Stability of Inviscid Parallel Flows between Two Parallel Walls

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Abstract: In this paper, the stability problem of inviscid parallel flow between two parallel walls is studied. Firstly, it is obtained that the base flow for this classical problem is a uniform flow. Secondly, it is shown that the solution of the disturbance equation is c=U, i.e., the propagation speed of the disturbance equals the flow velocity. The disturbance in this flow is neutral. Finally, it is suggested that the classical Rayleigh Theorem on inflectional velocity instability is incorrect which states that the necessary condition for instability of inviscid flow is the existence of an inflection point on the velocity profile.

Keywords: Flow instability; Linear disturbance; Inviscid flow; Rayleigh theorem; Inflection.

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47.15.Fe	Stability of laminar flows
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1. Introduction

Stability of parallel flows is the basis of modern flow stability theory. The Rayleigh theorem on inflectional instability is a fundamental theorem on inviscid stability theory. This theorem is found in many text books and employed in the scientific community since Rayleigh published his classical work in 1880. However, this theorem is still facing challenge today as it contradicts to some observations.

In the classical theory for flow instability, Rayleigh (1880) first developed a general linear stability theory for inviscid parallel shear flows, and showed that a necessary condition for instability is that the velocity profile has a point of inflection [1]. Heisenberg (1924) showed that if a velocity distribution allows an inviscid neutral disturbance with finite wave-length and non-vanishing phase velocity, the disturbance with the same wave-length is unstable in the real fluid when the Reynolds number is sufficiently large [2]. Later, Tollmien (1935) succeeded in showing that Rayleigh's criterion also constitutes a sufficient condition for the amplification of disturbances for velocity distributions of the symmetrical type or of the boundary-layer type [3]. Lin (1944) mentioned that he has shown where a point of inflexion exists in the velocity curve, but a neutral disturbance does not exist [4]. In other word, Rayleigh's criterion is not a sufficient condition for instability (the reason was further clarified by Fjϕrtoft (1950)). About the dual roles of viscosity, Lin was able to demonstrate the different influences of viscosity on the disturbance amplification at low Re and high Re. His conclusions are as follows. For small viscosity, the effect of viscosity is essentially destabilizing and an increase of Re gives more stability. For large viscosity (low Re), viscosity plays a stabilizing role by the dissipation of energy. Fjϕrtoft (1950) gave a further necessary condition for inviscid instability, that there is a maximum of vorticity for instability; he also gave the second further necessary condition for inviscid instability, $U''(U-U_{IP})<0$ (see Fig.1) [5]. Therefore, it is well known that inviscid flow with inflectional velocity profile is unstable, while inviscid flow with no inflectional velocity profile is stable [6-10]. However, when compared with experiments, contradictory results were obtained [6-10]. The associated analysis showed that the effect of viscosity is far more complex, and it may play dual roles to the flow instability for some type of flows [6]. The linear stability theory of small disturbances with Orr-Sommerfeld equation has been confirmed by the famous experiment of Schubauer and Skramstad [11], which proved that there exists a 2D wave under low noised environment and this wave was named as Tollmien-Schlichting (T-S) wave [6-10].

Recently, we proposed a new theory, named as ***energy gradient theory***, to explain the flow instability and transition to turbulence [12-17]. The critical condition calculated at turbulent transition determined by experiments obtains consistent agreement with the available experimental data for parallel flows and Taylor-Couette flows [14]. When the theory is considered for both parallel and curved shear flows, three important theorems have been deduced [16]. These theorems are: (1) Potential flow (inviscid and irrotational) is stable; (2) Inviscid rotational (inviscid and nonzero vorticity) flow is unstable; (3) Velocity profile with an inflectional point is unstable when there is no work input or output to the system, for both inviscid and viscous flows. From the theorem (3), it is demonstrated that the existence of an inflection point on velocity profile is a sufficient condition for pressure driven flows, for both the inviscid and viscous flows. As a necessary condition, the existence of an inflection point on velocity profile for pressure driven flows, this has been proved when considering the effect of disturbance [17]. Following these results, it is suggested that the classical Rayleigh theorem is not quite truly correct which states that a necessary condition for inviscid flow instability is the existence of an inflection point on the velocity profile.

In present study, the stability problem of inviscid parallel flow between two parallel walls is studied. Finally, it is shown that the classical Rayleigh Theorem on inflectional velocity instability is incorrect.

