THE RCCN CRITERION OF SEPARABILITY FOR STATES IN INFINITE-DIMENSIONAL QUANTUM SYSTEMS

YU GUO AND JINCHUAN HOU

Abstract. In this paper, the realignment criterion and the RCCN criterion of separability for states in infinite-dimensional bipartite quantum systems are established. Let H_A and H_B be complex Hilbert spaces with $\dim H_A \otimes H_B = +\infty$. Let ρ be a state on $H_A \otimes H_B$ and $\{\delta_k\}$ be the Schmidt coefficients of ρ as a vector in the Hilbert space $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$. We introduce the realignment operation ρ^R and the computable cross norm $\|\rho\|_{\rm CCN}$ of ρ and show that, if ρ is separable, then $\|\rho^R\|_{\rm Tr} = \|\rho\|_{\rm CCN} = \sum\limits_k \delta_k \leq 1$. In particular, if ρ is a pure state, then ρ is separable if and only if $\|\rho^R\|_{\rm Tr} = \|\rho\|_{\rm CCN} = \sum\limits_k \delta_k = 1$.

1. Introduction

The quantum entanglement is one of the most striking features of the quantum mechanics and it is used as a physical resource for communication information processing [1]. Consequently, the detection of entanglement, that is, distinguishing separable and entangled states, has been investigated extensively [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. However, in spite of the considerable effort, no necessary-sufficient criterion that is practically implementable is known so far even though in finite-dimensional bipartite quantum systems. The case of infinite-dimensional systems can't be neglected since they do exist in the quantum world [14, 15]. Therefore, how to recognize the separability of states in infinite-dimensional systems is a more difficult problem that is of both fundamental and practical importance within quantum mechanics and quantum information theory.

It is known that, a density operator ρ (i.e., a positive trace-one operator) acting on a separable Hilbert space $H = H_A \otimes H_B$ describing the state of two quantum systems A and B, is called *separable* if it can be written as a convex combination

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1, \ p_i \geq 0 \tag{1}$$

or can be approximated in the trace norm by the states of the above form [16, 17], where ρ_i^A and ρ_i^B are (pure) states in the subsystems A and B which are described by the complex Hilbert spaces H_A and H_B, respectively. Otherwise, ρ is called *entangled*. Let $\mathcal{S}_{s-p}$ be the set of all

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separable pure states. It is shown in [18] that, any separable state ρ admits a representation of the Bochner integral

$$\rho = \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B) d\mu(\rho^A \otimes \rho^B), \tag{2}$$

where μ is a Borel probability measure on $\mathcal{S}_{s-p}$, $\rho^A \otimes \rho^B \in \mathcal{S}_{s-p}$ and $\varphi : \mathcal{S}_{s-p} \to \mathcal{S}_{s-p}$ is a measurable function. Particularly, if $\dim H_A \otimes H_B < +\infty$, then a state ρ acting on $H_A \otimes H_B$ is separable if and only if ρ can be written as [16]

$$\rho = \sum_{i=1}^{n} p_i \rho_i^A \otimes \rho_i^B, \tag{3}$$

where ρ_i^A and ρ_i^B are pure states in the subsystems A and B, respectively, and where $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $n \leq (\dim H_A \otimes H_B)^2$. In the infinite-dimensional case, there exists separable state that can not be written in the form $\sum_{i=1}^{+\infty} p_i \rho_i^A \otimes \rho_i^B$ with $\sum_{i=1}^{+\infty} p_i = 1$ [18].

For the finite-dimensional bipartite quantum systems, K. Chen and L.-A. Wu proposed the realignment criterion in [2], which reads as: if ρ is a separable state of the bipartite quantum system, then the trace norm of the realignment matrix of ρ is not larger than 1. A short later, O.Rudolph proved in [11] that if ρ is a state of the bipartite quantum system, then the computable cross norm of ρ equals the trace norm of the realignment matrix of ρ. This result, combining the result in [2], is called the *realignment criterion or computable cross norm criterion* (or RCCN criterion briefly) [2, 10, 11]. Then, a natural problem is arisen: whether or not there is a counterpart result for the infinite-dimensional bipartite quantum systems? We find that the answer is 'yes'. The aim of the present paper is to establish the realignment criterion and the RCCN criterion for the infinite-dimensional bipartite quantum systems.

The paper is organized as follows. In section 2, we summarize the studies on the realignment criterion and the RCCN criterion for finite-dimensional bipartite quantum systems, which enlightens the way how to generalize the conception of realignment to the infinite-dimensional case. Section 3 devotes to generalizing the notion of the realignment operation to the infinite-dimensional systems, and presenting the realignment criterion and the RCCN criterion for infinite-dimensional bipartite quantum systems. Let H_A and H_B be Hilbert spaces. We introduce three equivalent definitions of the realignment operation from the Hilbert-Schmidt class $C_2(H_A \otimes H_B)$ into $C_2(H_B \otimes H_B, H_A \otimes H_A)$ and reveal that the realignment operation

$$T \mapsto T^R \tag{4}$$

is an isometry with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$. Let ρ be a state on $H_A \otimes H_B$ and $\{\delta_k\}$ be the Schmidt coefficients of ρ regarded as a vector in the Hilbert space $C_2(H_A) \otimes C_2(H_B)$. We show that, if ρ is separable, then $\|\rho^R\|_{\mathrm{Tr}} = \|\rho\|_{\mathrm{CCN}} = \sum_k \delta_k \leq 1$. In particular, if ρ is a pure state, then ρ is separable if and only if $\|\rho^R\|_{\mathrm{Tr}} = \|\rho\|_{\mathrm{CCN}} = \sum_k \delta_k = 1$ (Criteria 3.3 and 3.7). Thus $\|\rho^R\|_{\mathrm{Tr}} = \|\rho\|_{\mathrm{CCN}} = \sum_k \delta_k > 1$ signals the entanglement of ρ. The RCCN criterion just provides a necessary condition for separability [11]. However, the RCCN criterion can detect many states with positive partial transpose (PPT) [10, 11], i.e., the so-called PPT states (which are bound entangled states). Several examples are given to illustrate the relations between the RCCN criterion and the PPT criterion. They show that the infinite-dimensional RCCN criterion can also detect some PPT states as desired (see Examples 3.8-3.10). A final conclusion is included in the last section.

We fix some notations. Throughout the paper we use the Dirac's symbols. $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ stand for the set of all real numbers, the set of all complex numbers and the set of all nonnegative integers, respectively. The bra-ket notation, $\langle\cdot|\cdot\rangle$ stands for the inner product in the given Hilbert spaces, i.e., $H_A \otimes H_B$, H_A, or H_B. The set of all bounded linear operators on some Hilbert space H is denoted by $\mathcal{B}(H)$, the set of trace class operators on H is denoted by $\mathcal{T}(H)$ and the set of all Hilbert-Schmidt class operators on H is denoted by $\mathcal{C}_2(H)$. $A \in \mathcal{B}(H)$ is self-adjoint if $A^\dagger = A$ ($A^\dagger$ stands for the adjoint operator of A); A is said to be positive, denoted by $A \geq 0$, if $A^\dagger = A$ and $\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle \in H$. A^T stands for the transpose of the operator A, $\|\cdot\|_{\mathrm{Tr}}$ denotes the trace norm and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, i.e., $\|A\|_{\mathrm{Tr}} = \mathrm{Tr}((A^\dagger A)^{\frac{1}{2}})$ and $\|A\|_2 = (\mathrm{Tr}(A^\dagger A))^{\frac{1}{2}}$. By $\mathcal{S}(H_A)$, $\mathcal{S}(H_B)$ and $\mathcal{S}(H_A \otimes H_B)$ we denote the sets of states on H_A, H_B and $H_A \otimes H_B$, respectively. By $\mathcal{S}_{sep}$ we denote the set of all separable states in $\mathcal{S}(H_A \otimes H_B)$. A state ρ is called a pure state if $\mathrm{Tr}(\rho^2) = 1$ and is called a mixed state if $\mathrm{Tr}(\rho^2) < 1$ as usual. We also call a unit vector $|\psi\rangle \in H_A \otimes H_B$ a pure state which is corresponding to the density operator $\rho = |\psi\rangle\langle\psi|$. We fix in the 'local Hilbert space' H_A, H_B orthonormal bases $\{|m\rangle\}_{m=1}^{N_A}$ and $\{|\mu\rangle\}_{\mu=1}^{N_B}$, where $N_A = \dim H_A$ and $N_B = \dim H_B$, respectively (note that we use Latin indices for the subsystem A and the Greek indices for the subsystem B. Also, N_A and N_B may be $+\infty$). Then, a vector $|\psi\rangle \in H_A \otimes H_B$ can be written as $|\psi\rangle = \sum_{m,\mu} d_{m\mu}|m\rangle|\mu\rangle \in H_A \otimes H_B$. Let $D_\psi = (d_{m\mu})$ (or $[d_{m\mu}]$) be the coefficient operator of $|\psi\rangle$. Remark that $D_\psi = (d_{m\mu})$ can be regarded as an operator from H_B into H_A and it is a Hilbert-Schmidt class operator with the Hilbert-Schmidt norm $\|D_\psi\|_2 = \||\psi\rangle\|$. We write $\overline{D} = (\overline{d_{m\mu}})$, where $\overline{d_{m\mu}}$ is the complex conjugation of $d_{m\mu}$. The partial transpose of $\rho \in \mathcal{S}(H_A \otimes H_B)$ with respect to the subsystem B (resp. A) is denoted by ρ^{T_B} (resp. ρ^{T_A}), that is, $\rho^{T_B} = (I \otimes \mathbf{T})\rho$ (resp. $\rho^{T_A} = (\mathbf{T} \otimes I)\rho$), where $\mathbf{T}$ is the map of taking transpose with respect to the given orthonormal basis.

