Random Spherical Triangles

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Abstract. Let Δ be a random spherical triangle (meaning that vertices are independent and uniform on the unit sphere). A closed-form expression for the area density of Δ has been known since 1867; a complicated integral expression for the perimeter density was found in 1994. Does there exist a closed-form expression for the latter? We attempt to answer this question from several directions. An outcome of our work is the exact value of the perimeter density at the point π.

A spherical triangle Δ is a region enclosed by three great circles on the unit sphere; a great circle is a circle whose center is at the origin. The sides of Δ are arcs of great circles and have length a, b, c. Each of these is $\leq \pi$. The angle α opposite side a is the dihedral angle between the two planes passing through the origin and determined by arcs b, c. The angles β, γ opposite sides b, c are similarly defined. Each of these is $\leq \pi$ too.

Define a **primal triangle** to be a random spherical triangle, obtained by selecting three independent uniformly distributed points A, B, C on the sphere to be vertices. Define a **dual triangle** to be a random spherical triangle, obtained by selecting three independent uniformly distributed great circles on the sphere to be sides. More precisely, starting with independent uniform points A', B', C' on the sphere, a dual triangle has vertices

$$A = \frac{B' \times C'}{\|B' \times C'\|}, \qquad B = \frac{A' \times C'}{\|A' \times C'\|}, \qquad C = \frac{A' \times B'}{\|A' \times B'\|}.$$

Hence, while its vertices are not independent, the poles of a dual triangle are.

Let Δ be a primal triangle. The area $\sigma = \alpha + \beta + \gamma - \pi$ of Δ satisfies $0 \leq \sigma \leq 2\pi$. The perimeter $\tau = a + b + c$ of Δ satisfies $0 \leq \tau \leq 2\pi$. Expressions for the trivariate density of (α, β, γ) and the trivariate density of (a, b, c) are known [1] but do not give useful insight into the distributions of σ and τ. Crofton & Exhumatus [2] determined the density for σ:

$$-\frac{(x^2 - 4\pi x + 3\pi^2 - 6)\cos(x) - 6(x - 2\pi)\sin(x) - 2(x^2 - 4\pi x + 3\pi^2 + 3)}{16\pi\cos(x/2)^4}$$

for $0 < x < 2\pi$; this formula remained obscure until it was cited in a recent paper [3]. Unpublished work of J. N. Boots (mentioned in [1]) was also apparently relevant. Jones & Benyon-Tinker [4, 5] determined the density for τ:

$$\frac{1}{4\pi}\int_{0}^{x/2}\frac{E\left(\sin\left(\frac{t}{2}\right)\right)-\cos\left(\frac{x-t}{2}\right)^{2}K\left(\sin\left(\frac{t}{2}\right)\right)}{\sqrt{\cos\left(\frac{t}{2}\right)^{2}-\cos\left(\frac{x-t}{2}\right)^{2}}}\sin(t)\,dt$$

for $0 < x < 2\pi$, where

$$K(\zeta)=\int_{0}^{\pi/2}\frac{1}{\sqrt{1-\zeta^{2}\sin(\theta)^{2}}}\,d\theta=\int_{0}^{1}\frac{1}{\sqrt{(1-t^{2})(1-\zeta^{2}t^{2})}}\,dt,$$

$$E(\zeta)=\int_{0}^{\pi/2}\sqrt{1-\zeta^{2}\sin(\theta)^{2}}\,d\theta=\int_{0}^{1}\sqrt{\frac{1-\zeta^{2}t^{2}}{1-t^{2}}}\,dt$$

are complete elliptic integrals of the first and second kind. We wonder: does there exist a closed-form expression for this latter density? A direct evaluation of the integral does not seem possible, yet conceivably a different geometric argument might yield a more accessible formula. We attempt to answer this question from several directions. Also, it is clear that the density is zero at $x = 0$ and diverges to infinity at $x = 2\pi$. Numerically the density $\approx 3\sqrt{2}/32$ to high precision at $x = \pi$, but a proof via the preceding is not known. An outcome of our work is a new formula that gives the exact value as predicted.

1. Two Coordinate Systems

We define two coordinate systems on the unit sphere that will help in our study of triangular area and density.

1.1. Primal Coordinates. Without loss of generality, let $A = (1, 0, 0)$ and $B = (\cos(\kappa), \sin(\kappa), 0)$ in xyz coordinates. We wish to locate the unique point C in the upper hemisphere so that the triangle ABC satisfies $\alpha = \theta$, $b = \rho$, $c = \kappa$. See Figure 1. The parameters ρ, θ are regarded as varying while the parameter κ is fixed. Think of rotating the equatorial disk in space so that the vector $(1, 0, 0)$ remains fixed and the vector $(0, 1, 0)$ moves toward $(0, 0, 1)$ through the angle θ. The rotation matrix performing this motion is [6]

$$R=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{pmatrix}$$

and

$$R\begin{pmatrix}\cos(\rho)\\ \sin(\rho)\\ 0\end{pmatrix}=\begin{pmatrix}\cos(\rho)\\ \sin(\rho)\cos(\theta)\\ \sin(\rho)\sin(\theta)\end{pmatrix},$$

which gives the point C. The three-dimensional transformation

$$\begin{pmatrix}r\\ \rho\\ \theta\end{pmatrix}\mapsto\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}r\cos(\rho)\\ r\sin(\rho)\cos(\theta)\\ r\sin(\rho)\sin(\theta)\end{pmatrix}$$

has Jacobian determinant

$$\left|\frac{\partial(x,y,z)}{\partial(r,\rho,\theta)}\right|=\begin{vmatrix}\cos(\rho) & -r\sin(\rho) & 0\\ \sin(\rho)\cos(\theta) & r\cos(\rho)\cos(\theta) & -r\sin(\rho)\sin(\theta)\\ \sin(\rho)\sin(\theta) & r\cos(\rho)\sin(\theta) & r\sin(\rho)\cos(\theta)\end{vmatrix}=r^2\sin(\rho)$$

which implies that the area element in primal $\rho\theta$ coordinates is $\sin(\rho)d\rho\, d\theta$.

1.2. Dual Coordinates. Without loss of generality, let $A=(1,0,0)$ and $B=(\cos(\rho),\sin(\rho),0)$ in xyz coordinates. It seems (at first glance) that we should locate the unique point C in the upper hemisphere so that the triangle ABC satisfies $\alpha=\kappa$, $\beta=\theta$, $c=\rho$. See Figure 2. The parameters ρ, θ are regarded as varying while the parameter κ is fixed.

Let us examine the great circle containing A, C. It must also contain the point $(0,\cos(\kappa),\sin(\kappa))$, since this is the image of $(0,1,0)$ after rotation through angle κ. Hence a normal vector is $V=(1,0,0)\times(0,\cos(\kappa),\sin(\kappa))=(0,-\sin(\kappa),\cos(\kappa))$.

Let us examine the great circle containing B, C. Think of rotating the equatorial disk in space so that the vector $(\cos(\rho),\sin(\rho),0)$ remains fixed and the vector $(\sin(\rho),-\cos(\rho),0)$ moves toward $(0,0,1)$ through the angle θ. The rotation matrix performing this motion is [6]

$$S=\begin{pmatrix}\cos(\rho)^2+(1-\cos(\rho)^2)\cos(\theta) & \cos(\rho)\sin(\rho)(1-\cos(\theta)) & -\sin(\rho)\sin(\theta)\\ \cos(\rho)\sin(\rho)(1-\cos(\theta)) & \sin(\rho)^2+(1-\sin(\rho)^2)\cos(\theta) & \cos(\rho)\sin(\theta)\\ \sin(\rho)\sin(\theta) & -\cos(\rho)\sin(\theta) & \cos(\theta)\end{pmatrix}$$

and

$$S\begin{pmatrix}\sin(\rho)\\ -\cos(\rho)\\ 0\end{pmatrix}=\begin{pmatrix}\sin(\rho)\cos(\theta)\\ -\cos(\rho)\cos(\theta)\\ \sin(\theta)\end{pmatrix}.$$

For example, if $\rho=\pi/2$, the image of $(1,0,0)$ after rotation through angle θ is $(\cos(\theta),0,\sin(\theta))$. As another example, if $\rho=0$, the image of $(0,-1,0)$ after rotation through angle θ is $(0, -\cos(\theta), \sin(\theta))$. Hence the great circle must contain the point $(\sin(\rho)\cos(\theta), -\cos(\rho)\cos(\theta), \sin(\theta))$ and a normal vector is

$$W = \begin{pmatrix} \cos(\rho) \\ \sin(\rho) \\ 0 \end{pmatrix} \times \begin{pmatrix} \sin(\rho)\cos(\theta) \\ -\cos(\rho)\cos(\theta) \\ \sin(\theta) \end{pmatrix} = \begin{pmatrix} \sin(\rho)\sin(\theta) \\ -\cos(\rho)\sin(\theta) \\ -\cos(\theta) \end{pmatrix}.$$

The point C is orthogonal to the two normal vectors and at unit distance from the origin, equivalently, $C = (V \times W)/\|V \times W\|$.

Now, in fact, this is more than what is required. We need only (on second glance) specify the great circle containing B, C and this is done via locating W or $-W$. The three-dimensional transformation

$$\begin{pmatrix} r \\ \rho \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -r\sin(\rho)\sin(\theta) \\ r\cos(\rho)\sin(\theta) \\ r\cos(\theta) \end{pmatrix}$$

has Jacobian determinant $r^2\sin(\theta)$, which implies that the area element in dual $\rho\theta$ coordinates is $\sin(\theta)d\rho\,d\theta$. Perhaps this is obvious by duality. It is good, however, to see the supporting geometric details.

2. Four Approaches

We illustrate using four different trigonometric identities and the above two coordinate systems. More possible approaches will be mentioned in a later section.

