On a possible definition of the moving preferred basis

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Abstract

There are many formalisms to describe quantum decoherence. However, many of them give a non general and ad hoc definition of "pointer basis" or "moving preferred basis", and this fact is a problem for the decoherence program. In this paper we will consider quantum systems under a general theoretical framework for decoherence and present a tentative very general definition of the moving preferred basis. which is implemented in a well known open system model. The obtained decoherence and the relaxation times are defined and compared with those of this model.

Contents

I. INTRODUCTION

From the appearance of the quantum mechanics many attempts have been made to recover the laws of the classic mechanics through some classic limit. The more common scheme of this type includes the *quantum decoherence*[1]. This process is in charge to erase the terms of interference of the density matrix, that are classically inadmissible, since they prevent the use of a classical (boolean) logic. In addition, decoherence leads to the rule that selects the candidates for classic states.

As it is pointed out in the brief historical summary of paper [1], three periods can be schematically identified in the development of the general program of decoherence [2]. A first period, when the arrival to the equilibrium of irreversible systems was studied. During this period, authors as van Kampen, van Hove, Daneri, et al. developed a formalism for explaining the decoherence phenomenon that was not successful at the time but it established the bases of this study. The main problem of this period was that too long *decoherence times* t_D were found, if compared with the experimental ones (which was of the order of the time in which the systems reach equilibrium, i. e. *relaxation time t_R*). In a second period the decoherence in open systems was studied, the main characters of this period were Zeh and Zurek. In their works, the decoherence is an interaction process between an open quantum system and its environment. This process, called *Environment-Induced Decoherence* (EID), determines, case by case, which is the privileged basis, usually called *moving preferred basis* where decoherence takes place in a decoherence time $t_D \ll t_R$ and it defines the observables that acquire classic characteristics and they could be interpreted in some particular cases as properties that obey a Boolean logic. This is the orthodox position in the subject [3]. So decoherence times in this period were much smaller, solving the problem of the first period. Recently, in a third period it becomes evident that dissipation was not a necessary condition for decoherence [4] and the study of the arrival to equilibrium of closed systems was also considered. We will not discuss closed systems in this paper but for the sake of completeness we will make only some comments. Closed system will be discussed at large elsewhere.

In this work we focus the attention on EID, which is a well known theory, with well established experimental verifications, which makes unnecessary any further explanation.

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[1] We will call *decoherence* to the vanishing of the off-diagonal terms in a properly specified basis. We will call *relaxation* to the decoherence in a final equilibrium basis, i.e. typical equilibrium

On the contrary other formalisms are not so well established, but they must be taken into account for the sake of completeness ([5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

In this paper, we will introduce a tentative definition of the moving preferred basis where the state decoheres in a very short time t_D, So the main problem of the first period is solved in a convenient and general way. Our main aim is to present a new conceptual perspective that will clarify some points that still remain rather obscure in the literature on the subject, (e. g. the definition of the moving preferred basis and the use of the pole technique)

A. The General Theoretical Framework for Decoherence

In previous works we have resumed the common characteristics of the different approaches of decoherence, which suggest the existence of a general framework for decoherence within which these approaches can all be framed (see [1], [14] and [15]). According to this general framework, that was developed in [14], and will be completed in future papers, decoherence is just a particular case of the general problem of irreversibility in quantum mechanics. Since the quantum state $\rho(t)$ follows a unitary evolution, it cannot reach a final equilibrium state for $t \to \infty$. Therefore, if the non-unitary evolution towards equilibrium is to be accounted for, a further element has to be added to this unitary evolution. The way to introduce this non-unitary evolution must include the splitting of the whole space of observables $\mathcal{O}$ into the relevant subspace $\mathcal{O}_R \subset \mathcal{O}$ and the irrelevant subspace. Once the essential role played by the selection of the relevant observables is clearly understood, the phenomenon of decoherence can be explained in four general steps:

1. **First step:** The space $\mathcal{O}_R$ of relevant observables is defined .

2. **Second step:** The expectation value $\langle O_R \rangle_{\rho(t)}$, for any $O_R \in \mathcal{O}_R$, is obtained. This step can be formulated in two different but equivalent ways:

   - A coarse-grained state $\rho_R(t)$ is defined by

   $$\langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_R(t)} \tag{1}$$

   for any $O_R \in \mathcal{O}$, and its non-unitary evolution (governed by a master equation) is computed (this step is typical in EID).