2. The RCCN criterion for finite-dimensional systems

To find a way of generalizing the notion of the realignment of a block matrix to that of an operator matrix acting on an infinite-dimensional Hilbert space, in this section, we summarize some known facts about the realignment criterion and the related RCCN criterion for finite-dimensional bipartite quantum systems in references [2, 11, 19, 20, 21] and discuss them briefly. Assume that $\dim H_A = N_A$ and $\dim H_B = N_B$ are finite throughout this section.

Firstly, we recall the definition of the realignment operation for the $N_AN_B\times N_AN_B$ matrices, i.e., the $N_A\times N_A$ block matrices with each block is of size $N_B\times N_B$. Recalling that, for a $N_A\times N_A$ block matrix $T = (B_{ij})_{N_A\times N_A}$ with each block B_{ij} of the size $N_B \times N_B$, $1 \le i,j \le N_A$, the row realignment matrix T^R of T is defined as

$$\begin{aligned} T^R = \ & [(\mathrm{vec}(B_{11}))^T, \dots, (\mathrm{vec}(B_{1N_A}))^T, \dots, \\ & (\mathrm{vec}(B_{N_A1}))^T, \dots, (\mathrm{vec}(B_{N_AN_A}))^T]^T, \end{aligned} \tag{5}$$

which is a $N_A^2 \times N_B^2$ matrix, where for a given $X = [x_{ij}]$ with $1 \le i \le s$ and $1 \le j \le t$, $\mathrm{vec}(X)$ is defined by

$$\mathrm{vec}(X) = [x_{11}, \dots, x_{1t}, x_{21}, \dots, x_{2t}, \dots, x_{s1}, \dots, x_{st}].$$

For example, in the case of a two-qubit system, let

$$\rho = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right),$$

where B_{ij}s are operators on the space associated with the second system. Then the row realignment matrix of ρ (ref. [20]) is

$$\rho^R = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{21} & \rho_{22} \\ \rho_{13} & \rho_{14} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{41} & \rho_{42} \\ \rho_{33} & \rho_{34} & \rho_{43} & \rho_{44} \end{pmatrix}.$$

It is clear that the realignment operation $T \mapsto T^R$ is a linear map, that is, $(\alpha T + \beta S)^R = \alpha T^R + \beta S^R$, $\alpha, \beta \in \mathbb{C}$.

The so-called realignment criterion due to Chen and Wu [2] is the following

The realignment criterion for finite-dimensional bipartite systems. *Assume that H_A and H_B are of finite-dimensions and $\rho \in \mathcal{S}(H_A \otimes H_B)$ is a state. If ρ is separable, then* $\|\rho^R\|_{\mathrm{Tr}} \le 1$.

The realignment criterion presents a quite strong necessary condition for separability which is easily performed and independent to the well-known PPT criterion. However, the above definition of the realignment operation cannot be generalized to the infinite-dimensional cases. Fortunately, there are several different definitions of the realignment operation that are equivalent to each other. This allows us to find ways of generalizing the realignment operation to infinite-dimensional cases.

With respect to a fixed product basis $\{|m\rangle|\mu\rangle\}$ of $H_A \otimes H_B = \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$, every operator $A \in \mathcal{B}(H_A \otimes H_B)$ can be written in the form $A = [a_{m\mu,n\nu}]$, where the entry $a_{m\mu,n\nu} = \langle m|\langle\mu|A|n\rangle|\nu\rangle$, the double indices $(m\mu) \leftrightarrow (m-1)N_B + \mu$ and $(n\nu) \leftrightarrow (n-1)N_B + \nu$ refer respectively to rows and columns of matrix A. Then we have [20]

$$A^R = [\tilde{a}_{mn,\mu\nu}],\ \tilde{a}_{mn,\mu\nu} = a_{m\mu,n\nu}, \tag{6}$$

where the double indices $(mn) \leftrightarrow (m-1)N_A + n$ and $(\mu\nu) \leftrightarrow (\mu-1)N_B + \nu$ refer respectively to rows and columns of matrix A^R. For the above example ρ in the case of a two-qubit system, using the double indices, we may write

$$\rho = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{11,21} & \rho_{11,22} \\ \rho_{12,11} & \rho_{12,12} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{21,21} & \rho_{21,22} \\ \rho_{22,11} & \rho_{22,12} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}$$

and then

$$\rho^R = \begin{pmatrix} \tilde{\rho}_{11,11} & \tilde{\rho}_{11,12} & \tilde{\rho}_{11,21} & \tilde{\rho}_{11,22} \\ \tilde{\rho}_{12,11} & \tilde{\rho}_{12,12} & \tilde{\rho}_{12,21} & \tilde{\rho}_{12,22} \\ \tilde{\rho}_{21,11} & \tilde{\rho}_{21,12} & \tilde{\rho}_{21,21} & \tilde{\rho}_{21,22} \\ \tilde{\rho}_{22,11} & \tilde{\rho}_{22,12} & \tilde{\rho}_{22,21} & \tilde{\rho}_{22,22} \end{pmatrix} = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{12,11} & \rho_{12,12} \\ \rho_{11,21} & \rho_{11,22} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{22,11} & \rho_{22,12} \\ \rho_{21,21} & \rho_{21,22} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}.$$

The operation of realignment can also be defined in another alternative way [11]. For a $N_A \times N_A$ matrix $A = [a_{mn}] \in \mathcal{B}(H_A)$ (resp. $N_B \times N_B$ matrix $B = [b_{\mu\nu}] \in \mathcal{B}(H_B)$) in terms of the basis $\{|m\rangle\}$ (resp. $\{|\mu\rangle\}$), regard A (resp. B) as a vector $|A\rangle = \sum_{m,n} a_{mn}|m\rangle|n\rangle$ in $\mathbb{C}^{N_A^2}$ (resp. $|B\rangle = \sum_{\mu,\nu} b_{\mu\nu}|\mu\rangle|\nu\rangle$ in $\mathbb{C}^{N_B^2}$). If $\rho = \sum_{k=1}^{s} A_k \otimes B_k \in \mathcal{S}(H_A \otimes H_B)$, then

$$\rho^R = \sum_{k=1}^{s} |A_k\rangle\langle B_k|, \tag{7}$$

where $\langle B_k|$ *denotes the transpose of* $|B_k\rangle$ (not the conjugate transpose as usual), $k = 1, 2, \ldots, s$. In particular, for any pure state $\rho_\psi = |\psi\rangle\langle\psi|$, write $|\psi\rangle = \sum_{m,\mu} d_{m\mu}|m\rangle|\mu\rangle$ and $D = [d_{m\mu}]$, then by [20],

$$\rho_\psi^R = D \otimes \overline{D}. \tag{8}$$

It follows that, for any mixed state $\rho = \sum_{i=1}^{t} p_i\rho_i$, where $p_i \geq 0$, $\sum_{i=1}^{t} p_i = 1$ and ρ_i are pure states of the bipartite system, $i = 1, 2, \ldots, t$, ρ^R can be defined to be

$$\rho^R = \sum_{i=1}^{t} p_i\rho_i^R = \sum_{i=1}^{t} p_i D_i \otimes \overline{D}_i, \tag{9}$$

where $\rho_i = |\psi_i\rangle\langle\psi_i|$ with $|\psi_i\rangle = \sum d_{m\mu}^{(i)}|m\rangle|\mu\rangle$ and $D_i = [d_{m\mu}^{(i)}]$ is the coefficient matrix of $|\psi_i\rangle$ [20].