2.1. Primal Area. As in section [1.1], assume that the triangle ABC satisfies $\alpha = \theta$, $b = \rho$, $c = \kappa$. These three parameters are related to area σ as follows:

$$\tan\left(\frac{\rho}{2}\right) = \cot\left(\frac{\kappa}{2}\right)\csc\left(\theta - \frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right).$$

A proof appears in section [6.1]. For fixed σ and κ, define

$$f(\theta) = \begin{cases} \pi & \text{if } 0 \le \theta < \sigma/2, \\ 2\arctan\left[\cot\left(\frac{\kappa}{2}\right)\csc\left(\theta - \frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right)\right] & \text{if } \sigma/2 \le \theta \le \pi \end{cases}$$

then the conditional probability, given c, is

$$\begin{aligned} \mathrm{P}\{\text{area} \le \sigma \,|c = \kappa\} &= \frac{1}{2\pi}\int_0^{\pi}\int_0^{f(\theta)} \sin(\rho)d\rho\,d\theta \\ &= \frac{1}{2\pi}\int_0^{\sigma/2}\int_0^{\pi} \sin(\rho)d\rho\,d\theta + \frac{1}{2\pi}\int_{\sigma/2}^{\pi}\int_0^{f(\theta)} \sin(\rho)d\rho\,d\theta \end{aligned}$$

$$= \frac{1}{2\pi}\left(\sigma + \int\limits_{\sigma/2}^{\pi}(1-\cos(f(\theta)))d\theta\right).$$

This result can be experimentally verified by generating many primal triangles ABC with $c = \kappa$, and then plotting all pairs (θ, ρ) corresponding to triangles with area $\leq \sigma$. The scatterplot fills the region $[0, \pi] \times [0, \pi]$ except for the portion lying above the curve $\rho = f(\theta)$.

We will later discuss [3.1] how the unconditional probability $\mathrm{P}\{\text{area} \leq \sigma\}$ is evaluated exactly, for arbitrary σ. The method is quite long and intricate.

Here is a quick method for evaluating not probability, but instead density, at $\sigma = \pi$. We start with the conditional density

$$\begin{aligned}\frac{d}{d\sigma}\frac{1}{2\pi}\left(\sigma + \int\limits_{\sigma/2}^{\pi}(1-\cos(f(\theta)))d\theta\right) &= \frac{1}{2\pi}\left(1 - \tfrac{1}{2}\left[1-\cos\left(f\left(\tfrac{\sigma}{2}\right)\right)\right] + \int\limits_{\sigma/2}^{\pi}\frac{d}{d\sigma}(1-\cos(f(\theta)))d\theta\right) \\ &= \frac{1}{2\pi}\int\limits_{\sigma/2}^{\pi}\sin(f(\theta))g(\theta)d\theta\end{aligned}$$

where

$$\sin(f(\theta)) = \frac{2\tan\left(\frac{\kappa}{2}\right)\sin\left(\theta-\frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right)}{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\theta-\frac{\sigma}{2}\right)^2+\sin\left(\frac{\sigma}{2}\right)^2}$$

since

$$\sin(2\arctan(\zeta)) = \frac{2\zeta}{1+\zeta^2},$$

and where

$$g(\theta) = \frac{df}{d\sigma} = \frac{\tan\left(\frac{\kappa}{2}\right)\sin(\theta)}{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\theta-\frac{\sigma}{2}\right)^2+\sin\left(\frac{\sigma}{2}\right)^2}.$$

It follows that the unconditional density is

$$\frac{1}{2\pi}\int\limits_{0}^{\pi}\int\limits_{\sigma/2}^{\pi}\frac{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\theta-\frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right)\sin(\theta)}{\left[\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\theta-\frac{\sigma}{2}\right)^2+\sin\left(\frac{\sigma}{2}\right)^2\right]^2}\sin(\kappa)d\theta\,d\kappa$$

because the density for κ is $\sin(\kappa)/2$. By the half-angle formula for tangent, this is the same as

$$\frac{1}{2\pi}\int\limits_{0}^{\pi}\int\limits_{\sigma/2}^{\pi}\frac{(1-\cos(\kappa))(1+\cos(\kappa))\sin\left(\theta-\frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right)\sin(\theta)}{\left[(1-\cos(\kappa))\sin\left(\theta-\frac{\sigma}{2}\right)^2+(1+\cos(\kappa))\sin\left(\frac{\sigma}{2}\right)^2\right]^2}\sin(\kappa)d\theta\,d\kappa.$$

In the special case when $\sigma = \pi$, this becomes

$$-\frac{1}{2\pi}\int_{0}^{\pi}\int_{\pi/2}^{\pi}\frac{\cos(\theta)\sin(\theta)\left(1-\cos(\kappa)^2\right)\sin(\kappa)}{\left[1+\cos(\kappa)+(1-\cos(\kappa))\cos(\theta)^2\right]^2}d\theta\,d\kappa$$

$$= \quad -\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{0}\frac{u\left(1-v^2\right)}{\left[1+v+(1-v)\,u^2\right]^2}du\,dv = \frac{1}{4\pi}$$

consistent with Crofton & Exhumatus. No analogous simplication seems to occur, for example, when $\sigma = \pi/2$ or $\sigma = 3\pi/2$.

2.2. Dual Perimeter. As in section [1.2], assume that the triangle ABC satisfies $\alpha = \kappa$, $\beta = \theta$, $c = \rho$. These three parameters are related to perimeter τ as follows:

$$\tan\left(\frac{\theta}{2}\right) = \cot\left(\frac{\kappa}{2}\right)\sin\left(\frac{\tau}{2}-\rho\right)\csc\left(\frac{\tau}{2}\right).$$

A proof appears in section [6.3]. For fixed τ and κ, define

$$f(\rho) = \begin{cases} 2\arctan\left[\cot\left(\frac{\kappa}{2}\right)\sin\left(\frac{\tau}{2}-\rho\right)\csc\left(\frac{\tau}{2}\right)\right] & \text{if } 0 \le \rho \le \tau/2, \\ 0 & \text{if } \tau/2 < \rho \le \pi \end{cases}$$

then the conditional probability, given α, is

$$\begin{aligned} \mathrm{P}\left\{\text{perimeter} \le \tau \,|\alpha = \kappa\right\} &= \frac{1}{2\pi}\int_{0}^{\pi}\int_{0}^{f(\rho)}\sin(\theta)d\theta\,d\rho \\ &= \frac{1}{2\pi}\int_{0}^{\tau/2}\int_{0}^{f(\rho)}\sin(\theta)d\theta\,d\rho + \frac{1}{2\pi}\int_{\tau/2}^{\pi}\int_{0}^{0}\sin(\theta)d\theta\,d\rho \\ &= \frac{1}{2\pi}\int_{0}^{\tau/2}(1-\cos(f(\rho)))d\rho. \end{aligned}$$

This result can be experimentally verified by generating many dual triangles ABC with $\alpha = \kappa$, and then plotting all pairs (ρ, θ) corresponding to triangles with perimeter $\le \tau$. The scatterplot fills the region $[0, \pi] \times [0, \pi]$ except for the portion lying above the curve $\theta = f(\rho)$.

Since (dual perimeter) = ($2\pi-$ primal area), it is not surprising that conditional probabilities are so similar.

For completeness' sake, let us compute the conditional density

$$\begin{aligned}\frac{d}{d\tau}\frac{1}{2\pi}\int_0^{\tau/2}(1-\cos(f(\rho)))d\rho &= \frac{1}{2\pi}\left(\frac{1}{2}\left[1-\cos\left(f\left(\frac{\tau}{2}\right)\right)\right]+\int_0^{\tau/2}\frac{d}{d\tau}(1-\cos(f(\rho)))d\rho\right)\\ &= \frac{1}{2\pi}\int_0^{\tau/2}\sin(f(\rho))g(\rho)d\rho\end{aligned}$$

where

$$\sin(f(\rho)) = \frac{2\tan\left(\frac{\kappa}{2}\right)\sin\left(\frac{\tau}{2}-\rho\right)\sin\left(\frac{\tau}{2}\right)}{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\frac{\tau}{2}\right)^2+\sin\left(\frac{\tau}{2}-\rho\right)^2},$$

$$g(\rho) = \frac{df}{d\tau} = \frac{\tan\left(\frac{\kappa}{2}\right)\sin(\rho)}{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\frac{\tau}{2}\right)^2+\sin\left(\frac{\tau}{2}-\rho\right)^2}.$$

It follows that the unconditional density is

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\pi}\int_0^{\tau/2}\frac{\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\frac{\tau}{2}-\rho\right)\sin\left(\frac{\tau}{2}\right)\sin(\rho)}{\left[\tan\left(\frac{\kappa}{2}\right)^2\sin\left(\frac{\tau}{2}\right)^2+\sin\left(\frac{\tau}{2}-\rho\right)^2\right]^2}\sin(\kappa)d\rho\,d\kappa\\ =\ &\frac{1}{2\pi}\int_0^{\pi}\int_0^{\tau/2}\frac{(1-\cos(\kappa))(1+\cos(\kappa))\sin\left(\frac{\tau}{2}-\rho\right)\sin\left(\frac{\tau}{2}\right)\sin(\rho)}{\left[(1-\cos(\kappa))\sin\left(\frac{\tau}{2}\right)^2+(1+\cos(\kappa))\sin\left(\frac{\tau}{2}-\rho\right)^2\right]^2}\sin(\kappa)d\rho\,d\kappa\end{aligned}$$

because the density for κ is $\sin(\kappa)/2$. In the special case when $\tau=\pi$, this becomes

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{\pi}\int_0^{\pi/2}\frac{\cos(\rho)\sin(\rho)(1-\cos(\kappa)^2)\sin(\kappa)}{\left[1-\cos(\kappa)+(1+\cos(\kappa))\cos(\rho)^2\right]^2}d\rho\,d\kappa\\ =\ &\frac{1}{2\pi}\int_{-1}^{1}\int_0^{1}\frac{u(1-v^2)}{\left[1-v+(1+v)u^2\right]^2}du\,dv=\frac{1}{4\pi}\end{aligned}$$

consistent with Crofton & Exhumatus.

