the diagonal. Let $\mathfrak{gl}_3(\mathbb{Q})$ be the Lie algebra of 3×3-matrices with coefficients in $\mathbb{Q}$, and set

$$\mathfrak{g}_A = \{X \in \mathfrak{gl}_3(\mathbb{Q}) \mid \text{ there is a } \lambda \in F \text{ with } X^T A + AX = \lambda A\}.$$

Then $\mathfrak{g}_A$ is a 4-dimensional Lie algebra, spanned by the identity matrix and

$$X_1 = \begin{pmatrix} 0 & -\alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & 0 & -\beta \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\alpha & 0 \end{pmatrix}.$$

The subalgebra spanned by X_1, X_2, X_3 is simple; we denote it by $\mathfrak{g}_A^0$.

Consider

$$C_1 = \begin{pmatrix} 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 & \alpha\beta \\ 0 & 0 & 1 & 0 \\ 0 & -\beta & 0 & 0 \\ -\frac{1}{\alpha} & 0 & 0 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & -\alpha\beta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then the C_i together with the 4×4 identity matrix span a 4-dimensional central simple associative algebra over $\mathbb{Q}$, denoted $\mathcal{A}_A$. The map sending X_i to $\frac{1}{2}C_i$ is an isomorphism of $\mathfrak{g}_A^0$ into the Lie algebra corresponding to $\mathcal{A}_A$ (i.e., with Lie product $[x, y] = xy - yx$).

Let B now be the matrix corresponding to $\theta_{\gamma,\delta}$. Note that $\theta_{\alpha,\beta} \sim \theta_{\gamma,\delta}$ is equivalent to the existence of $M \in \mathrm{GL}_3(\mathbb{Q})$ with $M^T AM = \lambda B$. Futhermore, if such M exists then $M^{-1}\mathfrak{g}_A M = \mathfrak{g}_B$. But then also $\mathcal{A}_A$ and $\mathcal{A}_B$ are isomorphic over $\mathbb{Q}$. It follows that any maximal orders of these algebras will have the same discriminant.

As a consequence, if this does not happen then $\theta_{\alpha,\beta} \not\sim \theta_{\gamma,\delta}$. With MAGMA it is possible to compute maximal orders and their discriminants. Here are some cocycles that are *not* equivalent with respect to $\sim$ (the numbers in brackets are the discriminants): $\theta_{1,-1}$ (1), $\theta_{1,1}$ (4), $\theta_{3,3}$ (9), $\theta_{3,-2}$ (36), $\theta_{5,-2}$ (100), $\theta_{5,2}$ (25), $\theta_{7,2}$ (49), $\theta_{7,13}$ (169), $\theta_{-7,13}$ (676), $\theta_{-7,-13}$ (8281), $\theta_{-19,-13}$ (61009).

Furthermore, we have that $\theta_{\gamma,\delta} \sim \theta_{1,-1}$ if and only if the curve given by $x^2 + \gamma y^2 + \delta z^2 = 0$ has a rational point over $\mathbb{Q}$, if and only if the discriminant of a maximal order of $\mathcal{A}_B$ is 1.

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