Similarly, the column realignment matrix of ρ, denoted by ρ^{R^c}, was defined in [2]. For the two-qubit state ρ mentioned above, we have

$$\rho^{R^c}=\begin{pmatrix}\rho_{11}&\rho_{21}&\rho_{12}&\rho_{22}\\ \rho_{31}&\rho_{41}&\rho_{32}&\rho_{42}\\ \rho_{13}&\rho_{23}&\rho_{14}&\rho_{24}\\ \rho_{33}&\rho_{43}&\rho_{34}&\rho_{44}\end{pmatrix}.$$

It is easy to check that,

$$\rho^{R^c}=[\tilde{\rho}_{\nu\mu,nm}]^T,\ \tilde{\rho}_{\nu\mu,nm}=\rho_{m\mu,n\nu}; \tag{10}$$

if $\rho=\sum\limits_k A_k\otimes B_k$ with $A_k=\sum\limits_{m,n}a_{mn}^{(k)}|m\rangle\langle n|$ and $B_k=\sum\limits_{\mu,\nu}b_{\mu\nu}^{(k)}|\mu\rangle\langle\nu|$, then

$$\rho^{R^c}=\sum_{k=1}^{s}|\tilde{A}_k\rangle\langle\tilde{B}_k|, \tag{11}$$

where $|\tilde{A}_k\rangle=\sum\limits_{m,n}a_{mn}^{(k)}|n\rangle|m\rangle$ and $|\tilde{B}_k\rangle=\sum\limits_{\mu,\nu}b_{\mu\nu}^{(k)}|\nu\rangle|\mu\rangle$, $k=1,2,\dots,s$; if $\rho=\sum\limits_{i=1}^{t}p_i\rho_i$, then

$$\rho^{R^c}=\sum_{i=1}^{t}p_i\rho_i^{R'}=\sum_{i=1}^{t}p_iD_i^T\otimes\bar{D}_i^T, \tag{12}$$

where ρ_i, p_i and D_i defined as in Eq.(9). For instance, using the double indices, the column realignment matrix of the example ρ mentioned above is

$$\rho^{R^c}=\begin{pmatrix}\tilde{\rho}_{11,11}&\tilde{\rho}_{11,12}&\tilde{\rho}_{11,21}&\tilde{\rho}_{11,22}\\ \tilde{\rho}_{12,11}&\tilde{\rho}_{12,12}&\tilde{\rho}_{12,21}&\tilde{\rho}_{12,22}\\ \tilde{\rho}_{21,11}&\tilde{\rho}_{21,12}&\tilde{\rho}_{21,21}&\tilde{\rho}_{21,22}\\ \tilde{\rho}_{22,11}&\tilde{\rho}_{22,12}&\tilde{\rho}_{22,21}&\tilde{\rho}_{22,22}\end{pmatrix}^T=\begin{pmatrix}\rho_{11,11}&\rho_{12,11}&\rho_{11,12}&\rho_{12,12}\\ \rho_{21,11}&\rho_{22,11}&\rho_{21,12}&\rho_{22,12}\\ \rho_{11,21}&\rho_{12,21}&\rho_{11,22}&\rho_{12,22}\\ \rho_{21,21}&\rho_{22,21}&\rho_{21,22}&\rho_{22,22}\end{pmatrix}.$$

The singular values of ρ^R and ρ^{R^c} are equal [21]. In fact, let $F_A=\sum\limits_{m,n=1}^{N_A}|m\rangle\langle n|\otimes|n\rangle\langle m|$ and $F_B=\sum\limits_{\mu,\nu=1}^{N_B}|\mu\rangle\langle\nu|\otimes|\nu\rangle\langle\mu|$; then F_A (resp. F_B) is a unitary matrix of size $N_A^2\times N_A^2$ (resp. $N_B^2\times N_B^2$). F_A and F_B are the so-called swap operators or the flip operators [22]. It is easily checked that $F_A|A_k\rangle=|\tilde{A}_k\rangle$ and $F_B|B_k\rangle=|\tilde{B}_k\rangle$, $k=1,2,\dots$. It turns out that

$$\rho^R=F_A\rho^{R^c}F_B. \tag{13}$$

Therefore, we need to consider the row realignment only.

In the following, the realignment of a matrix always refers to the row realignment of the matrix unless specified.

Note that, for any state $\rho\in\mathcal{S}(H_A\otimes H_B)$, one has

$$\|\rho^R\|_2=\|\rho\|_2\le\|\rho\|_{\mathrm{Tr}}=1. \tag{14}$$

For any $C \in \mathcal{B}(H_A \otimes H_B)$, the *computable cross norm* of C, $||C||_{\mathrm{CCN}}$, is defined by:

$$\begin{aligned} ||C||_{\mathrm{CCN}} := \quad & \inf\{\sum_{k=1}^{s} ||A_k||_2 ||B_k||_2 : C = \sum_{k=1}^{s} A_k \otimes B_k, \\ & A_k \in \mathcal{B}(H_A), B_k \in \mathcal{B}(H_B)\}, \end{aligned} \tag{15}$$

where the infimum is taken over all finite decompositions of C into a finite sum of simple tensors [11].

Notice that the linear space $\mathcal{B}(H_A \otimes H_B)$ can be considered as a Hilbert space if it is equipped with the (complex) Hilbert-Schmidt scalar product:

$$\langle A|B\rangle := \mathrm{Tr}(A^\dagger B), A, B \in \mathcal{B}(H_A \otimes H_B),$$

the Hilbert-Schmidt norm, $|| \cdot ||_2$, reads as

$$||A||_2 := (\mathrm{Tr}(A^\dagger A))^{1/2}.$$

Then, every $\rho \in \mathcal{S}(H_A \otimes H_B)$ can be regarded as a 'vector' in the Hilbert space $\mathcal{B}(H_A \otimes H_B)$ equipped with the Hilbert-Schmidt inner product. It follows that there is a Schmidt decomposition of ρ:

$$\rho = \sum_{k=1}^{r} \delta_k E_k \otimes F_k,$$

where the coefficients $\{\delta_k\}$ are positive, $\{E_k\}$, $\{F_k\}$ are orthonormal sets of Hilbert spaces $\mathcal{B}(H_A)$, $\mathcal{B}(H_B)$, respectively, and r is the Schmidt number of ρ. The set of the positive numbers $\{\delta_k\}$ is uniquely determined by the corresponding vector ρ, and they are called the Schmidt coefficients of ρ [21].

It is showed in [11, 20, 21] that

$$||\rho||_{\mathrm{CCN}} = ||\rho^R||_{\mathrm{Tr}} = \sum_{k=1}^{r} \delta_k, \tag{16}$$

where δ_k is the Schmidt coefficients of ρ. From this point of view, this cross norm $||\rho||_{\mathrm{CCN}}$ of ρ is called computable cross norm of ρ, and the following criterion is called *the realignment or computable cross norm criterion* (the RCCN criterion for short) due to [11, 19, 20].