2.3. Primal Perimeter. As in section [1.1], assume that the triangle ABC satisfies $\alpha=\theta$, $b=\rho$, $c=\kappa$. These three parameters are related to perimeter τ as follows:

$$\cos(\theta)=\frac{\sin(\tau-\kappa)}{\sin(\kappa)}+\frac{\cos(\tau-\kappa)-\cos(\kappa)}{\sin(\kappa)}\cot(\rho).$$

A proof appears in section [6.2]. For fixed τ and κ, define

$$f(\rho)=\begin{cases}\pi & \text{if } 0\le\rho<\tau/2-\kappa,\\ \arccos\left[\frac{\sin(\tau-\kappa)}{\sin(\kappa)}+\frac{\cos(\tau-\kappa)-\cos(\kappa)}{\sin(\kappa)}\cot(\rho)\right] & \text{if } \tau/2-\kappa\le\rho\le\tau/2,\\ 0 & \text{if } \tau/2<\rho\le\pi\end{cases}$$

assuming $\kappa\le\tau/2$; otherwise $f(\rho)=0$. Then the conditional probability is

$$\begin{aligned}\mathrm{P}\left\{\text{perimeter}\le\tau\,|c=\kappa\right\} &= \frac{1}{2\pi}\int_0^{\pi}\int_0^{f(\rho)}\sin(\rho)d\theta\,d\rho\\ &= \frac{1}{2\pi}\left(\int_0^{\tau/2-\kappa}\int_0^{\pi}\sin(\rho)d\theta\,d\rho+\int_{\tau/2-\kappa}^{\tau/2}\int_0^{f(\rho)}\sin(\rho)d\theta\,d\rho+\int_{\tau/2}^{\pi}\int_0^{0}\sin(\rho)d\theta\,d\rho\right)\\ &= \frac{1}{2\pi}\left(\pi\left[1-\cos\left(\frac{\tau}{2}-\kappa\right)\right]+\int_{\tau/2-\kappa}^{\tau/2}f(\rho)\sin(\rho)d\rho\right).\end{aligned}$$

This result can be experimentally verified by generating many primal triangles ABC with $c=\kappa$, and then plotting all pairs (ρ,θ) corresponding to triangles with perimeter $\le\tau$. The scatterplot fills the region $[0,\pi]\times[0,\pi]$ except for the portion lying above the curve $\theta=f(\rho)$.

An exact evaluation of the unconditional probability $\mathrm{P}\left\{\text{perimeter}\le\tau\right\}$, for arbitrary τ, remains open [3.2].

Here is a quick method for evaluating not probability, but instead density, at $\tau=\pi$. We start with the conditional density

$$\begin{aligned}&\frac{d}{d\tau}\frac{1}{2\pi}\left(\pi\left[1-\cos\left(\frac{\tau}{2}-\kappa\right)\right]+\int_{\tau/2-\kappa}^{\tau/2}f(\rho)\sin(\rho)d\rho\right)\\ &= \frac{1}{2\pi}\left(\frac{\pi}{2}\sin\left(\frac{\tau}{2}-\kappa\right)+\frac{1}{2}f\left(\frac{\tau}{2}\right)\sin\left(\frac{\tau}{2}\right)-\frac{1}{2}f\left(\frac{\tau}{2}-\kappa\right)\sin\left(\frac{\tau}{2}-\kappa\right)+\int_{\tau/2-\kappa}^{\tau/2}\frac{d}{d\tau}f(\rho)\sin(\rho)d\rho\right)\\ &= \frac{1}{2\pi}\int_{\tau/2-\kappa}^{\tau/2}g(\rho)\sin(\rho)d\rho\end{aligned}$$

where

$$g(\rho)=\frac{df}{d\tau}=\frac{\sin(\tau-\kappa-\rho)\sin(\rho)}{\sqrt{\sin(\kappa)^2\sin(\rho)^2-\left[\cos(\kappa)\cos(\rho)-\cos(\tau-\kappa-\rho)\right]^2}}.$$

It follows that the unconditional density is

$$\frac{1}{4\pi}\int_{0}^{\tau/2}\int_{\tau/2-\kappa}^{\tau/2}\frac{\sin(\tau-\kappa-\rho)\sin(\rho)}{\sqrt{\sin(\kappa)^2\sin(\rho)^2-[\cos(\kappa)\cos(\rho)-\cos(\tau-\kappa-\rho)]^2}}\sin(\kappa)d\rho\,d\kappa$$

because the density for κ is $\sin(\kappa)/2$. In the special case when $\tau=\pi$, this becomes

$$\begin{aligned}&\frac{1}{4\pi}\int_{0}^{\pi/2}\int_{\pi/2-\kappa}^{\pi/2}\frac{\sin(\kappa+\rho)\sin(\kappa)\sin(\rho)}{\sqrt{\sin(\kappa)^2\sin(\rho)^2-[\cos(\kappa)\cos(\rho)+\cos(\kappa+\rho)]^2}}d\rho\,d\kappa\\
=\;&\frac{1}{4\pi}\int_{0}^{\pi/2}\int_{\pi/2-\kappa}^{\pi/2}\frac{\sin(\kappa+\rho)\sin(\kappa)\sin(\rho)}{\sqrt{-4\cos(\kappa)\cos(\rho)\cos(\kappa+\rho)}}d\rho\,d\kappa\\
=\;&\frac{1}{8\pi}\int_{0}^{\pi/2}\int_{\pi/2-\kappa}^{\pi/2}\frac{\sin(\kappa+\rho)}{\sqrt{-\cos(\kappa+\rho)}}\frac{\sin(\kappa)}{\sqrt{\cos(\kappa)}}\frac{\sin(\rho)}{\sqrt{\cos(\rho)}}d\rho\,d\kappa\\
=\;&\frac{1}{4\pi}\int_{\pi/4}^{\pi/2}\int_{\pi/2-\kappa}^{\kappa}\frac{\sin(\kappa+\rho)}{\sqrt{-\cos(\kappa+\rho)}}\frac{\sin(\kappa)\sin(\rho)}{\sqrt{\cos(\kappa)\cos(\rho)}}d\rho\,d\kappa\\
=\;&\frac{1}{4\sqrt{2}\pi}\int_{\pi/4}^{\pi/2}\int_{\pi/2-\kappa}^{\kappa}\frac{\sin(\kappa+\rho)}{\sqrt{-\cos(\kappa+\rho)}}\frac{\cos(\kappa-\rho)-\cos(\kappa+\rho)}{\sqrt{\cos(\kappa-\rho)+\cos(\kappa+\rho)}}d\rho\,d\kappa.\end{aligned}$$

Let $u=\kappa+\rho$, $v=\kappa-\rho$. Then $|\partial(u,v)/\partial(\kappa,\rho)|=2$ and the integral is transformed to

$$\frac{1}{8\sqrt{2}\pi}\int_{0}^{\pi/2}\int_{\pi/2}^{\pi-v}\frac{\sin(u)}{\sqrt{-\cos(u)}}\frac{\cos(v)-\cos(u)}{\sqrt{\cos(v)+\cos(u)}}du\,dv=\frac{3\sqrt{2}}{32}$$

as promised.

2.4. Dual Area. As in section [1.2], assume that the triangle ABC satisfies $\alpha=\kappa$, $\beta=\theta$, $c=\rho$. These three parameters are related to area σ as follows:

$$-\cos(\rho)=\frac{\sin(\sigma-\kappa)}{\sin(\kappa)}+\frac{\cos(\sigma-\kappa)-\cos(\kappa)}{\sin(\kappa)}\cot(\theta).$$

A proof appears in section [6.4]. For fixed σ and κ, define

$$f(\theta)=\begin{cases}\pi & \text{if } 0\le\theta<\sigma/2,\\ \pi-\arccos\left[\frac{\sin(\sigma-\kappa)}{\sin(\kappa)}+\frac{\cos(\sigma-\kappa)-\cos(\kappa)}{\sin(\kappa)}\cot(\theta)\right] & \text{if } \sigma/2\le\theta\le\pi-(\kappa-\sigma/2),\\ 0 & \text{if } \pi-(\kappa-\sigma/2)<\theta\le\pi\end{cases}$$

assuming $\kappa \geq \sigma/2$; otherwise $f(\theta) = \pi$. Then the conditional probability is

$$\begin{aligned} \mathrm{P}\left\{\text{area} \leq \sigma \,|\alpha = \kappa\right\} &= \frac{1}{2\pi} \int_0^{\pi} \int_0^{f(\theta)} \sin(\theta) d\rho \, d\theta \\ &= \frac{1}{2\pi} \left(\int_0^{\sigma/2} \int_0^{\pi} \sin(\theta) d\rho \, d\theta + \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} \int_0^{f(\theta)} \sin(\theta) d\rho \, d\theta + \int_{\pi-(\kappa-\sigma/2)}^{\pi} \int_0^{0} \sin(\theta) d\rho \, d\theta \right) \\ &= \frac{1}{2\pi} \left(\pi \left[1 - \cos\left(\frac{\sigma}{2}\right) \right] + \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} f(\theta) \sin(\theta) d\theta \right). \end{aligned}$$

This result can be experimentally verified by generating many dual triangles ABC with $\alpha = \kappa$, and then plotting all pairs (θ, ρ) corresponding to triangles with area $\leq \sigma$. The scatterplot fills the region $[0, \pi] \times [0, \pi]$ except for the portion lying above the curve $\rho = f(\theta)$.

Since (dual area) = ($2\pi-$ primal perimeter), it is not surprising that conditional probabilities are so similar.