2. Rayleigh Equation

It should be distinguished between the base flow and the mean flow. However, for linear disturbance, since the perturbation is infinitesmall, the mean flow is the same as the base flow, but, the concept is different.

Let the base flow, which may be regarded as steady, be described by its Cartesian velocity components U,V,W and its pressure P, the corresponding quantities for the disturbance will be denoted by u', v', w' (u' in streamwise, v' in transverse, and w' in spanwise directions) and p', respectively. Hence, in the resultant motion the velocity components and the pressure are

$$u = U + u', \; v = V + v', \; w = W + w', \; p = P + p'. \tag{1}$$

Substituting the above expressions into the Euler equation for inviscid flow and substracting the equation for the base flow, the linearized equation of disturbance can be obtained [1, 6-10].

It is assumed that the disturbance is two-dimensional (2D), then a stream function is introduced. The stream function representing a single oscillation of the disturbance is assumed to be of the form

$$\psi(x, y, t) = \phi(y)e^{i(\alpha x - \beta t)}, \tag{2}$$

where α is a real quantity and β is a complex quantity, $\beta = \beta_r + i\beta_i$. Dividing β by α, a complex quantity c is obtained, $c = \beta / \alpha = c_r + ic_i$. Here, c_r is the speed of the wave propagating and c_i expresses the degree of damping or amplification of the disturbance (c_i =0, neutral disturbance; c_i <0, the disturbance decays; c_i >0, the disturbance amplified). Thus,

$$u' = \frac{\partial \psi}{\partial y} = \phi'(y)e^{i(\alpha x - \beta t)}, \tag{3}$$

$$v' = -\frac{\partial \psi}{\partial x} = -i\alpha\phi(y)e^{i(\alpha x - \beta t)}. \tag{4}$$

Introducing these values into the linearized equation of the disturbance, the following ordinary differential equation is obtained [1, 6-10],

$$(U - c)(\phi'' - \alpha^2 \phi) - U''\phi = 0, \tag{5}$$

which is known as the frictionless stability equation, or Rayleigh's equation.

3. Rayleigh's necessary condition for instability of inviscid flows

Re-writing Eq.(5) as

$$\phi''-\alpha^2\phi-\frac{U''}{U-c}\phi=0. \qquad (6)$$

Next, we multiply Eq.(6) by its complex conjugate, then obtain [1, 6-10]

$$\phi^*\phi''-\alpha^2\phi\phi^*-\frac{U''}{U-c}\phi\phi^*=0. \qquad (7)$$

Then, integrating the above equation by part over y, the imaginary part of the resulting equation is

$$c_i\int_{y_1}^{y_2}\frac{U''|\phi|^2}{|U-c|^2}dy=0. \qquad (8)$$

If the disturbance is amplified, c_i is larger than zero. It can be seen that for the equality to be valid U'' has to change sign over the integration space. Thus, there should be at least one point over the distance between y_1 and y_2 at which U''=0. In other words, it is necessary that there is an inflection point on the velocity profile for flow instability. This is the famous **Rayleigh Theorem [1].**

4. Re-visiting: Solution of Rayleigh Equation

In the solution of Eqs.(6) to (8), it has been assumed (or implicitly assumed) that the shape of the velocity profile of the base flow is curved (see Fig.1). This assumption is not consistent with the governing equation of inviscid flow, i.e., Euler equation.

Firstly, before analyzing the stability of a linear disturbance, the base flow should be first solved. This can be done by the following. Applying the Euler equation and the inviscid wall boundary condition (slip condition) to the given geometry, the solution of the inviscid base flow between two parallel walls is a uniform flow (see Fig.2),

$U=C$ (C is a constant). (9)

Thus, we obtain $U'=0; \quad and \quad U''=0$. (10)

Introducing $U''=0$ into Eq.(5), we have

$$(U-c)(\phi''-\alpha^2\phi)=0. \qquad (11)$$

There are two possible solutions for Eq.(11),

$$(U - c) = 0, \quad (12)$$

and

$$(\phi'' - \alpha^2 \phi) = 0. \quad (13)$$

For Eq.(12), we have the solution

$$c = U = C \qquad \text{(C is a constant).} \quad (14)$$

For Eq.(13), $\phi'' = \alpha^2 \phi$, the general solution to this differential equation is then

$$\phi(y) = B_1 e^{\alpha y} + B_1 e^{-\alpha y} \quad (B_1 \text{ and } B_2 \text{ are constants).} \quad (15)$$

Apply the boundary condition, y=0, $\phi = 0$; y=2h, $\phi = 0$ (see Fig.2); then, it is found that Eq.(15) has no solution except $\phi(y) \equiv 0$. Thus, it is concluded that Eq.(13) has no solution under the given boundary conditions.