- $\langle O_R \rangle_{\rho(t)}$ is computed and studied as the expectation value of O_R in the state $\rho(t)$. This is the generic case for other formalisms.

3. **Third step:** It is proved that $\langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_R(t)}$ reaches a final equilibrium value $\langle O_R \rangle_{\rho_*}$, then

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_*}, \qquad \forall O_R \in \mathcal{O}_R \tag{2}$$

This also means that the coarse-grained state $\rho_R(t)$ evolves towards a final equilibrium state:

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho_R(t)} = \langle O_R \rangle_{\rho_{R*}}, \qquad \forall O_R \in \mathcal{O}_R \tag{3}$$

The characteristic time for these limits is the t_R, the *relaxation time.*

4. **Fourth step:** Also a *moving preferred basis* $\{|\widetilde{j(t)}\rangle\}$ must be defined as we will see in section I.B. This basis is the eigen basis of certain state $\rho_P(t)$ such that

$$\lim_{t \to \infty} \langle O_R \rangle_{(\rho_R(t) - \rho_P(t))} = 0, \qquad \forall O_R \in \mathcal{O}_R \tag{4}$$

The characteristic time for this limit is the t_D, the *decoherence time.*

The final equilibrium state ρ_* is obviously diagonal in its own eigenbasis, which turns out to be the final preferred basis. But, from eqs. (2) or (3) we cannot say that $\lim_{t \to \infty} \rho(t) = \rho_*$ or $\lim_{t \to \infty} \rho_R(t) = \rho_{R*}$. Then, the mathematicians say that the unitarily evolving quantum state $\rho(t)$ of the whole system *only has a weak limit,* symbolized as:

$$W - \lim_{t \to \infty} \rho(t) = \rho_* \tag{5}$$

equivalent to eq. (2). As a consequence, the coarse-grained state $\rho_R(t)$ also has a weak limit, as follows from eq.(3):

$$W - \lim_{t \to \infty} \rho_R(t) = \rho_{R*} \tag{6}$$

equivalent to eq. (3). Also

$$W - \lim_{t \to \infty} (\rho_R(t) - \rho_P(t)) = 0 \tag{7}$$

These weak limits mean that, although the off-diagonal terms of $\rho(t)$ never vanish through the unitary evolution, the system decoheres *from an observational point of view,* that is, from the viewpoint given by any relevant observable $O_R \in \mathcal{O}_R$.

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From this general perspective, the phenomenon of destructive interference, that produced the decoherence phenomenon is relative, because the off-diagonal terms of $\rho(t)$ and $\rho_R(t)$ vanish only from the viewpoint of the relevant observables $O_R \in \mathcal{O}_R$, and the superselection rule that precludes superpositions only retains the states defined by the corresponding decoherence bases as we will see. The only difference between EID and other formalisms for decoherence is the selection of the relevant observables (see [1] for details):

. In EID the relevant observables are those having the following form:

$$O_R = O_S \otimes I_E \in \mathcal{O}_R \tag{8}$$

where O_S are the observables of the system and I_E is the identity operator of the environment. Then eq. (1) reads

$$\langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_R(t)} = \langle O_S \rangle_{\rho_S(t)}, \text{ where } \rho_S(t) = Tr_E \rho(t)$$

where in $Tr_E \rho(t)$ we have "traced away" the environment. In the other formalisms other restriction in the set of observables can be introduced

B. The definition of moving preferred basis

The moving preferred basis was introduced, case by case in several papers (see [4]) in a non systematic way. On the other hand in references [16] and [17] Roland Omnès introduces a rigorous and almost general definition of the moving preferred basis based in a reasonable choice of the relevant observables, and other physical considerations.

In this paper we will introduce an alternative general definition to define this basis: As it is well known the eigen values of the Hamiltonian are the inverse of the characteristic frequencies of the unitary evolution of an oscillatory system. Analogously, for non-unitary evolutions, the poles of the complex extension of the Hamiltonian are the *catalogue* of the decaying modes of these non-unitary evolutions towards equilibrium (see [18]). This will be the main idea to implement the definition of our moving preferred basis. i. e. we will use only the poles nearest to the real axis and we will eliminate the other poles in order to obtain an adiabatic-like definition.

We will compare and try to unify these two methods in the future. Really we already have begin this approach with Omnès in section III.