For completeness' sake, let us compute the conditional density

$$\begin{aligned} & \frac{d}{d\sigma} \frac{1}{2\pi} \left(\pi \left[1 - \cos\left(\frac{\sigma}{2}\right) \right] + \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} f(\theta) \sin(\theta) d\theta \right) \\ = & \frac{1}{2\pi} \left(\frac{\pi}{2} \sin\left(\frac{\sigma}{2}\right) + \frac{1}{2} f\left(\pi - \left(\kappa - \frac{\sigma}{2}\right)\right) \sin\left(\pi - \left(\kappa - \frac{\sigma}{2}\right)\right) - \frac{1}{2} f\left(\frac{\sigma}{2}\right) \sin\left(\frac{\sigma}{2}\right) \right. \\ & \left. + \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} \frac{d}{d\sigma} f(\theta) \sin(\theta) d\theta \right) \\ = & \frac{1}{2\pi} \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} g(\theta) \sin(\theta) d\theta \end{aligned}$$

where

$$g(\theta) = \frac{df}{d\sigma} = -\frac{\sin(\sigma - \kappa - \theta) \sin(\theta)}{\sqrt{\sin(\kappa)^2 \sin(\theta)^2 - \left[\cos(\kappa)\cos(\theta) - \cos(\sigma - \kappa - \theta)\right]^2}}.$$

It follows that the unconditional density is

$$-\frac{1}{4\pi} \int_{\sigma/2}^{\pi} \int_{\sigma/2}^{\pi-(\kappa-\sigma/2)} \frac{\sin(\sigma - \kappa - \theta) \sin(\theta)}{\sqrt{\sin(\kappa)^2 \sin(\theta)^2 - \left[\cos(\kappa)\cos(\theta) - \cos(\sigma - \kappa - \theta)\right]^2}} \sin(\kappa) d\theta \, d\kappa$$

because the density for κ is $\sin(\kappa)/2$. In the special case when $\sigma = \pi$, this becomes

$$
\begin{aligned}
&-\frac{1}{4\pi}\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2-\kappa}\frac{\sin(\kappa+\theta)\sin(\kappa)\sin(\theta)}{\sqrt{\sin(\kappa)^2\sin(\theta)^2-\left[\cos(\kappa)\cos(\theta)+\cos(\kappa+\theta)\right]^2}}d\theta\,d\kappa \\
=\;& -\frac{1}{4\pi}\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2-\kappa}\frac{\sin(\kappa+\theta)\sin(\kappa)\sin(\theta)}{\sqrt{-4\cos(\kappa)\cos(\theta)\cos(\kappa+\theta)}}d\theta\,d\kappa \\
=\;& -\frac{1}{8\pi}\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2-\kappa}\frac{\sin(\kappa+\theta)}{\sqrt{-\cos(\kappa+\theta)}}\frac{\sin(\kappa)}{\sqrt{-\cos(\kappa)}}\frac{\sin(\theta)}{\sqrt{-\cos(\theta)}}d\theta\,d\kappa \\
=\;& -\frac{1}{4\pi}\int_{\pi/2}^{3\pi/4}\int_{\kappa}^{3\pi/2-\kappa}\frac{\sin(\kappa+\theta)}{\sqrt{-\cos(\kappa+\theta)}}\frac{\sin(\kappa)\sin(\theta)}{\sqrt{\cos(\kappa)\cos(\theta)}}d\theta\,d\kappa \\
=\;& -\frac{1}{4\sqrt{2}\pi}\int_{\pi/2}^{3\pi/4}\int_{\kappa}^{3\pi/2-\kappa}\frac{\sin(\kappa+\theta)}{\sqrt{-\cos(\kappa+\theta)}}\frac{\cos(\kappa-\theta)-\cos(\kappa+\theta)}{\sqrt{\cos(\kappa-\theta)+\cos(\kappa+\theta)}}d\theta\,d\kappa \\
=\;& -\frac{1}{8\sqrt{2}\pi}\int_{0}^{\pi/2}\int_{\pi+v}^{3\pi/2}\frac{\sin(u)}{\sqrt{-\cos(u)}}\frac{\cos(v)-\cos(u)}{\sqrt{\cos(v)+\cos(u)}}du\,dv = \frac{3\sqrt{2}}{32}
\end{aligned}
$$

as promised.

3. Two Evaluations

3.1. Successful Evaluation for Primal Area. Starting from the half-angle formula for tangent

$$\tan\left(\frac{\rho}{2}\right)^2 = \frac{1-\cos(\rho)}{1+\cos(\rho)}$$

we deduce that

$$\cot\left(\frac{\rho}{2}\right)^2 + 1 = \frac{1+\cos(\rho)}{1-\cos(\rho)} + 1 = \frac{2}{1-\cos(\rho)}$$

hence

$$\frac{1-\cos(\rho)}{2} = \frac{1}{\cot\left(\frac{\rho}{2}\right)^2+1} = \frac{1}{\Omega^2\sin\left(\theta-\frac{\sigma}{2}\right)^2+1}$$

by [2.1], where

$$\Omega = \tan\left(\frac{\kappa}{2}\right)\csc\left(\frac{\sigma}{2}\right).$$

It follows that

$$\int_{\sigma/2}^{\pi} \frac{1-\cos(f(\theta))}{2} d\theta = \int_{\sigma/2}^{\pi} \frac{1}{\Omega^2 \sin\left(\theta - \frac{\sigma}{2}\right)^2 + 1} d\theta$$
$$= \begin{cases} \dfrac{\pi - \arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}} & \text{if } 0 \le \sigma < \pi, \\ -\dfrac{\arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}} & \text{if } \pi \le \sigma \le 2\pi \end{cases}$$

and thus

$$\mathrm{P}\left\{\text{area} \le \sigma\right\} = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{\pi}\left[\frac{\sigma}{2} + \frac{\pi - \arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}}\right]\sin(\kappa)d\kappa & \text{if } 0 \le \sigma < \pi, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{\pi}\left[\frac{\sigma}{2} - \frac{\arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}}\right]\sin(\kappa)d\kappa & \text{if } \pi \le \sigma \le 2\pi. \end{cases}$$

The area density is therefore

$$\begin{cases} \dfrac{1}{2\pi}\left[1 + \dfrac{d}{d\sigma}\displaystyle\int_0^{\pi}\frac{\pi - \arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}}\sin(\kappa)d\kappa\right] & \text{if } 0 \le \sigma < \pi, \\ \dfrac{1}{2\pi}\left[1 - \dfrac{d}{d\sigma}\displaystyle\int_0^{\pi}\frac{\arctan\left(\sqrt{\Omega^2+1}\tan(\sigma/2)\right)}{\sqrt{\Omega^2+1}}\sin(\kappa)d\kappa\right] & \text{if } \pi \le \sigma \le 2\pi \end{cases}$$

which, as outlined in [7.1], gives rise to the Crofton/Exhumatus expression.

3.2. Unsuccessful Evaluation for Primal Perimeter. There does not seem to be an analogous approach for computing

$$\int_{\tau/2-\kappa}^{\tau/2} f(\rho)\sin(\rho)d\rho$$

from [2.3] in closed-form. We suspect that elliptic integrals will arise, but have not yet found a method for demonstrating this. See [7.2] for more details.

4. Two More Approaches

4.1. Median Area. Assume that the triangle ABC satisfies $c = \kappa$. A **median** in ABC is the great circle drawn from vertex C to the midpoint P of side c. Let ρ

denote the spherical distance between P and C, and θ denote the angle between PB and PC. These three parameters are related to primal area σ as follows:

$$\tan\left(\frac{\sigma}{2}\right) = \frac{\sin(\kappa/2)\sin(\rho)\sin(\theta)}{\cos(\kappa/2)+\cos(\rho)}.$$

A proof appears in section [6.5]. For fixed σ and κ, define

$$f(\theta) = \arccos\left[\frac{\cos\left(\frac{\kappa}{2}\right)\tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2 \mp \sin\left(\frac{\kappa}{2}\right)^2\sqrt{1+\tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2}}{\cos\left(\frac{\kappa}{2}\right)^2 - \tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2 - 1}\right]$$

where $-$ is chosen if $\sigma \leq \pi$ and $+$ is chosen if $\sigma > \pi$; then the conditional probability is

$$\begin{aligned} \mathrm{P}\left\{\text{area} \leq \sigma \,|c = \kappa\right\} &= \frac{1}{2\pi}\int_0^\pi\int_0^{f(\theta)} \sin(\rho)d\rho\, d\theta \\ &= \frac{1}{2\pi}\int_0^\pi (1-\cos(f(\theta)))d\theta \\ &= \frac{1}{2\pi}\int_0^\pi \left(1-\frac{\cos\left(\frac{\kappa}{2}\right)\tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2 \mp \sin\left(\frac{\kappa}{2}\right)^2\sqrt{1+\tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2}}{\cos\left(\frac{\kappa}{2}\right)^2 - \tan\left(\frac{\sigma}{2}\right)^2\csc(\theta)^2 - 1}\right)d\theta. \end{aligned}$$

This result can be experimentally verified by generating many primal triangles ABC with $c = \kappa$, and then plotting all pairs (θ, ρ) corresponding to triangles with area $\leq \sigma$. The scatterplot fills the region $[0,\pi]\times[0,\pi]$ except for the portion lying above the curve $\rho = f(\theta)$. This approach is believed to be the same as Crofton & Exhumatus (details in [2] are rather thin). Not seeing any advantage over our approach in [2.1], we stop here.