The RCCN criterion for finite-dimensional bipartite quantum systems *Let H_A and H_B be finite-dimensional Hilbert spaces and $\rho \in \mathcal{S}(H_A \otimes H_B)$ be a state. Let $\rho = \sum_{k=1}^{r} \delta_k E_k \otimes F_k$ be the Schmidt decomposition of ρ as a vector of $C_2(H_A) \otimes C_2(H_B)$. If ρ is separable, then*

$$||\rho||_{\mathrm{CCN}} = ||\rho^R||_{\mathrm{Tr}} = \sum_{k=1}^{r} \delta_k \leq 1. \tag{17}$$

In particular, if ρ is a pure state, then ρ is separable if and only if

$$||\rho||_{\mathrm{CCN}} = ||\rho^R||_{\mathrm{Tr}} = \sum_{k=1}^{r} \delta_k = 1. \tag{18}$$

It is known that the RCCN criterion is neither weaker nor stronger than the PPT criterion [11]. Namely, there exist PPT entangled states which can be detected by the RCCN criterion, while there are non-PPT entangled states which can not be detected by the RCCN criterion (for instance, certain $d\times d$ Werner states, ref. [10, 11, 16]).

3. The RCCN criterion for infinite-dimensional systems

In this section, we will establish the realignment criterion and the RCCN criterion for infinite-dimensional bipartite quantum systems. Unless specifically stated, we assume that at least one of H_A and H_B is of infinite dimension throughout this section.

In [3], we proposed a so-called realignment operation for a given pure state in infinite-dimensional bipartite quantum systems. For a given fixed product basis $\{|m\rangle|\mu\rangle\}$ of $H_A\otimes H_B$, every unit vector $|\psi\rangle$ can be written in $|\psi\rangle=\sum\limits_{m,\mu}d_{m\mu}|m\rangle|\mu\rangle$. Write $D=(d_{m\mu})$ and $\bar{D}=(\overline{d_{m\mu}})$. Then the realignment operator of the pure state $\rho_\psi=|\psi\rangle\langle\psi|$ is defined to be

$$\rho_\psi^{R'}=D\otimes\bar{D}. \tag{19}$$

It is straightforward that $\|\rho_\psi^{R'}\|_2=\|D\otimes\bar{D}\|_2=\|D\|_2\cdot\|\bar{D}\|_2=1$, $\rho_\psi^{R'}\in C_2(H_B\otimes H_B,H_A\otimes H_A)$. As the realignment operation must be linear, we can define a realignment operation for a mixed state $\rho=\sum_{i=1}^k p_i\rho_i$ by $\rho^{R'}=\sum_{i=1}^k p_i\rho_i^{R'}$, where ρ_is are pure states, $k\in\mathbb{N}$ or $k=+\infty$. This definition obviously coincides with that for finite-dimensional systems.

Like the case of finite-dimensions, for an arbitrarily fixed product basis $\{|m\rangle|\mu\rangle\}$, ρ_ψ can be written in an infinite matrix of double indices

$$\rho_\psi=(\rho^\psi_{m\mu,n\nu}),\ \rho^\psi_{m\mu,n\nu}:=\langle m|\langle\mu|\rho_\psi|n\rangle|\nu\rangle \tag{20}$$

and we have

$$\rho_\psi^{R'}=D\otimes\bar{D}=(\tilde{\rho}^\psi_{mn,\mu\nu}),\ \tilde{\rho}^\psi_{mn,\mu\nu}=\langle m|\langle n|\rho_\psi^{R'}|\mu\rangle|\nu\rangle. \tag{21}$$

It is easy to check that

$$\tilde{\rho}^\psi_{mn,\mu\nu}=\rho^\psi_{m\mu,n\nu}. \tag{22}$$

Inspired by Eq.(22), we now give a definition of the realignment operation. As usual, we denote by $C_2(H,K)$ ($C_2(H)$ if $H=K$) the set of all Hilbert-Schmidt operator from the Hilbert space H into the Hilbert space K. That is, $C_2(H,K)$ is a Hilbert space with respect to the complex scalar product $\langle A|B\rangle:=\mathrm{Tr}(A^\dagger B), A,B\in C_2(H,K)$. The Hilbert-Schmidt norm of A is $\|A\|_2=(\mathrm{Tr}(A^\dagger A))^{1/2}$.

Definition 3.1. *Let $T\in C_2(H_A\otimes H_B)$ and $Z\in C_2(H_B\otimes H_B,H_A\otimes H_A)$. Let $\{|m\rangle\}$ and $\{|\mu\rangle\}$ be arbitrarily given orthonormal bases of H_A and H_B, respectively. Then T and Z can be written respectively in*

$$T=(t_{m\mu,n\nu}),\ t_{m\mu,n\nu}=\langle m|\langle\mu|T|n\rangle|\nu\rangle$$

and

$$Z = (z_{mn,\mu\nu}),\ z_{mn,\mu\nu} = \langle m|\langle n|Z|\mu\rangle|\nu\rangle.$$

If

$$z_{mn,\mu\nu} = t_{m\mu,n\nu}, \tag{23}$$

we say that Z is the realignment operator of T, denoted by $T^R = Z$, with respect to the given bases.

The realignment operation $\mathcal{R} : C_2(H_A \otimes H_B) \to C_2(H_B \otimes H_B, H_A \otimes H_A)$ defined by $\mathcal{R}T = T^R$ as in the Definition 3.1 is an isometry, namely, $\mathcal{R}$ is linear and $\|\mathcal{R}T\|_2 = \|T^R\|_2 = \|T\|_2$ for every T. Particularly, for any $\rho \in \mathcal{S}(H_A \otimes H_B)$, we have

$$\|\rho^R\|_2 = \|\rho\|_2 \le \|\rho\|_{\rm Tr} = 1. \tag{24}$$

By Eqs.(20)-(22) and Definition 3.1, it follows that, for any mixed state $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ where $\sum_i p_i = 1$, $p_i \ge 0$, $\rho_i = |\psi_i\rangle\langle\psi_i|$ are pure states, we have

$$\rho^R = \sum_i p_i D_i \otimes \bar{D}_i = \rho^{R'}, \tag{25}$$

where the series converges in Hilbert-Schmidt norm on $C_2(H_B\otimes H_B, H_A\otimes H_A)$, and $D_i = (d^{(i)}_{m\mu})$ whenever $|\psi_i\rangle = \sum_{m,\mu} d^{(i)}_{m,\mu}|m\rangle|\mu\rangle$. Thus the realignment operation defined in Definition 3.1 coincides with that introduced in [3]. It is also clear that ρ^R is independent on the decomposition of ρ, that is, if $\rho = \sum_j q_j|\phi_j\rangle\langle\phi_j|$ is another decomposition of ρ into an infinite convex combination, then $\sum_i p_i D_i \otimes \bar{D}_i = \sum_j q_j D'_j \otimes \bar{D}'_j$, where $D'_j = (d^{(j)}_{m\mu})$ whenever $|\phi_j\rangle = \sum_{m,\mu} d^{(j)}_{m,\mu}|m\rangle|\mu\rangle$.

Remark 3.2. (1) It is clear that the realignment operation in Definition 3.1 is an infinite-dimensional generalization of the row realignment of a matrix for finite-dimensional case as discussed in section 2.

(2) The trace norm and the Hilbert-Schmidt norm of the realignment operator of a state is independent on the choice of the bases of H_A and H_B.

(3) Similarly, we can define the column realignment operation $T \mapsto T^{R^c}$ by Eq.(10). Then, if $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, we have $\rho^{R^c} = \sum_i p_i D_i^T \otimes \bar{D}_i^T$, where p_i, D_i are the same as the ones mentioned above. Let $F_A := \sum_{m,n} |m\rangle|n\rangle\langle n|\langle m|$ and $F_B := \sum_{\mu,\nu} |\mu\rangle|\nu\rangle\langle\nu|\langle\mu|$. It turns out that $F^\dagger_{A/B} = F_{A/B}$, $F_{A/B}F^\dagger_{A/B} = I_{A/B}$ and $\rho^{R^c} = F_A\rho^R F_B$, where $I_{A/B}$ is the identity operator on $H_{A/B}$. It follows that $\|\rho^R\|_{\rm Tr} = \|\rho^{R^c}\|_{\rm Tr}$. Thus, it is sufficient to discuss the row realignment operation only.

In what follows, we will show that the set of the realignment operators of the separable states are trace class operators from $H_B \otimes H_B$ into $H_A \otimes H_A$. In fact, we have

Criterion 3.3. (**The realignment criterion for infinite-dimensional bipartite quantum systems**) *If $\rho \in \mathcal{S}(H_A \otimes H_B)$ is separable, then*

$$\|\rho^R\|_{\mathrm{Tr}} \leq 1. \tag{26}$$

In particular, if ρ is a pure state, then ρ is separable if and only if

$$\|\rho^R\|_{\mathrm{Tr}} = 1. \tag{27}$$

Proof. The last assertion was already proved in [3], that is, for the case that ρ is a pure state, ρ is separable if and only if $\|\rho^R\|_{\mathrm{Tr}} = 1$.

