# A secret sharing scheme using groups

Dimitrios Panagopoulos
Presefonis 3 N. Iraklio Attikis 14121 Greece
dpanagop@yahoo.com

24 August 2010

**Abstract**

In this paper a secret sharing scheme based on the word problem in groups is introduced. The security of the scheme and possible variations are discussed in section 2. The article concludes with the suggestion of two categories of platform groups for the implementation of the scheme.

## 1 Introduction

The problem of distributing a secret among a group of n persons in such a way that it can be reconstructed only if at least t of them combine their shares was solved independently by A. Shamir [5] and G. Blakley [1] in 1979. During the recent years several cryptographic methods used group theoretic machinery (see e.g. [4]). In the present article, combining these two fields, we use group presentations and the word problem in groups in order to develop a new secret sharing scheme. It's main advantage to the schemes mentioned before is that it does not require the secret message to be determined before each individual person receives his share of the secret.

In the section following the introduction the scheme is introduced. The article ends with a general discussion about the scheme and some suggestions concerning the platform groups which could be used for its implementation.

## 2 The scheme

Suppose that a binary sequence must be distributed among n persons in such a way that at least t of them must cooperate in order to obtain the whole sequence. The secret sharing scheme consists of the following steps:

Step 1 A group G with finite presentation $G =< x_1, x_2, \ldots, x_k / r_1, \ldots, r_m >$ and soluble word problem is chosen. We require that $m = \begin{pmatrix} n \\ t-1 \end{pmatrix}$.

Step 2 Let $A_1, \ldots, A_m$ be an enumeration of the subsets of $\{1, \ldots, n\}$ with t-1 elements. Define n subsets of $\{r_1, \ldots, r_n\}$, $R_1, \ldots, R_n$ with $r_j \in R_i$ if and only if $i \notin A_j$, $j = 1, \ldots, m$, $i = 1, \ldots, n$.

Then for every $j = 1, \ldots, m$, $r_j$ is not contained in exactly t-1 of the subsets $R_1, \ldots, R_n$. It follows that $r_j$ is contained in any union of t of them whereas if we take any t-1 of the $R_1, \ldots, R_n$ there exists a j such that $r_j$ is not contained in their union.

Step 3 Distribute to each of the n persons one of the sets $R_1, \ldots, R_n$. The set $\{x_1, \ldots, x_k\}$ is known to all of them.

Step 4 If the binary sequence to be distributed is $a_1 \cdots a_l$ construct and distribute a sequence of elements $w_1, \ldots, w_l$ of G such that $w_i =_G 1$ if and only if $a_i = 1$, $i = 1, \ldots, l$. The word $w_i$ must involve most of the relations $r_1, \ldots, r_m$ if $w_i = 1$. Furthermore, all of the relations must be used at some point in the construction of some element.

Any t of the n persons can obtain the sequence $a_1 \cdots a_l$ by taking the union of the subsets of the relations of G that they possess and thus obtaining the presentation $G =< x_1, x_2, \ldots, x_k / r_1, r_2, \ldots, r_m >$ and solving the word problem $w_i =_G 1$ in G for $i = 1, \ldots, l$.

A coalition of fewer than t persons cannot decode correctly the message since the union of fewer than t of the sets $R_1, \ldots, R_n$ contains some but not all of the relations $r_1, \ldots, r_m$. Thus such a coalition could obtain a group presentation $G' =< x_1, \ldots, x_k / r'_1, \ldots, r'_p >$ with $p < m$ and $G \neq G'$, where $w_i =_G 1$ is not equivalent to $w_i =_{G'} 1$ in general.

## 3 Remarks and implementation

It should be pointed out that, contrary to other schemes (e.g. Shamir's, Blakley's scheme), the secret sequence to be shared is not needed until the final step. It is possible for someone to distribute the sets $R_1, \ldots, R_n$ and decide at a later time what the sequence would be. In that way the scheme can also be used so that t of the n persons can verify the authenticity of the message. In particular the binary sequence in step 4 could contain a predetermined subsequence (signature) along with the normal message. Then t persons may check whether this predetermined sequence is contained in the encoded message thus validating it. One word of caution though, such a use might make possible for less than t persons (or even a third party) to discover all of the relations $r_1, \ldots, r_m$. This can be made more difficult by not specifying where exactly the signature should appear.

One method of attack to this system is to search the pool of possible presentations of groups $G =< x_1, x_2, \ldots, x_k / r_1, \ldots, r_m >$ that are used in the first step and try to decode the transmitted message $w_1, \ldots w_l$. This task is easier if the attacker has some information concerning the encoded message (e.g. the attacker may knows that a certain block of the message contains a specific binary sequence/singature as discussed in the previous paragraph). Thus, this pool must contain a large number of groups. The reader may consult [4, 6.1.5] for further discussion on the efficiency of this type of attack.

The above line of attack is expedited if the attacker possesses some of the sets $R_1, \ldots, R_n$ (e.g. he might be one of the n persons sharing the secret). For this the reason we require in step 4 that a word $w$ encoding 1 must involve most of the relations. Because if someone possesses the relations $r'_1, \ldots, r'_p$ and only them are involved in a word $w =_G 1$ then he may decode correctly the word since $w =_{G'} 1$ for the group $G' =< x_1, \ldots, x_k / r'_1, \ldots, r'_p >$.

One way of creating a word representing 1 is by the product

$$\prod_j^l [r'_j, w_j]$$

where $r'_j$ is a relation, $w_j$ a random element, $l$ a (large) natural number and $[a, b] = aba^{-1}b^{-1}$ is the commutator of a and b. This kind of encoding might, also, render useless some of the quotient attacks [4, 6.1.6]. A larger set of relations in step 1 should make these attacks more difficult to use. One the other hand, the fact that by using only the relationships contained in $R_j$ for a word $w$ the person with this set can decode correctly the word, may be used to send messages to a specific person secretly from the rest of the group.

Finally we propose some categories of group presentations which could be used in step 1:

**Polycyclic groups:** polycyclic groups with presentation

$$< x_1, \ldots, x_k / x_j^{a_i} = w_{ij},\, a_j^{a_i^{-1}} = v_{ij},\, a_l^{r_l} = u_l \text{ for } 1 \le i < j \le k,\, l \in I >$$

where $I \subseteq \{1, \ldots, k\}$, $r_l \in \mathbb{N}$ for all $l \in I$, $w_{ij}, v_{ij}, u_j$ are words in $a_{j+1}, \ldots, a_k$ and $x^y = y^{-1}xy$. The interested reader may consult [3] for a discussion on the use of polycyclic groups.

**Coxeter groups:** Coxeter groups with presentation

$$< x_1, \ldots, x_k / \, (s_i s_j)^{m_{ij}} = 1, \, i, j = 1, \ldots, k >$$

where $m_{ij} \in \mathbb{N} \cup \{+\infty\}$, $m_{ij} \neq 0$, $m_{ii} = 1$. There exists extensive bibliography on Coxeter groups. A place to start is [2]. In there there is reference on the word problem in Coxeter groups.

## References


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