4.2. Bisector Perimeter. Assume that the triangle ABC satisfies $\alpha = \kappa$. An **angle bisector** in ABC is the great circle drawn from vertex A that splits angle α in half. Define Q to be the intersection point between this circle and side BC. Let ρ denote the spherical distance between Q and A, and θ denote the angle between QC and QA. These three parameters are related to dual perimeter τ as follows:

$$\tan\left(\frac{\tau}{2}\right) = -\frac{\cos(\kappa/2)\sin(\rho)\sin(\theta)}{\sin(\kappa/2)+\cos(\rho)\sin(\theta)}.$$

A proof appears in section [6.6]. For fixed τ and κ, define

$$f_{\text{lower}}(\rho) = \arcsin\left[-\frac{\tan(\tau/2)\sin(\kappa/2)}{\tan(\tau/2)\cos(\rho)+\cos(\kappa/2)\sin(\rho)}\right],$$

$$f_{\text{upper}}(\rho) = \pi - \arcsin\left[-\frac{\tan(\tau/2)\sin(\kappa/2)}{\tan(\tau/2)\cos(\rho) + \cos(\kappa/2)\sin(\rho)}\right]$$

assuming

$$\arccos\left(-\frac{\cos(\tau/2) + \sin(\kappa/2)}{1 + \cos(\tau/2)\sin(\kappa/2)}\right) = \rho_{\text{thres}} \le \rho \le \pi.$$

The region of all pairs (ρ, θ) corresponding to triangles with perimeter $\le \tau$ is more complicated than earlier examples. The scatterplot fills the region $[\rho_{\text{thres}}, \pi] \times [0, \pi]$ except for portions lying either above the curve $\theta = f_{\text{upper}}(\rho)$ or below the curve $\theta = f_{\text{lower}}(\rho)$. The conditional probability is

$$\begin{aligned}
\mathrm{P}\left\{\text{perimeter} \le \tau \,|\alpha = \kappa\right\} &= \frac{1}{2\pi}\int_{\rho_{\text{thres}}}^{\pi}\int_{f_{\text{lower}}(\rho)}^{f_{\text{upper}}(\rho)} \sin(\theta) d\theta\, d\rho \\
&= \frac{1}{2\pi}\int_{\rho_{\text{thres}}}^{\pi} \left(\cos(f_{\text{lower}}(\rho)) - \cos(f_{\text{upper}}(\rho))\right) d\rho \\
&= \frac{1}{\pi}\int_{\rho_{\text{thres}}}^{\pi} \sqrt{1 - \left(\frac{\tan(\tau/2)\sin(\kappa/2)}{\tan(\tau/2)\cos(\rho) + \cos(\kappa/2)\sin(\rho)}\right)^2} d\rho.
\end{aligned}$$

Due to the unanticipated complexity, we stop here.

5. Two More Coordinate Systems

The primal coordinate system [1.1] allows us to specify a triangle, given a fixed side κ, with an additional side ρ and an angle θ. Can we do as well with two angles instead? The dual coordinate system [1.2] allows us to likewise specify a triangle, given a fixed angle κ. Can we do as well with two sides instead?

5.1. Angle Coordinates. Without loss of generality, let $A = (1, 0, 0)$ and $B = (\cos(\kappa), \sin(\kappa), 0)$ in xyz coordinates. We wish to locate the unique point C in the hemisphere so that the triangle ABC satisfies $\alpha = \varphi$, $\beta = \psi$, $c = \kappa$. See Figure 3. The parameters φ, ψ are regarded as varying while the parameter κ is fixed.

Let us examine the great circle containing A, C. It must also contain the point $(0, \cos(\varphi), \sin(\varphi))$, since this is the image of $(0, 1, 0)$ after rotation through angle φ. Hence a normal vector is $V = (1, 0, 0) \times (0, \cos(\varphi), \sin(\varphi)) = (0, -\sin(\varphi), \cos(\varphi))$.

Let us examine the great circle containing B, C. Think of rotating the equatorial disk in space so that the vector $(\cos(\kappa), \sin(\kappa), 0)$ remains fixed and the vector $(\sin(\kappa), -\cos(\kappa), 0)$ moves toward $(0, 0, 1)$ through the angle ψ. The rotation matrix performing this motion is [6]

$$S=\begin{pmatrix} \cos(\kappa)^2+(1-\cos(\kappa)^2)\cos(\psi) & \cos(\kappa)\sin(\kappa)(1-\cos(\psi)) & -\sin(\kappa)\sin(\psi) \\ \cos(\kappa)\sin(\kappa)(1-\cos(\psi)) & \sin(\kappa)^2+(1-\sin(\kappa)^2)\cos(\psi) & \cos(\kappa)\sin(\psi) \\ \sin(\kappa)\sin(\psi) & -\cos(\kappa)\sin(\psi) & \cos(\psi) \end{pmatrix}$$

and

$$S\begin{pmatrix} \sin(\kappa) \\ -\cos(\kappa) \\ 0 \end{pmatrix}=\begin{pmatrix} \sin(\kappa)\cos(\psi) \\ -\cos(\kappa)\cos(\psi) \\ \sin(\psi) \end{pmatrix}.$$

Hence the great circle must contain the point $(\sin(\kappa)\cos(\psi),-\cos(\kappa)\cos(\psi),\sin(\psi))$ and a normal vector is

$$W=\begin{pmatrix} \cos(\kappa) \\ \sin(\kappa) \\ 0 \end{pmatrix}\times\begin{pmatrix} \sin(\kappa)\cos(\psi) \\ -\cos(\kappa)\cos(\psi) \\ \sin(\psi) \end{pmatrix}=\begin{pmatrix} \sin(\kappa)\sin(\psi) \\ -\cos(\kappa)\sin(\psi) \\ -\cos(\psi) \end{pmatrix}.$$

The point C is orthogonal to the two normal vectors and at unit distance from the origin, equivalently, $C=(V\times W)/\left\|V\times W\right\|$. We have

$$V\times W=\begin{pmatrix} \sin(\varphi)\cos(\psi)+\cos(\kappa)\cos(\varphi)\sin(\psi) \\ \sin(\kappa)\cos(\varphi)\sin(\psi) \\ \sin(\kappa)\sin(\varphi)\sin(\psi) \end{pmatrix},$$

$$\left\|V\times W\right\|=\sqrt{1-\left(\cos(\varphi)\cos(\psi)-\cos(\kappa)\sin(\varphi)\sin(\psi)\right)^2}$$

and thus the Jacobian determinant of $(r,\varphi,\psi)\mapsto(x,y,z)=-rC$ simplifies to

$$\frac{\sin(\kappa)^2\sin(\varphi)\sin(\psi)\left[\left(\sin(\varphi)\cos(\psi)+\cos(\kappa)\cos(\varphi)\sin(\psi)\right)^2+\sin(\kappa)^2\sin(\psi)^2\right]}{\left[1-\left(\cos(\varphi)\cos(\psi)-\cos(\kappa)\sin(\varphi)\sin(\psi)\right)^2\right]^{5/2}}.$$

5.2. Side Coordinates. Without loss of generality, let $A=(1,0,0)$ and $B=(\cos(\xi),\sin(\xi),0)$ in xyz coordinates. It seems (at first glance) that we should locate the unique point C in the upper hemisphere so that the triangle ABC satisfies $\alpha=\kappa$, $c=\xi$, $b=\eta$. See Figure 4. The parameters ξ, η are regarded as varying while the parameter κ is fixed. Think of rotating the equatorial disk in space so that the vector $(1,0,0)$ remains fixed and the vector $(0,1,0)$ moves toward $(0,0,1)$ through the angle κ. The rotation matrix performing this motion is [6]

$$R=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\kappa) & -\sin(\kappa) \\ 0 & \sin(\kappa) & \cos(\kappa) \end{pmatrix}$$

and

$$R\begin{pmatrix}\cos(\eta)\\ \sin(\eta)\\ 0\end{pmatrix}=\begin{pmatrix}\cos(\eta)\\ \cos(\kappa)\sin(\eta)\\ \sin(\kappa)\sin(\eta)\end{pmatrix},$$

which gives the point C.

Now, in fact, this is less than what is required. We must (on second glance) specify the great circle containing B, C. This is done via a normal vector $U = (B \times C)/\left\|B \times C\right\|$, where

$$\begin{aligned}B\times C &= \begin{pmatrix}\cos(\xi)\\ \sin(\xi)\\ 0\end{pmatrix}\times\begin{pmatrix}\cos(\eta)\\ \cos(\kappa)\sin(\eta)\\ \sin(\kappa)\sin(\eta)\end{pmatrix}\\ &= \begin{pmatrix}\sin(\kappa)\sin(\xi)\sin(\eta)\\ -\sin(\kappa)\cos(\xi)\sin(\eta)\\ -\sin(\xi)\cos(\eta)+\cos(\kappa)\cos(\xi)\sin(\eta)\end{pmatrix},\end{aligned}$$

$$\left\|B\times C\right\|=\sqrt{1-(\cos(\xi)\cos(\eta)+\cos(\kappa)\sin(\xi)\sin(\eta))^2}.$$

Thus the Jacobian determinant of $(r,\xi,\eta)\mapsto(x,y,z)=rU$ simplifies to

$$\frac{\sin(\kappa)^2\sin(\xi)\sin(\eta)\left[(\sin(\xi)\cos(\eta)-\cos(\kappa)\cos(\xi)\sin(\eta))^2+\sin(\kappa)^2\sin(\eta)^2\right]}{\left[1-(\cos(\xi)\cos(\eta)+\cos(\kappa)\sin(\xi)\sin(\eta))^2\right]^{5/2}}.$$

5.3. Possible Applications. Combining an identity in [7] with the Law of Cosines for Angles, we obtain

$$\tan\left(\frac{\tau}{2}\right)=\frac{\sin(\varphi)\sin(\psi)\sin(\kappa)}{\cos(\varphi)+\cos(\psi)-\cos(\varphi)\cos(\psi)+\cos(\kappa)\sin(\varphi)\sin(\psi)-1}.$$