If ρ is a separable mixed state, then by Eq.(2), there exist a Borel probability measure μ on $\mathcal{S}_{s-p}$ and a measurable function $\varphi : \mathcal{S}_{s-p} \to \mathcal{S}_{s-p}$ such that ρ has a representation of the Bochner integral

$$\rho = \int_{\mathcal{S}_{s-p}} \varphi(\rho^A \otimes \rho^B) d\mu(\rho^A \otimes \rho^B), \quad \rho^A \otimes \rho^B \in \mathcal{S}_{s-p}. \tag{28}$$

It is known that, from the definition of the Bochner integral, there exists a sequence of step functions $\{\varphi_n\}$ such that

$$\varphi(\rho^A \otimes \rho^B) = \lim_{n\to\infty} \varphi_n(\rho^A \otimes \rho^B)$$

with respect to the trace norm, where

$$\varphi_n(\rho^A \otimes \rho^B) = \sum_{i=1}^{k_n} \chi_{E_i}(\rho^A \otimes \rho^B)\rho_i^A \otimes \rho_i^B,$$

$\chi_{E_i}(\cdot)$ is the characteristic function of E_i and $\{E_i\}_{i=1}^{k_n}$ is a partition of $\mathcal{S}_{s-p}$. Thus

$$\rho = \lim_{n\to\infty} \sum_{i=1}^{k_n} \mu(E_i)\rho_i^A \otimes \rho_i^B$$

with respect to the trace norm, as well as with respect to the Hilbert-Schmidt norm. Because the realignment operation is continuous, we have

$$\begin{aligned} \rho^R &= \lim_{n\to\infty} \sum_{i=1}^{k_n} \mu(E_i)(\rho_i^A \otimes \rho_i^B)^R \\ &= \lim_{n\to\infty} \int_{\mathcal{S}_{s-p}} (\varphi_n(\rho_i^A \otimes \rho_i^B))^R d\mu(\rho^A \otimes \rho^B) \\ &= \int_{\mathcal{S}_{s-p}} (\varphi(\rho_i^A \otimes \rho_i^B))^R d\mu(\rho^A \otimes \rho^B) \end{aligned} \tag{29}$$

with respect to the Hilbert-Schmidt norm. Therefore,

$$\begin{aligned} \|\rho^R\|_{\mathrm{Tr}} &\leq \int_{\mathcal{S}_{s-p}} \|(\varphi(\rho^A \otimes \rho^B))^R\|_{\mathrm{Tr}} d\mu(\rho^A \otimes \rho^B) \\ &= \int_{\mathcal{S}_{s-p}} 1 d\mu(\rho^A \otimes \rho^B) = 1 \end{aligned}$$

as $\|(\varphi(\rho^A \otimes \rho^B))^R\|_{\mathrm{Tr}} = 1$ by Eq.(27). This completes the proof. □

Let $\mathcal{S}_{sep}^R = \{\rho^R : \rho \in \mathcal{S}_{sep}\}$. The previous criterion shows that $\mathcal{S}_{sep}^R \subset \mathcal{T}_1(H_B \otimes H_B, H_A \otimes H_A)$, where $\mathcal{T}_1(H_B \otimes H_B, H_A \otimes H_A)$ denotes the set of all trace-class operators from $H_B \otimes H_B$ into $H_A \otimes H_A$ with the trace-norm not greater than 1.

Next we will show that, there is another alternative way to perform realignment operation for all states which is equivalent to the operation proposed as in Definition 3.1.

Let $A \in \mathcal{B}(H_A)$, $B \in \mathcal{B}(H_B)$. For given bases $\{|m\rangle\}$ and $\{|\mu\rangle\}$ of H_A and H_B, respectively, A and B can be written in the form $A = \sum\limits_{m,n} a_{mn}|m\rangle\langle n|$ and $B = \sum\limits_{\mu,\nu} b_{\mu\nu}|\mu\rangle\langle\nu|$. Regard A as a vector $|A\rangle = \sum\limits_{m,n} a_{mn}|m\rangle|n\rangle$ in the Hilbert space $C_2(H_A)$ and B as $|B\rangle = \sum\limits_{\mu,\nu} b_{\mu\nu}|\mu\rangle|\nu\rangle$ in the Hilbert space $C_2(H_B)$, respectively. Let $\langle B|$ denote the transpose of $|B\rangle$. Let ρ be a separable state with $\rho = \sum\limits_i p_i\rho_i^A \otimes \rho_i^B$, where $p_i \geq 0$, $\sum\limits_i p_i = 1$, ρ_i^A and ρ_i^B are pure states in $\mathcal{S}(H_A)$ and $\mathcal{S}(H_B)$, respectively. It is easy to see that $\rho^R = \sum\limits_i p_i D_i \otimes \overline{D_i} = \sum\limits_i p_i|\rho_i^A\rangle\langle\rho_i^B|$ as $|\rho_i^A\rangle\langle\rho_i^B| = D_i \otimes \overline{D_i}$. This motivates the possibility of generalizing Eq.(7) to infinite-dimensional cases.

To do this, notice that $\mathcal{S}(H_A \otimes H_B) \subset \mathcal{T}(H_A \otimes H_B) \subset C_2(H_A \otimes H_B) = C_2(H_A) \otimes C_2(H_B)$. So each state ρ can be regarded as a "vector" of the Hilbert space $C_2(H_A) \otimes C_2(H_B)$. Considering the Fourier representation of $T \in C_2(H_A) \otimes C_2(H_B)$ with respect to a product basis of $C_2(H_A) \otimes C_2(H_B)$, we see that T can be written in the form

$$T = \sum_k A_k \otimes B_k, \tag{30}$$

where $\{A_k\} \subset C_2(H_A)$, $\{B_k\} \subset C_2(H_B)$ and the series converges in the Hilbert-Schmidt norm.

Proposition 3.4. *Let $T \in C_2(H_A \otimes H_B)$. Write $T = \sum\limits_k A_k \otimes B_k$ as in Eq.(30). Then, with respect to given bases $\{|m\rangle\}$ and $\{|\mu\rangle\}$ of H_A and H_B, respectively, we have*

$$T^R = \sum_k |A_k\rangle\langle B_k|, \tag{31}$$

where the series converges in Hilbert-Schmidt norm on $C_2(H_B \otimes H_B, H_A \otimes H_A)$, $|A_k\rangle = \sum\limits_{m,n} a_{mn}^{(k)}|m\rangle|n\rangle$ if $A_k = \sum\limits_{m,n} a_{mn}^{(k)}|m\rangle\langle n|$, $|B_k\rangle = \sum\limits_{\mu,\nu} b_{\mu\nu}^{(k)}|\mu\rangle|\nu\rangle$ if $B_k = \sum\limits_{\mu,\nu} b_{\mu\nu}^{(k)}|\mu\rangle\langle\nu|$ and $\langle B_k|$ denotes the transpose of $|B_k\rangle$, $k = 1, 2, \ldots$.

Proof. Write $\mathcal{R}'(T) = \sum\limits_k |A_k\rangle\langle B_k|$ whenever $T = \sum\limits_k A_k \otimes B_k$ as in Eq.(30). We show that

$$\mathcal{R}'(T) = \mathcal{R}(T) = T^R \tag{32}$$

holds for all $T \in C_2(H_A \otimes H_B)$. It is easy to check that, if $T = \sum\limits_k A_k \otimes B_k = (t_{m\mu,n\nu})$, then $T^{R'} = \sum\limits_k |A_k\rangle\langle B_k| = (\tilde{t}_{mn,\mu\nu})$ with $\tilde{t}_{mn,\mu\nu} = t_{m\mu,n\nu}$. Thus, $T^{R'} = T^R$ is well defined. It remains to show that the series $\sum\limits_k |A_k\rangle\langle B_k|$ converges to T in the Hilbert-Schmidt norm. Let $T_n = \sum\limits_{k=1}^n A_k \otimes B_k$ and $T_n^R = \sum\limits_{k=1}^n |A_k\rangle\langle B_k|$; then $\|T^R - T_n^R\|_2 = \|\sum\limits_{k=n+1}^\infty |A_k\rangle\langle B_k|\|_2 = \|\sum\limits_{k=n+1}^\infty A_k \otimes B_k\|_2 \to 0$ $(n \to +\infty)$ since $\|T - T_n\|_2 \to 0$ $(n \to +\infty)$. $\square$

By now, for any state $\rho \in \mathcal{S}(H_A \otimes H_B)$, we have three equivalent definitions of the realignment operator of it.