Therefore, the linear perturbation equation of inviscid flow between two parallel walls has only one solution $c = U = C$. There is no information on the variation of the amplitude of the disturbance. This means that the amplitude of the disturbance is kept constant in the uniform flow, and it is neutral.

As a result, since the base flow is $U = C$ in the whole domain, Eq.(6) can not be obtained for inviscid parallel flows. Thus, the Rayleigh criterion may not exist. As such, the Rayleigh theorem on the inflectional instability of inviscid parallel flow is incorrect.

When one analyzes the flow stability, the base profile should be given first in terms of the governing equation. Then, the corresponding perturbation equation should be studied at the appropriate boundary conditions to observe the amplification or decay of the imposed disturbance [9]. If the flow between the two parallel walls is inviscid, the base flow is only a uniform flow. If the shape of the velocity profile is curved as shown in Fig.1, then there must be variation of pressure along y direction, $\partial p / \partial y \neq 0$. This would violate the Rayleigh equation which is based on the assumption of parallel flows. Only the viscous flow which is governed by Navier-Stokes equation can produce the velocity profile with curvature as shown in Fig.1. In viscous parallel flows, the variation of velocity along y direction is balanced by the viscous friction force in the Navier-Stokes equation. In inviscid parallel flows, the variation of velocity along y direction must be balanced by pressure in the Euler equation.

In the derivation of the Rayleigh theorem [1-10], the base flow is assumed a viscous profile, while the linear disturbance equation employs inviscid linear equation. This is obviously contradictable.

Strictly, the existence of inflection point on the velocity profile is the necessary and sufficient condition for turbulent transition, for pressure driven flows [17]. For shear driven flows, this is neither a necessary condition, nor a sufficient condition for turbulent transition [17].

5. Conclusions

The base flow of inviscid flow between two parallel walls is the uniform flow. The first and second derivatives are both zero everywhere. The solution of the linear perturbed equation (Rayleigh equation) is c=U. That is, the propagating speed of disturbance is same as U. The disturbance is neutral. As a result, it is suggested that the classical Rayleigh theorem is incorrect.

The key problem should be pointed out that when one analyzes the flow stability, the profile of the base flow and the disturbance should be governed by the same basic equation, i.e., Euler, or Navier-Stokes equation.