Let us solve for ψ as follows:

$$[\cos(\varphi)+\cos(\psi)-\cos(\varphi)\cos(\psi)-1]\tan\left(\frac{\tau}{2}\right)=\sin(\varphi)\sin(\psi)\left[\sin(\kappa)-\cos(\kappa)\tan\left(\frac{\tau}{2}\right)\right]$$

hence

$$-\left[1-\cos(\varphi)\right]\left[1-\cos(\psi)\right]\tan\left(\frac{\tau}{2}\right)=\sin(\varphi)\sin(\psi)\left[\sin(\kappa)-\cos(\kappa)\tan\left(\frac{\tau}{2}\right)\right]$$

hence

$$-\frac{1-\cos(\varphi)}{\sin(\varphi)}\tan\left(\frac{\tau}{2}\right)=\frac{\sin(\psi)}{1-\cos(\psi)}\left[\sin(\kappa)-\cos(\kappa)\tan\left(\frac{\tau}{2}\right)\right]$$

hence

$$\tan\left(\frac{\varphi}{2}\right)\csc\left(\frac{\tau}{2}-\kappa\right)\sin\left(\frac{\tau}{2}\right)=\frac{\sin(\psi)}{1-\cos(\psi)}$$

hence

$$\cos(\psi)=\frac{\tan\left(\frac{\varphi}{2}\right)^2\csc\left(\frac{\tau}{2}-\kappa\right)^2\sin\left(\frac{\tau}{2}\right)^2-1}{\tan\left(\frac{\varphi}{2}\right)^2\csc\left(\frac{\tau}{2}-\kappa\right)^2\sin\left(\frac{\tau}{2}\right)^2+1}$$

because $y=\sqrt{1-x^2}/(1-x)$ has inverse $x=(y^2-1)\,/\,(y^2+1)$. For fixed τ and κ, define

$$f(\varphi)=\begin{cases}\arccos\left[\frac{\tan\left(\frac{\varphi}{2}\right)^2\csc\left(\frac{\tau}{2}-\kappa\right)^2\sin\left(\frac{\tau}{2}\right)^2-1}{\tan\left(\frac{\varphi}{2}\right)^2\csc\left(\frac{\tau}{2}-\kappa\right)^2\sin\left(\frac{\tau}{2}\right)^2+1}\right] & \text{if } 0\leq\kappa<\tau/2,\\ 0 & \text{if } \tau/2\leq\kappa\leq\pi\end{cases}$$

then the conditional probability, given c, is

$$\mathrm{P}\left\{\text{perimeter}\leq\tau\,|c=\kappa\right\}=\frac{1}{2\pi}\int\limits_0^{\pi}\int\limits_0^{f(\theta)}\frac{\sin(\kappa)^2\sin(\varphi)\sin(\psi)\left[(\sin(\varphi)\cos(\psi)+\cos(\kappa)\cos(\varphi)\sin(\psi))^2+\sin(\kappa)^2\sin(\psi)^2\right]}{\left[1-(\cos(\varphi)\cos(\psi)-\cos(\kappa)\sin(\varphi)\sin(\psi))^2\right]^{5/2}}d\psi\,d\varphi.$$

Similarly, combining an identity in [7] with the Law of Cosines for Sides, we obtain

$$\tan\left(\frac{\sigma}{2}\right)=\frac{\sin(\xi)\sin(\eta)\sin(\kappa)}{1+\cos(\xi)+\cos(\eta)+\cos(\xi)\cos(\eta)+\cos(\kappa)\sin(\xi)\sin(\eta)}.$$

Let us solve for η as follows:

$$\left[1+\cos(\xi)+\cos(\eta)+\cos(\xi)\cos(\eta)\right]\tan\left(\frac{\sigma}{2}\right)=\sin(\xi)\sin(\eta)\left[\sin(\kappa)-\cos(\kappa)\tan\left(\frac{\sigma}{2}\right)\right]$$

hence

$$\frac{1+\cos(\xi)}{\sin(\xi)}\tan\left(\frac{\sigma}{2}\right)=\frac{\sin(\eta)}{1+\cos(\eta)}\left[\sin(\kappa)-\cos(\kappa)\tan\left(\frac{\sigma}{2}\right)\right]$$

hence

$$\cos(\eta)=\frac{1-\cot\left(\frac{\xi}{2}\right)^2\csc\left(\kappa-\frac{\sigma}{2}\right)^2\sin\left(\frac{\sigma}{2}\right)^2}{1+\cot\left(\frac{\xi}{2}\right)^2\csc\left(\kappa-\frac{\sigma}{2}\right)^2\sin\left(\frac{\sigma}{2}\right)^2}$$

because $y=\sqrt{1-x^2}/(1+x)$ has inverse $x=(1-y^2)\,/\,(1+y^2)$. For fixed σ and κ, define

$$f(\xi)=\begin{cases}\pi & \text{if } 0\leq\kappa<\sigma/2,\\ \arccos\left[\frac{1-\cot\left(\frac{\xi}{2}\right)^2\csc\left(\kappa-\frac{\sigma}{2}\right)^2\sin\left(\frac{\sigma}{2}\right)^2}{1+\cot\left(\frac{\xi}{2}\right)^2\csc\left(\kappa-\frac{\sigma}{2}\right)^2\sin\left(\frac{\sigma}{2}\right)^2}\right] & \text{if } \sigma/2\leq\kappa\leq\pi\end{cases}$$

then the conditional probability, given α, is

$$\mathrm{P}\left\{\text{area} \le \sigma \,|\alpha = \kappa\right\} = \frac{1}{2\pi}\int_0^\pi \int_0^{f(\xi)} \frac{\sin(\kappa)^2\sin(\xi)\sin(\eta)\left[(\sin(\xi)\cos(\eta)-\cos(\kappa)\cos(\xi)\sin(\eta))^2+\sin(\kappa)^2\sin(\eta)^2\right]}{\left[1-(\cos(\xi)\cos(\eta)+\cos(\kappa)\sin(\xi)\sin(\eta))^2\right]^{5/2}}d\eta\, d\xi.$$

We have not further pursued this direction of inquiry.

6. Trigonometric Identities

The following formulas are required in the main text.

6.1. Primal Case i. To prove

$$\tan\left(\frac{b}{2}\right) = \cot\left(\frac{c}{2}\right)\csc\left(\alpha - \frac{\sigma}{2}\right)\sin\left(\frac{\sigma}{2}\right)$$

we expand $\cos(\sigma/2)$ and make use of Delambre's analogies [8]:

$$\begin{aligned}
\cos\left(\frac{\sigma}{2}\right) &= \cos\left(\frac{\beta+\gamma}{2} - \frac{\pi-\alpha}{2}\right)\\
&= \cos\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\
&= \left[\cos\left(\frac{b+c}{2}\right)\sin\left(\frac{\alpha}{2}\right)^2 + \cos\left(\frac{b-c}{2}\right)\cos\left(\frac{\alpha}{2}\right)^2\right]\sec\left(\frac{a}{2}\right)\\
&= \left[\left(\cos\frac{b}{2}\cos\frac{c}{2} - \sin\frac{b}{2}\sin\frac{c}{2}\right)\frac{1-\cos\alpha}{2} + \left(\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\sin\frac{c}{2}\right)\frac{1+\cos\alpha}{2}\right]\sec\frac{a}{2}\\
&= \left(\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\sin\frac{c}{2}\cos\alpha\right)\sec\frac{a}{2}.
\end{aligned}$$

Also

$$\begin{aligned}
\sin\left(\frac{\sigma}{2}\right) &= \sin\left(\frac{\beta+\gamma}{2} - \frac{\pi-\alpha}{2}\right)\\
&= -\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\\
&= \left[-\cos\left(\frac{b+c}{2}\right)\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) + \cos\left(\frac{b-c}{2}\right)\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right]\sec\left(\frac{a}{2}\right)\\
&= \left[-\left(\cos\frac{b}{2}\cos\frac{c}{2} - \sin\frac{b}{2}\sin\frac{c}{2}\right) + \left(\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\sin\frac{c}{2}\right)\right]\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sec\frac{a}{2}\\
&= 2\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sec\frac{a}{2} = \sin\frac{b}{2}\sin\frac{c}{2}\sin\alpha\sec\frac{a}{2}.
\end{aligned}$$

Dividing, we obtain

$$\cot\frac{\sigma}{2} = \frac{\cos\frac{b}{2}\cos\frac{c}{2} + \sin\frac{b}{2}\sin\frac{c}{2}\cos\alpha}{\sin\frac{b}{2}\sin\frac{c}{2}\sin\alpha}$$

hence

$$\cos\frac{\sigma}{2}\sin\alpha = \left(\cot\frac{b}{2}\cot\frac{c}{2} + \cos\alpha\right)\sin\frac{\sigma}{2}$$

and therefore

$$\sin\left(\alpha - \frac{\sigma}{2}\right) = \sin\alpha\cos\frac{\sigma}{2} - \cos\alpha\sin\frac{\sigma}{2} = \cot\frac{b}{2}\cot\frac{c}{2}\sin\frac{\sigma}{2}$$

as was to be shown.

6.2. Primal Case ii.

To prove

$$\cos(\alpha) = \frac{\sin(\tau - c)}{\sin(c)} + \frac{\cos(\tau - c) - \cos(c)}{\sin(c)}\cot(b)$$

we expand $\cos(a)$:

$$\begin{aligned}\cos(a) &= \cos(\tau - b - c)\\ &= \cos\left(-b + (\tau - c)\right)\\ &= \cos(b)\cos(\tau - c) + \sin(b)\sin(\tau - c)\end{aligned}$$

and make use of the Law of Cosines for Sides:

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(\alpha)$$

thus

$$\sin(b)\sin(c)\cos(\alpha) = \sin(\tau - c)\sin(b) + [\cos(\tau - c) - \cos(c)]\cos(b).$$

Alternatively, we have

$$\sin(c)\cos(\alpha) = \sin(\tau - c) + (\cos(\tau - c) - \cos(c))\cot(b)$$

hence

$$-\tan(b) = \frac{\cos(\tau - c) - \cos(c)}{\sin(\tau - c) - \sin(c)\cos(\alpha)}$$

but solving for b turns out to be more complicated than our strategy of solving for α.