C. Organization of the paper.

. In Section I we have introduced a general framework for decoherence. Some candidates for moving preferred basis are introduced in section II, which is implemented in two models and the times of decoherence and relaxation and the moving preferred basis in these models are defined. In principle these definitions can be used in EID and probably for other formalisms. In Section III we will define the moving preferred basis of our formalism and we will present the paradigmatic EID: Omnès (or Lee-Friedrich) model. and show that the pole method yields the usual results. Finally in Section IV we will draw our conclusions. An appendix completes this paper.

II. TOWARDS A GENERAL DEFINITION FOR THE MOVING PREFERRED BASIS.

A. Introduction and review

In this section we will try to introduce a very general theory for the moving preferred basis in the case of a general distribution of poles and for *any relevant observable space* $\mathcal{O}_R$. Then it is necessary to endow the coordinates of observables and states in the Hamiltonian basis $\{|\omega\rangle\}$ (i.e. the functions $O(\omega, \omega')$ and $\rho(\omega.\omega')$) with extra analytical properties in order to find the definition of a moving preferred basis in the most, general, convincing, and simplest way. It is well known that this move is usual in many chapters of physic e. g. in the scattering theory (see [19]).

It is also well known that evolution towards equilibrium has two phases.

i.- A exponential dumping phase that can be described studying the analytical continuation of the Hamiltonian into the complex plane of the energy (see [18], [20], [21], [22], [23], [24]), a fact which is also well known in the scattering theory.

ii.- A final decaying inverse-polynomial in t^{-1} known as the long time of Khalfin effect (see [25], [26]), which is very weak and difficult to detect experimentally (see [27])[2].

These two phases will play an important role in the definition of the moving preferred basis. They can be identified by the theory of analytical continuation of vectors, observables

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[2] There is also an initial ($t = 0$) non exponential Zeno-period which is unimportant for this paper

and states. To introduce the main equations we will make a short abstract of papers [18] and [23].

B. Analytic continuations in the bra-ket language.

We begin reviewing the analytical continuation for pure states. Let the Hamiltonian be $H = H_0 + V$ where the free Hamiltonian H_0 satisfies ([18]. eq. (8) or [23])

$$H_0|\omega\rangle = \omega|\omega\rangle, \ \langle\omega|H_0 = \omega\langle\omega|, \quad 0 \le \omega < \infty$$

and (see [18]. eq. (9))

$$I = \int_0^\infty d\omega|\omega\rangle\langle\omega|, \ \langle\omega|\omega'\rangle = \delta(\omega - \omega') \tag{9}$$

Then (see [18]. eq. (10))

$$H_0 = \int_0^\infty \omega|\omega\rangle\langle\omega|d\omega$$

and (see [18]. eq. (11))

$$H = H_0 + V = \int_0^\infty \omega|\omega\rangle\langle\omega|d\omega + \int_0^\infty d\omega \int_0^\infty d\omega' V_{\omega\omega'}|\omega\rangle\langle\omega'| = \int_0^\infty \omega|\omega^+\rangle\langle\omega^+|d\omega \tag{10}$$

where the $|\omega^+\rangle$ are the eigenvectors of H, that also satisfy eq. (9). The eigen vectors of H are given by the Lippmann-Schwinger equations (see [18]. eq. (12) and (13))

$$\langle\psi|\omega^+\rangle = \langle\psi|\omega\rangle + \langle\psi|\frac{1}{\omega + i0 - H}V|\omega\rangle, \quad \langle\omega^+|\varphi\rangle = \langle\omega|\varphi\rangle + \langle\omega|V\frac{1}{\omega - i0 - H}|\varphi\rangle \tag{11}$$

Let us now endow the function of ω with adequate analytical properties (see [19]). E.g. let us consider that the state $|\varphi\rangle$ (resp. $\langle\psi|$) is such that it does not create poles in the complex extension of $\langle\omega|\varphi\rangle$ (resp. in $\langle\psi|\omega\rangle$) and therefore this function is analytic in the whole complex plane. This is a simplification that we will be forced to abandon in some cases as we will see in remark 1 of section II.E. Moreover we will consider that the complex extensions of function $\langle\omega^+|\varphi\rangle$ (resp. $\langle\psi|\omega^+\rangle$) is analytic but with just one simple pole at z_0 $= \omega_0 - \frac{i}{2}\gamma_0, \gamma_0 > 0$ in the lower halfplane (resp. another pole $z_0^* = \omega_0 + \frac{i}{2}\gamma_0, \gamma_0 > 0$ on the upper halfplane) (see [10] for details [3]). There can be many of such poles but , by now, we