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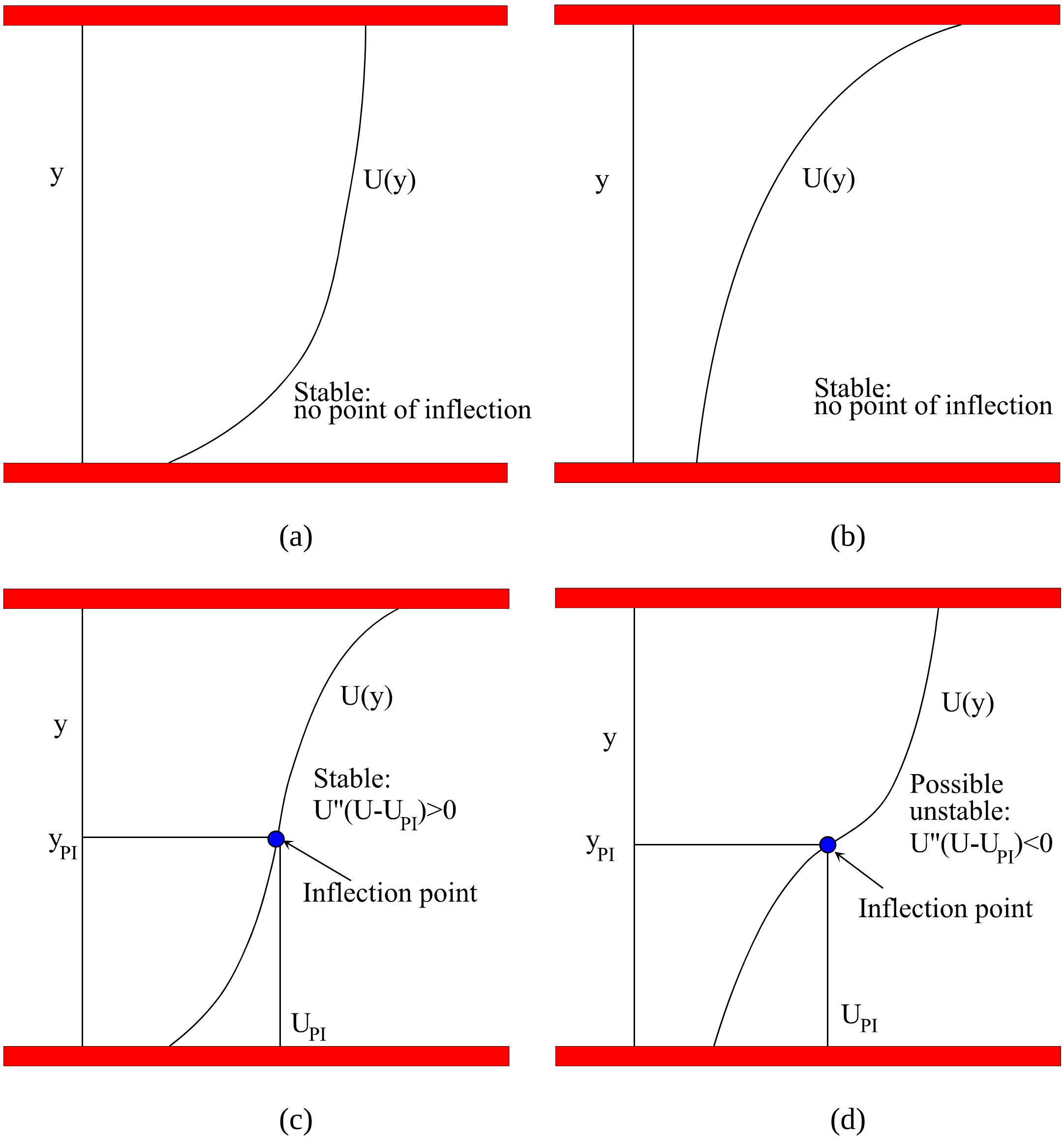

Fig.1 Four candidate inviscid velocity profiles evaluated from Rayleigh Theorem (1880) and Fjϕrtoft Theorem (1950) (adapted from White, 1991; and Drazin and Reid, 2004).

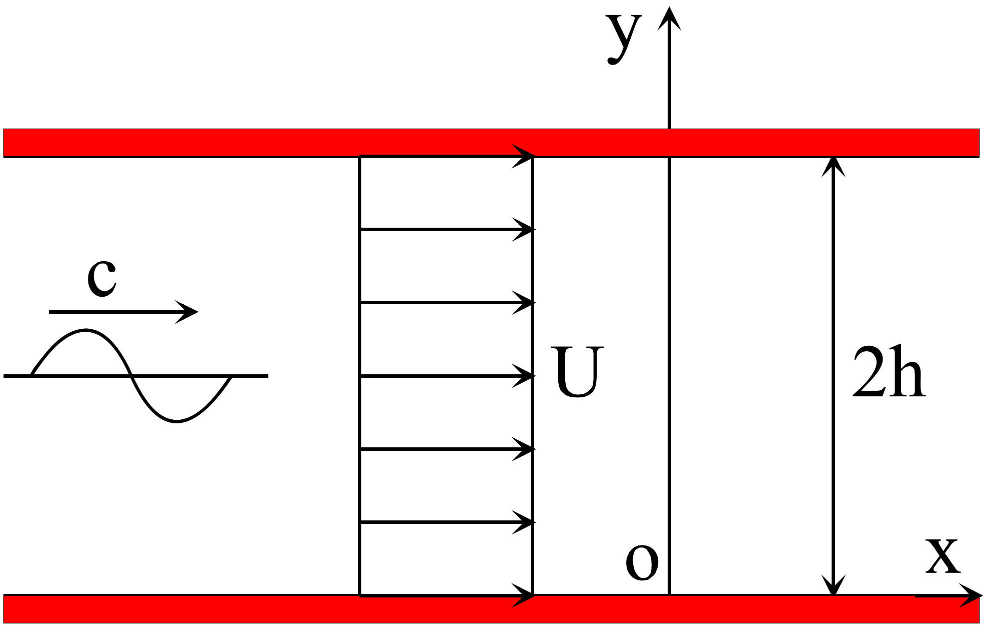

Fig.2 Inviscid parallel flow between two parallel walls; the base flow is a uniform flow.