6.3. Dual Case i. To prove

$$\tan\left(\frac{\beta}{2}\right) = \cot\left(\frac{\alpha}{2}\right)\sin\left(\frac{\tau}{2} - c\right)\csc\left(\frac{\tau}{2}\right)$$

we expand $\cos(\tau/2)$ and make use of Delambre's analogies [8]:

$$\begin{aligned}
\cos\left(\frac{\tau}{2}\right) &= \cos\left(\frac{a+b}{2} + \frac{c}{2}\right) \\
&= \cos\left(\frac{a+b}{2}\right)\cos\left(\frac{c}{2}\right) - \sin\left(\frac{a+b}{2}\right)\sin\left(\frac{c}{2}\right) \\
&= \left[\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{c}{2}\right)^2 - \cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{c}{2}\right)^2\right]\csc\left(\frac{\gamma}{2}\right) \\
&= \left[\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\frac{1+\cos c}{2} - \left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\frac{1-\cos c}{2}\right]\csc\frac{\gamma}{2} \\
&= \left(-\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos c\right)\csc\frac{\gamma}{2}.
\end{aligned}$$

Also

$$\begin{aligned}
\sin\left(\frac{\tau}{2}\right) &= \sin\left(\frac{a+b}{2} + \frac{c}{2}\right) \\
&= \cos\left(\frac{a+b}{2}\right)\sin\left(\frac{c}{2}\right) + \sin\left(\frac{a+b}{2}\right)\cos\left(\frac{c}{2}\right) \\
&= \left[\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{c}{2}\right)\sin\left(\frac{c}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{c}{2}\right)\cos\left(\frac{c}{2}\right)\right]\csc\left(\frac{\gamma}{2}\right) \\
&= \left[\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) + \left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\right]\cos\frac{c}{2}\sin\frac{c}{2}\csc\frac{\gamma}{2} \\
&= 2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{c}{2}\sin\frac{c}{2}\csc\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin c\csc\frac{\gamma}{2}.
\end{aligned}$$

Dividing, we obtain

$$\cot\frac{\tau}{2} = \frac{-\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos c}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin c}$$

hence

$$\cos\frac{\tau}{2}\sin c = \left(-\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \cos c\right)\sin\frac{\tau}{2}$$

and therefore

$$\sin\left(\frac{\tau}{2}-c\right)=\sin\frac{\tau}{2}\cos c-\cos\frac{\tau}{2}\sin c=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\sin\frac{\tau}{2}$$

as was to be shown.

6.4. Dual Case ii. To prove

$$-\cos(c)=\frac{\sin(\sigma-\alpha)}{\sin(\alpha)}+\frac{\cos(\sigma-\alpha)-\cos(\alpha)}{\sin(\alpha)}\cot(\beta)$$

we expand $-\cos(\gamma)$:

$$\begin{aligned}-\cos(\gamma)&=-\cos(\sigma-\alpha-\beta+\pi)\\&=\cos\left(-\beta+(\sigma-\alpha)\right)\\&=\cos(\beta)\cos(\sigma-\alpha)+\sin(\beta)\sin(\sigma-\alpha)\end{aligned}$$

and make use of the Law of Cosines for Angles:

$$-\cos(\gamma)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\cos(c)$$

thus

$$-\sin(\alpha)\sin(\beta)\cos(c)=\sin(\sigma-\alpha)\sin(\beta)+[\cos(\sigma-\alpha)-\cos(\alpha)]\cos(\beta).$$

Alternatively, we have

$$-\sin(\alpha)\cos(c)=\sin(\sigma-\alpha)+(\cos(\sigma-\alpha)-\cos(\alpha))\cot(\beta)$$

hence

$$-\tan(\beta)=\frac{\cos(\sigma-\alpha)-\cos(\alpha)}{\sin(\sigma-\alpha)+\sin(\alpha)\cos(c)}$$

but solving for β turns out to be more complicated than our strategy of solving for c.

6.5. Median Case. Let ρ, θ be defined within triangle ABC as in [4.1]. Applying the Law of Cosines for Sides to both triangles CPB and CPA, we have

$$\cos(a)=\cos(\rho)\cos(c/2)+\sin(\rho)\sin(c/2)\cos(\theta), \tag{1}$$

$$\cos(b)=\cos(\rho)\cos(c/2)-\sin(\rho)\sin(c/2)\cos(\theta) \tag{2}$$

because $\cos(\pi-\theta)=-\cos(\theta)$; hence

$$\begin{aligned}1+\cos(a)+\cos(b)+\cos(c)&=1+2\cos(\rho)\cos(c/2)+\cos(c)\\&=2\cos(\rho)\cos(c/2)+2\cos(c/2)^2\\&=2\cos(c/2)\left(\cos(\rho)+\cos(c/2)\right).\end{aligned}$$

Applying the Law of Sines to both triangles CPB and ABC, we have

$$\frac{\sin(a)}{\sin(\theta)} = \frac{\sin(\rho)}{\sin(\beta)}, \qquad \frac{\sin(b)}{\sin(\beta)} = \frac{\sin(c)}{\sin(\gamma)}$$

hence

$$\begin{aligned} \sin(a)\sin(b)\sin(\gamma) &= \frac{\sin(\rho)\sin(\theta)}{\sin(\beta)}\frac{\sin(\beta)\sin(c)}{\sin(\gamma)}\sin(\gamma) \\ &= \sin(\rho)\sin(\theta)\sin(c) \\ &= 2\sin(\rho)\sin(\theta)\sin(c/2)\cos(c/2). \end{aligned}$$

Eriksson [7] proved that

$$\tan\left(\frac{\sigma}{2}\right) = \frac{\sin(a)\sin(b)\sin(\gamma)}{1+\cos(a)+\cos(b)+\cos(c)}$$

from which

$$\tan\left(\frac{\sigma}{2}\right) = \frac{\sin(c/2)\sin(\rho)\sin(\theta)}{\cos(c/2)+\cos(\rho)} \tag{3}$$

follows immediately.

Adding equation (2) to (1), we obtain [9]

$$\cos(\rho) = \frac{\cos(a)+\cos(b)}{2\cos\left(\frac{c}{2}\right)} = \frac{\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{c}{2}\right)};$$

subtracting equation (2) from (1), we obtain

$$\cos(\theta) = \frac{\cos(a)-\cos(b)}{2\sin\left(\frac{c}{2}\right)\sin(\rho)} = -\frac{\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{c}{2}\right)\sin(\rho)}.$$

Thus, given a, b, c, it is easy to compute ρ and then θ (in that order).

Rearranging equation (3) to

$$\frac{\tan(\sigma/2)}{\sin(\theta)} = \frac{\sin(c/2)\sin(\rho)}{\cos(c/2)+\cos(\rho)} = \frac{\sqrt{1-\cos(c/2)^2}\sqrt{1-\cos(\rho)^2}}{\cos(c/2)+\cos(\rho)},$$

that is,

$$z = \frac{\sqrt{1-y^2}\sqrt{1-x^2}}{y+x}$$

we solve for $x = \cos(\rho)$:

$$(y+x)^2 z^2 = \left(1-y^2\right)\left(1-x^2\right),$$

that is

$$\left(1-y^2+z^2\right)x^2 + \left(2yz^2\right)x - \left(1-y^2-y^2z^2\right) = 0$$

and obtain the expression for $\rho = f(\theta)$.

6.6. Angle Bisector Case. Let ρ, θ be defined within triangle ABC as in [4.2]**.** Applying the Law of Cosines for Angles to both triangles AQC and AQB, we have

$$-\cos(\gamma) = \cos(\theta)\cos(\alpha/2) + \sin(\theta)\sin(\alpha/2)\cos(\rho), \tag{4}$$

$$-\cos(\beta) = -\cos(\theta)\cos(\alpha/2) + \sin(\theta)\sin(\alpha/2)\cos(\rho) \tag{5}$$

because $\cos(\pi-\theta) = -\cos(\theta)$ and $\sin(\pi-\theta) = \sin(\theta)$; hence

$$\begin{aligned}\cos(\alpha)+\cos(\beta)+\cos(\gamma)-1 &= \cos(\alpha) - 2\sin(\theta)\sin(\alpha/2)\cos(\rho) - 1\\ &= -2\sin(\theta)\sin(\alpha/2)\cos(\rho) - 2\sin(\alpha/2)^2\\ &= -2\sin(\alpha/2)\left(\sin(\theta)\cos(\rho)+\sin(\alpha/2)\right).\end{aligned}$$

Applying the Law of Sines to triangle AQB, we have

$$\frac{\sin(c)}{\sin(\theta)} = \frac{\sin(\rho)}{\sin(\beta)}$$

hence

$$\begin{aligned}\sin(\alpha)\sin(\beta)\sin(c) &= \sin(\alpha)\frac{\sin(\rho)\sin(\theta)}{\sin(c)}\sin(c)\\ &= \sin(\alpha)\sin(\rho)\sin(\theta)\\ &= 2\sin(\alpha/2)\cos(\alpha/2)\sin(\rho)\sin(\theta).\end{aligned}$$

The dual of Eriksson's result is [7]

$$\tan\left(\frac{\tau}{2}\right) = \frac{\sin(\alpha)\sin(\beta)\sin(c)}{\cos(\alpha)+\cos(\beta)+\cos(\gamma)-1}$$

from which

$$\tan\left(\frac{\tau}{2}\right) = -\frac{\cos(\alpha/2)\sin(\rho)\sin(\theta)}{\sin(\alpha/2)+\cos(\rho)\sin(\theta)} \tag{6}$$

follows immediately.

Adding equation (5) to (4), we obtain [9]

$$\cos(\rho) \;=\; -\frac{\cos(\beta)+\cos(\gamma)}{2\sin\left(\frac{\alpha}{2}\right)\sin(\theta)} = -\frac{\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)\sin(\theta)};$$

subtracting equation (5) from (4), we obtain

$$\cos(\theta) = \frac{\cos(\beta)-\cos(\gamma)}{2\cos\left(\frac{\alpha}{2}\right)} = -\frac{\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}.$$

Thus, given α, β, γ, it is easy to compute θ and then ρ (in that order).