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[3] This is a toy model with just one pole and the Khalfin effect. More general models , with two poles, will be considered in the next subsection. The pole corresponds to the residue that we can compute with the curve C and the Khalfin effect to the integral along the curve Γ of Figure 1.
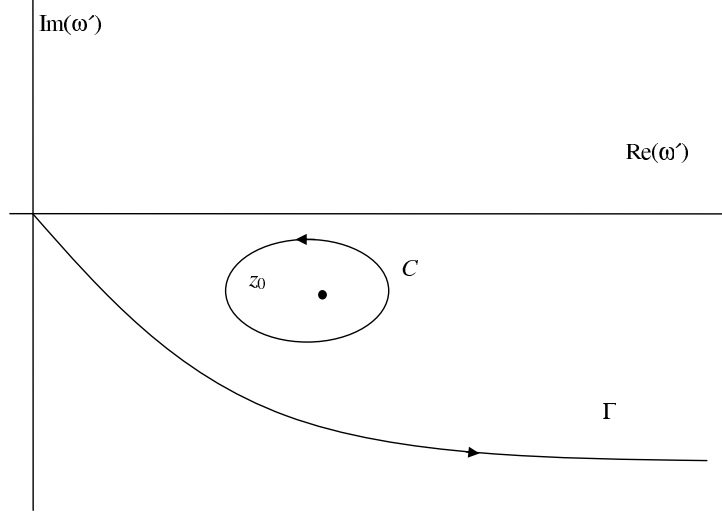
FIG. 1: Complex contour Γ on the lower complex energy plane usedin our evaluation of integrals. The "energy" z_0 is the pole that we assume to be simple.

will just consider one pole for simplicity, being the generalization straightforward. Then we make an analytic continuation of the positive ω axis to the curve Γ of the lower half-plane as in Figure 1[4].

Then (see [18]. eq. (29)) we can define

$$\langle \widetilde{f_0}|\varphi \rangle \equiv cont_{\omega' \to z_0} \langle \omega'^+|\varphi \rangle, \quad \langle \psi|f_0 \rangle \equiv (-2\pi i)cont_{\omega' \to z_0}(\omega' - z_0)\langle \psi|\omega^+ \rangle$$

$$\langle \widetilde{f_{z'}}|\varphi \rangle \equiv cont_{\omega' \to z'} \langle \omega'^+|\varphi \rangle, \quad \langle \psi|f_{z'} \rangle \equiv cont_{\omega' \to z} \langle \psi|\omega^+ \rangle, \ z' \in \Gamma, \forall \ |\varphi \rangle \ \langle \psi| \qquad (12)$$

and (see [18]. eq. (31))

$$\langle \psi|\widetilde{f_0} \rangle \equiv cont_{\omega \to z_0^*} \langle \psi|\omega^+ \rangle, \quad \langle f_0|\varphi \rangle \equiv (2\pi i)cont_{\omega' \to z_0^*}(\omega - z_0)\langle \omega^+|\varphi \rangle$$

$$\langle \psi|\widetilde{f_{z'}} \rangle \equiv cont_{\omega \to z} \langle \psi|\omega^+ \rangle, \quad \langle f_z|\varphi \rangle \equiv cont_{\omega \to z} \langle \omega^+|\varphi \rangle, \ z \in \Gamma, \forall \ |\varphi \rangle \ \langle \psi| \qquad (13)$$

where *cont* means analytic continuation.

Finally it can be proved that (see [18])

$$H = z_0|f_0 \rangle \langle \widetilde{f_0}| + \int_\Gamma z|f_z \rangle \langle \widetilde{f_z}|dz$$

a simple extension of the eigen-decomposition of H to the complex plane

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[4] All the figures are merely illustrative and not the numerical solution of some precise problem.

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C. Analytical continuation in the observables and states language.

We could repeat what we have said about the pure states and the Hamiltonian with the states, observables, and the Liouvillian operator L (see a review in [28]). But we prefer to follow the path of [18] and keep the Hamiltonian framework and discuss the analytical continuation of $\langle O \rangle_{\rho(t)}$, that we will also symbolize as $(\rho(t)|O)$. In fact from section I.A we know that this scalar is the main character of the play so we will study its analytical properties ad nauseam.