Inspired by Eq.(30) and an idea in [11], we generalize the notion of "*computable cross norm*" to the infinite-dimensional case.

Definition 3.5. *The computable cross norm $\|T\|_{\text{CCN}}$ of an arbitrary element $T \in \mathcal{C}_2(H_A \otimes H_B)$ is defined by*

$$\|T\|_{\text{CCN}} := \inf\{\sum_k \|A_k\|_2\|B_k\|_2 : T = \sum_k A_k \otimes B_k\}, \tag{33}$$

where the infimum runs over all decompositions of T into elementary tensors as that in Eq.(30).

It is evident that $\|\cdot\|_{\text{CCN}}$ is a cross norm on $\mathcal{C}_2(H_A \otimes H_B)$ since $\|A \otimes B\|_{\text{CCN}} = \|A\|_2\|B\|_2$ for all $A \in \mathcal{C}_2(H_A), B \in \mathcal{C}_2(H_B)$. Also, we may have $\|T\|_{\text{CCN}} = +\infty$ for some T.

Noticing that, every vector in the tensor product Hilbert space of two Hilbert spaces has a so-called Schmidt decomposition [3]. Together with the fact $\mathcal{C}_2(H_A \otimes H_B) = \mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$, we can derive that, for any state ρ on $H_A \otimes H_B$, ρ has a Schmidt decomposition as a vector in $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$, i.e.,

$$\rho = \sum_{k=1}^{N_\rho} \delta_k E_k \otimes F_k, \tag{34}$$

where $E_k \in \mathcal{C}_2(H_A)$, $F_k \in \mathcal{C}_2(H_B)$ satisfying $\text{Tr}(E_k^\dagger E_l) = \delta_{kl}$ and $\text{Tr}(F_k^\dagger F_l) = \delta_{kl}$, $k, l = 1, 2, \ldots, N_\rho$, the positive scalars $\delta_1 \geq \delta_2 \geq \cdots$ are uniquely determined by the corresponding vector ρ, and they are the so-called Schmidt coefficients of ρ [3], while N_ρ (may be $+\infty$) is called the Schmidt number of ρ. Since $\text{Tr}(\rho^2) = \langle\rho|\rho\rangle = \sum_k \delta_k^2$, we have $\sum_k \delta_k^2 = 1 \Leftrightarrow \rho$ is a pure state and $\sum_k \delta_k^2 < 1 \Leftrightarrow \rho$ is a mixed state.

The following lemma highlights the relations among the trace norm of the realignment operator, the computable cross norm and the sum of the Schmidt coefficients of a state. As one might expect, the result is the same as that for the finite-dimensional case.

Lemma 3.6. *Let ρ be a state in $\mathcal{S}(H_A \otimes H_B)$ and $\{\delta_k\}$ be the Schmidt coefficients of ρ as a vector in $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$. Then we have*

$$\|\rho^R\|_{\text{Tr}} = \|\rho\|_{\text{CCN}} = \sum_k \delta_k. \tag{35}$$

Proof. Let $\rho = \sum_k A_k \otimes B_k$ as in Eq.(30) and $\rho = \sum_k \delta_k E_k \otimes F_k$ be the Schmidt decomposition of ρ as a vector in $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$, where the series converges in Hilbert-Schmidt norm. Then $\rho^R = \sum_k |A_k\rangle\langle B_k|$, and

$$\|\rho^R\|_{\text{Tr}} = \sum_k \delta_k \tag{36}$$

since one can regard $\rho^R = \sum\limits_k \delta_k |E_k\rangle\langle F_k|$ as the singular value decomposition of ρ^R. Next, we show that

$$\|\rho^R\|_{\mathrm{Tr}} = \inf\{\sum_k \||A_k\rangle\| \cdot \||B_k\rangle\| : \rho = \sum_k A_k \otimes B_k\}. \tag{37}$$

On the one hand, we have $\|\rho^R\|_{\mathrm{Tr}} \leq \sum \||A_k\rangle\langle B_k|\|_{\mathrm{Tr}} = \sum\limits_k \||A_k\rangle\| \cdot \||B_k\rangle\|$. On the other hand, $\|\rho^R\|_{\mathrm{Tr}} = \sum\limits_k \delta_k \||E_k\rangle\| \cdot \||F_k\rangle\| = \sum\limits_k \delta_k$ since $\||E_k\rangle\| = \|E_k\|_2 = \||F_k\rangle\| = \|F_k\|_2 = 1$. Namely, the infimum is attained at the singular value decomposition of ρ. Now, we arrive at

$$\begin{aligned} \|\rho^R\|_{\mathrm{Tr}} = & \inf\{\sum_k \||A_k\rangle\| \cdot \||B_k\rangle\| : \rho = \sum_k A_k \otimes B_k\} \\ = & \inf\{\sum_k \|A_k\|_2 \|B_k\|_2 : \rho = \sum_k A_k \otimes B_k\} \\ = & \|\rho\|_{\mathrm{CCN}}, \end{aligned} \tag{38}$$

which completes the proof. □

For a pure state $\rho_\psi \in \mathcal{S}(H_A \otimes H_B)$, let $|\psi\rangle = \sum\limits_k \lambda_k |m_k\rangle|\mu_k\rangle$ be the Schmidt decomposition of $|\psi\rangle \in H \otimes K$, then it is straightforward that

$$\|\rho_\psi\|_{\mathrm{CCN}} = (\sum_k \lambda_k)^2. \tag{39}$$

From this it is obvious that a pure state ρ is separable if and only if $\|\rho^R\|_{\mathrm{Tr}} = \|\rho_\psi\|_{\mathrm{CCN}} = 1$. Further more, combining Criterion 3.3 and Lemma 3.6, we establish the RCCN criterion for the infinite-dimensional systems, that is, the criterion below is the main result of this paper.

Criterion 3.7. (**The RCCN criterion for infinite-dimensional bipartite quantum systems**) *Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ and $\{\delta_k\}$ be the Schmidt coefficients of ρ as a vector in $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$. If ρ is separable, then*

$$\|\rho^R\|_{\mathrm{Tr}} = \|\rho\|_{\mathrm{CCN}} = \sum_k \delta_k \leq 1. \tag{40}$$

In particular, assume that ρ is a pure state, then ρ is separable if and only if

$$\|\rho^R\|_{\mathrm{Tr}} = \|\rho\|_{\mathrm{CCN}} = \sum_k \delta_k = 1. \tag{41}$$

In what follows, we give some examples to illustrate that there exist PPT entangled states that can be detected by the RCCN criterion, there exist PPT entangled states that can not be detected by the realignment criterion and there exist non-PPT entangled state that can not be detected by the RCCN criterion. These examples imply that the RCCN criterion is neither 'weaker' nor 'stronger' than the PPT criterion. There also exist entangled states which can not be detected by any one of these two criteria. However, we can show that, for the so-called 'symmetric sates', the PPT criterion is equivalent to the RCCN criterion, namely, a symmetric state ρ satisfying $\|\rho^R\|_{\mathrm{Tr}} \leq 1$ if and only if it is a PPT state (see in Proposition 3.13).