Rearranging equation (6) to

$$\tan(\tau/2)\sin(\alpha/2) + (\tan(\tau/2)\cos(\rho) + \cos(\alpha/2)\sin(\rho))\sin(\theta) = 0,$$

that is,

$$\sin(\theta) = -\frac{\tan(\tau/2)\sin(\alpha/2)}{\tan(\tau/2)\cos(\rho) + \cos(\alpha/2)\sin(\rho)}$$

we obtain the expression for $\theta = f(\rho)$. The smallest admissible value $\rho = \rho_{\text{thres}}$ occurs when $\sin(\theta) = 1$, that is,

$$\tan(\tau/2)\cos(\rho) + \cos(\alpha/2)\sin(\rho) + \tan(\tau/2)\sin(\alpha/2) = 0.$$

Solving

$$y\,x + z\sqrt{1-x^2} + y\sqrt{1-z^2} = 0$$

is made possible via

$$y^2\left(x + \sqrt{1-z^2}\right)^2 = z^2\left(1-x^2\right),$$

hence

$$\left(y^2+z^2\right)x^2 + 2y^2\sqrt{1-z^2}x + \left(y^2 - z^2 - y^2z^2\right) = 0$$

hence

$$x = \frac{-y^2\sqrt{1-z^2} - z^2\sqrt{1+y^2}}{y^2+z^2} = -\frac{\frac{1}{\sqrt{1+y^2}} + \sqrt{1-z^2}}{1 + \frac{1}{\sqrt{1+y^2}}\sqrt{1-z^2}}$$

hence

$$\cos(\rho) = -\frac{\cos(\tau/2) + \sin(\alpha/2)}{1 + \cos(\tau/2)\sin(\alpha/2)}$$

gives the desired threshold.

7. Definite Integrals

7.1. Crofton/Exhumatus. We wish to evaluate

$$\begin{cases} \displaystyle\int_0^\pi \frac{\pi - \arctan\left(\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}\,\tan\frac{y}{2}\right)}{\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}}\sin x\,dx & \text{if } 0 \le y < \pi, \\ \displaystyle -\int_0^\pi \frac{\arctan\left(\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}\,\tan\frac{y}{2}\right)}{\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}}\sin x\,dx & \text{if } \pi \le y \le 2\pi. \end{cases}$$

A miraculous substitution

$$\cos z = \cos\tfrac{x}{2}\cos\tfrac{y}{2}, \qquad 0 \le z \le \pi$$

is due to Crofton & Exhumatus [2]; from

$$\sin z\, dz = \tfrac{1}{2}\sin\tfrac{x}{2}\cos\tfrac{y}{2}\, dx$$

we deduce that

$$\sin x\, dx = 2\sin\tfrac{x}{2}\cos\tfrac{x}{2}\frac{\sin z\, dz}{\frac{1}{2}\sin\frac{x}{2}\cos\frac{y}{2}} = 4\frac{\cos z}{\cos\frac{y}{2}}\frac{\sin z\, dz}{\cos\frac{y}{2}} = \frac{4\cos z\sin z}{\cos^2\frac{y}{2}}dz$$

and

$$\tan^2\tfrac{x}{2}\csc^2\tfrac{y}{2} + 1 = \frac{\tan^2 z}{\tan^2\frac{y}{2}}$$

because

$$\begin{aligned}
\tan^2 z + 1 &= \sec^2 z = \sec^2\tfrac{x}{2}\sec^2\tfrac{y}{2} = \left(\tan^2\tfrac{x}{2} + 1\right)\sec^2\tfrac{y}{2} \\
&= \tan^2\tfrac{x}{2}\sec^2\tfrac{y}{2} + \left(\tan^2\tfrac{y}{2} + 1\right) \\
&= \left(\tan^2\tfrac{x}{2}\sec^2\tfrac{y}{2} + \tan^2\tfrac{y}{2}\right) + 1 \\
&= \left(\tan^2\tfrac{x}{2}\csc^2\tfrac{y}{2} + 1\right)\tan^2\tfrac{y}{2} + 1.
\end{aligned}$$

Since $\cos(z)$, $\cos(y/2)$ obviously have the same sign, it follows that $\tan(z)$, $\tan(y/2)$ likewise have the same sign and

$$\tan z = \sqrt{\tan^2\tfrac{x}{2}\csc^2\tfrac{y}{2} + 1}\,\tan\tfrac{y}{2}.$$

If $0 \le y < \pi$, clearly $\cos(y/2) > 0$ and the range $0 \le x \le \pi$ maps to $y/2 \le z \le \pi/2$. Also, $\tan(y/2) > 0$, hence

$$z = \arctan\left(\sqrt{\tan^2\tfrac{x}{2}\csc^2\tfrac{y}{2} + 1}\,\tan\tfrac{y}{2}\right)$$

hence

$$\begin{aligned}
\frac{\pi - \arctan\left(\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2} + 1}\,\tan\frac{y}{2}\right)}{\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2} + 1}}\sin x\, dx &= \frac{\tan\frac{y}{2}}{\tan z}(\pi - z)\frac{4\cos z\sin z}{\cos^2\frac{y}{2}}dz \\
&= \frac{4\tan\frac{y}{2}}{\cos^2\frac{y}{2}}(\pi - z)\cos^2 z\, dz.
\end{aligned}$$

The required definite integral is thus

$$\frac{4\tan\frac{y}{2}}{\cos^2\frac{y}{2}}\int_{y/2}^{\pi/2}(\pi - z)\cos^2 z\, dz$$

which is elementary.

If $\pi < y \leq 2\pi$, clearly $\cos(y/2) < 0$ and the range $0 \leq x \leq \pi$ maps to $y/2 \geq z \geq \pi/2$. Also, $\tan(y/2) < 0$, hence

$$z = \pi + \arctan\left(\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}\,\tan\frac{y}{2}\right)$$

hence

$$\begin{aligned}-\frac{\arctan\left(\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}\,\tan\frac{y}{2}\right)}{\sqrt{\tan^2\frac{x}{2}\csc^2\frac{y}{2}+1}}\sin x\,dx &= -\frac{\tan\frac{y}{2}}{\tan z}(z-\pi)\frac{4\cos z\sin z}{\cos^2\frac{y}{2}}dz \\ &= \frac{4\tan\frac{y}{2}}{\cos^2\frac{y}{2}}(\pi - z)\cos^2 z\,dz.\end{aligned}$$

The required definite integral is thus identical to before (although here the lower limit $y/2$ is greater than the upper limit $\pi/2$).

Computer algebra swiftly gives

$$\begin{aligned}&1 + \frac{d}{dy}\left[\frac{4\tan\frac{y}{2}}{\cos^2\frac{y}{2}}\int\limits_{y/2}^{\pi/2}(\pi - z)\cos^2 z\,dz\right] \\ = &-\frac{(y^2-4\pi y+3\pi^2-6)\cos(y) - 6(y-2\pi)\sin(y) - 2(y^2-4\pi y+3\pi^2+3)}{8\cos(y/2)^4}\end{aligned}$$

and this is useful at the conclusion of [3.1].

7.2. Jones/Benyon-Tinker. Combining our results with those in [4, 5], we have

$$\int\limits_{\tau/2-\kappa}^{\tau/2}\frac{\sin(\tau-\kappa-\rho)\sin(\kappa)\sin(\rho)}{\sqrt{\sin(\kappa)^2\sin(\rho)^2 - [\cos(\kappa)\cos(\rho)-\cos(\tau-\kappa-\rho)]^2}}d\rho \qquad (7)$$

$$= \frac{E\left(\sin\left(\frac{\kappa}{2}\right)\right) - \cos\left(\frac{\tau-\kappa}{2}\right)^2 K\left(\sin\left(\frac{\kappa}{2}\right)\right)}{\sqrt{\cos\left(\frac{\kappa}{2}\right)^2 - \cos\left(\frac{\tau-\kappa}{2}\right)^2}}\sin(\kappa)$$

in connection with primal perimeter [2.3] and

$$-\int\limits_{\sigma/2}^{\pi-(\kappa-\sigma/2)}\frac{\sin(\sigma-\kappa-\theta)\sin(\kappa)\sin(\theta)}{\sqrt{\sin(\kappa)^2\sin(\theta)^2 - [\cos(\kappa)\cos(\theta)-\cos(\sigma-\kappa-\theta)]^2}}d\theta \qquad (8)$$

$$= \frac{E\left(\cos\left(\frac{\kappa}{2}\right)\right) - \sin\left(\frac{\sigma-\kappa}{2}\right)^2 K\left(\cos\left(\frac{\kappa}{2}\right)\right)}{\sqrt{\sin\left(\frac{\kappa}{2}\right)^2 - \sin\left(\frac{\sigma-\kappa}{2}\right)^2}}\sin(\kappa)$$

in connection with dual area [2.4]. A direct symbolic proof of these formulas is not known [3.2].

Consider the problem of integrating equation (7) with respect to κ, $0 \le \kappa \le \tau/2$ and of integrating equation (8) with respect to κ, $\sigma/2 \le \kappa \le \pi$. In (7), ρ is integrated out first, κ second. In (8), θ is integrated out first, κ second. By symmetry, we gain nothing by integrating out κ first, thus a closed-form expression for unconditional density would seem unlikely. Another miraculous change of variables might, however, be brought into play. Other approaches based on other coordinate systems exist [5.1, 5.2]. It is still too early to rule out the possibility of a breakthrough here.

8. Acknowledgement

We are grateful to M. Larry Glasser for a helpful discussion about integrals at the end of [2.3] & [2.4]. Much more relevant material can be found at [10, 11], including experimental computer runs that aided theoretical discussion here. The book [5] studies length distributions for open and closed random n-step tours ($n \ge 3$) on spheres, thus generalizing our discussion of triangle perimeters considerably.

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