Example 3.8. Let H_A and H_B be complex Hilbert spaces with orthonormal bases $\{|0\rangle, |1\rangle, \ldots\}$ and $\{|0'\rangle, |1'\rangle, \ldots\}$, respectively. Let

$$\rho_\alpha = \frac{2}{7}|w\rangle\langle w| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-,$$

where $|w\rangle = \frac{1}{\sqrt{3}}(|0\rangle|0'\rangle + |1\rangle|1'\rangle + |2\rangle|2'\rangle)$, $\sigma_+ = \frac{1}{3}(|0\rangle|1'\rangle\langle 0|\langle 1'| + |1\rangle|2'\rangle\langle 1|\langle 2'| + |2\rangle|0'\rangle\langle 2|\langle 0'|)$, $\sigma_- = \frac{1}{3}(|1\rangle|0'\rangle\langle 1|\langle 0'| + |2\rangle|1'\rangle\langle 2|\langle 1'| + |0\rangle|2'\rangle\langle 0|\langle 2'|)$ and $2 \leq \alpha \leq 5$. A straightforward calculation shows that

$$\|\rho_\alpha^R\|_{\mathrm{Tr}} = \frac{19}{21} + \frac{2}{21}\sqrt{19 - 15\alpha + 3\alpha^2}.$$

It is easy to check that $\|\rho_\alpha^R\|_{\mathrm{Tr}} \leq 1$ if and only if $2 \leq \alpha \leq 3$. Thus, by the RCCN criterion, $3 < \alpha \leq 5$ implies ρ_α is entangled.

Define

$$\sigma = \sum_{i=3}^{+\infty} p_i|i\rangle\langle i| \otimes |i'\rangle\langle i'|, \ p_i \geq 0, \ \sum_{i=3}^{\infty} p_i = 1.$$

Then σ is a separable state and $\|\sigma^R\|_{\mathrm{Tr}} = 1$. Let

$$\rho_{t,\alpha} = t\rho_\alpha + (1-t)\sigma, \ 0 < t \leq 1, 3 < \alpha \leq 4. \tag{42}$$

By [11, 23], it is easily checked that $\rho_{t,\alpha}$ is a PPT state whenever $3 < \alpha \leq 4$ since $\rho_\alpha^{T_B} \geq 0$ whenever $3 < \alpha \leq 4$. On the other hand,

$$\|\rho_{t,\alpha}^R\|_{\mathrm{Tr}} = \|t\rho_\alpha^R + (1-t)\sigma^R\|_{\mathrm{Tr}} = t\|\rho_\alpha^R\|_{\mathrm{Tr}} + (1-t)\|\sigma^R\|_{\mathrm{Tr}}.$$

It follows that $\|\rho_{t,\alpha}^R\|_{\mathrm{Tr}} > 1$ for all $0 \leq t < 1$ as $\|\sigma^R\|_{\mathrm{Tr}} = 1$ and $\|\rho_\alpha^R\| > 1$ whenever $3 < \alpha \leq 4$. So, there are PPT entangled states that can be detected by the RCCN criterion.

The example below is discussed in [24] and it illustrates particularly that there exist non-PPT states as well as entangled PPT states that con not be recognized by the RCCN criterion.

Example 3.9. Let H_A and H_B be complex Hilbert spaces of dimension ≥ 4 with orthonormal bases $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$ and $\{|0'\rangle, |1'\rangle, |2'\rangle, \ldots\}$, respectively. Let $|\omega\rangle = \frac{1}{\sqrt{4}}(|0\rangle|0'\rangle + |1\rangle|1'\rangle + |2\rangle|2'\rangle + |3\rangle|3'\rangle)$. Define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{4}(|0\rangle|1'\rangle\langle 0|\langle 1'| + |1\rangle|2'\rangle\langle 1|\langle 2'| + |2\rangle|3'\rangle\langle 2|\langle 3'| + |3\rangle|0'\rangle\langle 3|\langle 0'|)$, $\rho_3 = \frac{1}{4}(|0\rangle|2'\rangle\langle 0|\langle 2'| + |1\rangle|3'\rangle\langle 1|\langle 3'| + |2\rangle|0'\rangle\langle 2|\langle 0'| + |3\rangle|1'\rangle\langle 3|\langle 1'|)$ and $\rho_4 = \frac{1}{4}(|0\rangle|3'\rangle\langle 0|\langle 3'| + |1\rangle|0'\rangle\langle 1|\langle 0'| + |2\rangle|1'\rangle\langle 2|\langle 1'| + |3\rangle|2'\rangle\langle 3|\langle 2'|)$. Let

$$\rho = \sum_{i=1}^{4} q_i\rho_i \quad \text{and} \quad \rho_t = (1-t)\rho + t\rho_0, \tag{43}$$

where $q_i \geq 0$ for $i = 1, 2, 3, 4$ with $q_1 + q_2 + q_3 + q_4 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H_A \otimes H_B$. It was shown in [23] that, for sufficiently small t, or for ρ_0 with $|i\rangle|\mu'\rangle\langle j|\langle\nu'|\rho_0 = \rho_0|i\rangle|\mu'\rangle\langle j|\langle\nu'| = 0$ for any $i, j, \mu, \nu \in \{0, 1, 2, 3\}$, the following statements are true.

(1) If $q_i < q_1$ for some $i = 2, 3, 4$, then ρ_t is entangled.

(2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_2q_4 \geq q_1^2$ and $q_3 \geq q_1$. Thus, if $0 < q_i < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_j < 1$ with $q_iq_j \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$, where $i, j \in \{2, 4\}$ and $i \neq j$, then ρ_t is PPT entangled.

(3) The trace norm of the realignment operator of ρ is

$$\begin{aligned}\|\rho^R\|_{\mathrm{Tr}} = \; & \tfrac{3}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 - q_1q_2 - q_2q_3 - q_3q_4 - q_1q_4} \\ & + \tfrac{1}{4}\sqrt{\textstyle\sum_{i=1}^4 q_i^2 + 3(q_1q_2 + q_2q_3 + q_3q_4 + q_1q_4)} + 3q_1.\end{aligned}$$

Thus, if ρ_0 is PPT, and if $q_1 \geq \frac{1}{6}$, $q_i = \frac{1}{2}q_1$, $q_j = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_i$, where $i, j \in \{2, 4\}$ and $i \neq j$, then ρ_t is PPT entangled with $\|\rho^R\|_{\mathrm{Tr}} > 1$ for sufficient small t, that is, ρ_t is PPT entangled that can be detected by the RCCN criterion; if $q_1 \leq \frac{1}{7}$, $q_i = \frac{1}{2}q_1$, $q_j = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_i$, where $i, j \in \{2, 4\}$ and $i \neq j$, then, for sufficient small t, $\|\rho^R\|_{\mathrm{Tr}} < 1$ and ρ_t is PPT entangled but can not be detected by the RCCN criterion.

The following illustrates that how to find suitable ρ_t so that $\|\rho_t^R\|_{\mathrm{Tr}} < 1$ but ρ_t is not PPT.

If ρ_0 is not PPT, we choose $q_1 \leq \frac{1}{7}$, $q_i = \frac{1}{2}q_1$, $q_j = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_i$, where $i, j \in \{2, 4\}$ and $i \neq j$. Then, as mentioned above we have $\|\rho^R\|_{\mathrm{Tr}} < 1$. Thus, for sufficient small t, $\|\rho_t^R\|_{\mathrm{Tr}} < 1$. This means that ρ_t is not PPT but can not be recognized by the RCCN criterion.

If ρ_0 is PPT, we choose $q_2 = \frac{1}{2} - \frac{3}{2}q_1$, $q_3 = \frac{1}{2}q_1$ and $q_4 = \frac{1}{2}$. A computation shows that $\|\rho^R\|_{\mathrm{Tr}} < 1$ for $q_1 \leq \frac{1}{7}$. For instance, $\|\rho^R\|_{\mathrm{Tr}} = 0.9866$ if $q_1 = \frac{1}{7}$; $\|\rho^R\|_{\mathrm{Tr}} = 0.9496$ if $q_1 = \frac{1}{8}$; $\|\rho^R\|_{\mathrm{Tr}} = 0.7264$ if $q_1 = \frac{1}{100}$. Since $q_3 < q_1$, ρ is not PPT and thus ρ_t is not PPT. However, $\|\rho_t^R\|_{\mathrm{Tr}} < 1$ for sufficient small t.

Example 3.10. Let $H_A = H_B$ be complex Hilbert spaces with orthonormal bases $\{|0\rangle, |1\rangle, \ldots\}$. Fixing a positive number $3 \leq m \in \mathbb{N}$. Define

$$\rho_{m,c} := \frac{1}{m^3 - m}((m - c)P_m + (mc - 1)F_m),$$

where $P_m := \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} |i\rangle\langle i| \otimes |j\rangle\langle j|$ and $F_m := \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} |i\rangle\langle j| \otimes |j\rangle\langle i|$. It is easy to check that

$$\|\rho_{m,c}^R\|_{\mathrm{Tr}} = \begin{cases} \frac{2}{m} - c: & \text{if } \; -1 \leq c \leq \frac{1}{m}, \\ c: & \text{if } \; 1 \geq c \geq \frac{1}{m}. \end{cases}$$

If $\dim H_A = m$, then $\rho_{m,c} = \rho_c$ is the so-called Werner state [16]. It is shown in [16] that

$$\rho_c \text{ is separable} \Leftrightarrow \rho^{T_B} \geq 0 \Leftrightarrow 0 \leq c \leq 1.$$

We can derive that

$$\rho_{m,c} \text{ is separable} \Leftrightarrow \rho^{T_B} \geq 0 \Leftrightarrow 0 \leq c \leq 1.$$

Consequently, if $\frac{2}{m} - 1 \leq c < 0$, then $\rho_{m,c}$ is a non-PPT entangled state which satisfies $\|(\rho_{m,c})^R\| \leq 1$, that is, $\rho_{m,c}$ can not be recognized by the RCCN criterion.

Let

$$\varrho = \sum_{i=m}^{+\infty} p_i|i\rangle\langle i| \otimes |i\rangle\langle i|,\ p_i \geq 0,\ \sum_{i=m}^{+\infty} p_i = 1,$$

then ϱ is separable. Let

$$\rho_{\varepsilon,c} = \varepsilon\varrho + (1-\varepsilon)\rho_{m,c},\ 0 \leq \varepsilon < 1,\ \frac{2}{m} - 1 \leq c < 0. \tag{44}$$

It is straightforward that $\rho_{\varepsilon,c}$ is a state acting on $H_A \otimes H_B$ and $\rho_{\varepsilon,c}^{T_B}$ is not positive. Now, we can conclude that for any $\frac{2}{m} - 1 \leq c < 0$ and $0 \leq \varepsilon < 1$, $\rho_{\varepsilon,c}$ is a non-PPT entangled state satisfying $\|\rho_{\varepsilon,c}^R\|_{\rm Tr} \leq 1$ since $\|\varrho^R\|_{\rm Tr} = 1$ and $\|\rho_{m,c}^R\|_{\rm Tr} \leq 1$.

Now let us turn to another related topic. In [22], several entanglement criteria for the so-called 'symmetric states' in the finite-dimensional bipartite quantum systems are presented. Recall that, a state ρ on a finite dimensional bipartite system $H_A \otimes H_B$ is called a symmetric state if $\dim H_A = \dim H_B = N$ and $\rho = F\rho = \rho F$, where F is the flip operator, namely, F satisfies $F|\psi_A\rangle|\psi_B\rangle = |\psi_B\rangle|\psi_A\rangle$ for any $|\psi_A\rangle \in H_A$ and $|\psi_B\rangle \in H_B$. It is showed that, for a symmetric state ρ in a finite-dimensional bipartite quantum system, $\|\rho^R\|_{\rm Tr} \leq 1$ if and only if ρ is a PPT state. Inspired by [22], we can generalize the conception of the symmetric states to the infinite-dimensional case with the same spirit.

Definition 3.11. *Let H_A and H_B be Hilbert spaces with $\dim H_A = \dim H_B = +\infty$. Let $\{|m\rangle\}$ and $\{|\mu\rangle\}$ be orthonormal bases respectively of H_A and H_B. A state $\rho \in \mathcal{S}(H_A \otimes H_B)$ is said to be symmetric if*

$$\rho = F\rho = \rho F, \tag{45}$$

where $F = \sum_{m,\mu} |m\rangle|\mu\rangle\langle\mu|\langle m|$.

The operator F in the definition is called the flip operator. It is clear that $F|\psi_A\rangle|\psi_B\rangle = |\psi_B\rangle|\psi_A\rangle$ for any $|\psi_A\rangle \in H_A$, $|\psi_B\rangle \in H_B$.

Write $\rho = (\rho_{m\mu,n\nu})$, where $\rho_{m\mu,n\nu} = \langle m|\langle\mu|\rho|n\rangle|\nu\rangle$, one can obtain

Lemma 3.12. *If $\rho \in \mathcal{S}(H_A \otimes H_B)$ is a symmetric state, then*

$$\rho_{m\mu,n\nu} = \rho_{\mu m,n\nu} = \rho_{m\mu,\nu n} = \rho_{\mu m,\nu n}. \tag{46}$$

Moreover,

$$F\rho^R = \rho^{T_A}. \tag{47}$$

Proof. Write $\rho^R = (\hat{\rho}_{m\mu,n\nu})$, $F\rho^R = (\check{\rho}_{m\mu,n\nu})$. It turns out that $\hat{\rho}_{m\mu,n\nu} = \rho_{mn,\mu\nu}$, $\check{\rho}_{m\mu,n\nu} = \hat{\rho}_{\mu m,n\nu}$, and thus $\check{\rho}_{m\mu,n\nu} = \hat{\rho}_{\mu m,n\nu} = \rho_{\mu n,m\nu}$. On the other hand, writing $\rho^{T_A} = (\tilde{\rho}_{m\mu,n\nu})$, we have $\tilde{\rho}_{m\mu,n\nu} = \rho_{n\mu,m\nu}$. Therefore, $\rho^{T_A} = F\rho^R$ since $\rho_{\mu n,m\nu} = \rho_{n\mu,m\nu}$. □

As F is unitary, by Lemma 3.12, the singular values of ρ^R is equal to the singular values of ρ^{T_A}. Since $\mathrm{Tr}(\rho^{T_A}) = 1$, it follows that ρ^{T_A} is not positive if and only if ρ^{T_A} has at least one negative eigenvalue. Therefore, ρ is not PPT implies that $\|\rho^R\|_{\rm Tr} = \|\rho^{T_A}\|_{\rm Tr} > 1$ and vice versa. Thus we have proved the following

Proposition 3.13. *If $\rho \in \mathcal{S}(H_A \otimes H_B)$ is symmetric, then ρ is a PPT state if and only if $\|\rho^R\|_{\rm Tr} \leq 1$.*

4. Conclusion

In conclusion, we generalize the row realignment operation, the Computable Cross Norm to the states of infinite-dimensional bipartite quantum systems. The realignment operators of the states are Hilbert-Schmidt operators from $H_B \otimes H_B$ into $H_A \otimes H_A$, and moreover, the row realignment operation $T \mapsto T^R$ is an isometric linear map from $\mathcal{C}_2(H_A \otimes H_B)$ into $\mathcal{C}_2(H_B \otimes H_B, H_A \otimes H_A)$. Similar to that in the finite-dimensional bipartite quantum systems, there are two kinds of realignment operations, namely, the row realignment operation and the column realignment operation. These two realignment operations are equivalent up to the trace norm. So, it suffices to discuss the row realignment operation.

In fact, three equivalent definitions of the realignment operation are introduced. This allow us to establish the realignment criterion and the RCCN criterion of separability for states in infinite-dimensional bipartite systems. Thus, for both finite-dimensional and infinite-dimensional systems, if a state $\rho \in \mathcal{S}(H_A \otimes H_B)$ is separable, then $\|\rho^R\|_{\rm Tr} = \|\rho\|_{\rm CCN} = \sum_k \delta_k \leq 1$, where $\|\rho\|_{\rm CCN}$ is the computable cross norm of ρ and $\{\delta_k\}$ are the Schmidt coefficients of ρ as a vector in the Hilbert space $\mathcal{C}_2(H_A) \otimes \mathcal{C}_2(H_B)$. For the case that ρ is pure, ρ is separable if and only if $\|\rho^R\|_{\rm Tr} = \|\rho\|_{\rm CCN} = \sum_k \delta_k = 1$. Like the case of finite-dimension, the RCCN criterion and the PPT criterion are independent as illustrated by examples, and these two criteria are equivalent for the symmetric states.

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Department of Mathematics, Shanxi University, Taiyuan, 030006, China; Department of Mathematics, Shanxi Datong University, Datong, 037009, China.

E-mail address: guoyu3@yahoo.com.cn

Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, P. R. China; Department of Mathematics, Shanxi University, Taiyuan, 030006, P. R. China

E-mail address: jinchuanhou@yahoo.com.cn