So let us call (see [18]. eq. (42))

$$|\omega) = |\omega\rangle\langle\omega|, \text{ and } |\omega, \omega') = |\omega\rangle\langle\omega'|$$

Then a generic relevant observable is $O_R \in \mathcal{O}_R$ (see [23] eq. (42) or [18]. eq. (42))

$$O_R = |O_R) = \int d\omega O(\omega)|\omega) + \int d\omega \int d\omega' O(\omega, \omega')|\omega, \omega') \tag{14}$$

and the generic states is ([23] eq. (45) or [18]. eq. (45))

$$\rho_R = (\rho_R| = \int d\omega \rho(\omega)\widetilde{(\omega|} + \int d\omega \int d\omega' \rho(\omega, \omega')\widetilde{(\omega, \omega'|} \tag{15}$$

where $\widetilde{(\omega|}, \widetilde{(\omega, \omega'|}$ will be defined in eqs. (19) and (20) but in the particular case $V = 0$ (see also [23] eq. (44) or [18]. eq. (45)). Then

$$\widetilde{(\omega|}O_R) = O(\omega), \quad \widetilde{(\omega, \omega'|}O_R) = O(\omega, \omega')$$

We will keep the treatment as general as possible, i.e. O_R would be any observable such that $O_R \in \mathcal{O}_R$ and ρ_R any state $\rho_R \in \mathcal{O}'_R$ [5]. In fact, in the next subsection we will only consider the generic mean value $(\rho_R(t)|O_R)$ for two paradigmatic model below. Model 1 with just one pole and the Khalfin effect and Model 2 with two poles.

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[5] Namely, even more general than the choice of EID $O_R = O_S \otimes I_E$ and more general than those of other formalisms. This is why we can find the moving preferred basis in a general case containing EID as particular case. Anyhow the analyticity conditions must also be satisfied. In the case of EID we can substitute O_R by O_S and $\rho_R(t)$ by $\rho_S(t)$ in some formulae e.g. $(\rho_R(t)|O_R) = (\rho_S(t)|O_S)$

D. Model 1. One pole and the Khalfin term:

It can be proved (cf. ([18]) eq. (67)) that the evolution equation of the mean value $(\rho(t)|O)$ is

$$\langle O_R \rangle_{\rho(t)} = (\rho(t)|O_R) = (\rho_R(t)|O_R) = \int_0^\infty \rho^*(\omega)O(\omega)\,d\omega + \int_0^\infty \int_0^\infty \rho^*(\omega,\omega')O(\omega,\omega')\,e^{i\frac{\omega-\omega'}{\hbar}t}\,d\omega d\omega' \tag{16}$$

i.e. this mean value in the case $V \neq 0$ reads

$$(\rho_R(t)|O_R) = \int d\omega(\rho(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O_R) + \int d\omega \int d\omega' e^{\frac{i}{\hbar}(\omega-\omega')t}(\rho_R(0)|\Phi_{\omega\omega'})(\widetilde{\Phi_{\omega\omega'}}|O_R) \tag{17}$$

Where $O_\omega = (\widetilde{\Phi_\omega}|O_R)$, $O_{\omega\omega'} = (\widetilde{\Phi_{\omega\omega'}}|O_R)$, $\rho_\omega = (\rho_R(0)|\Phi_\omega)$, $\rho_{\omega\omega'} = (\rho_R(0)|\Phi_{\omega\omega'})$. These Φ vectors are defined in eqs. (19) and (20). Then, if we endow the functions with analytical properties of subsection B and there is just one pole z_0 in the lower halfplane, we can prove ([18] eq. (70)) that

$$(\rho_R(t)|O_R) = \int d\omega(\rho_R(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O_R) + e^{\frac{i}{\hbar}(z_0^*-z_0)t}(\rho_R(0)|\Phi_{00})(\widetilde{\Phi_{00}}|O_R)$$

$$+ \int_\Gamma dz' e^{\frac{i}{\hbar}(z_0^*-z')t}(\rho_R(0)|\Phi_{0z'})(\widetilde{\Phi_{0z'}}|O_R) + \int_{\Gamma^*} dz e^{\frac{i}{\hbar}(z-z_0)t}(\rho_R(0)|\Phi_{0z})(\widetilde{\Phi_{0z}}|O_R)$$

$$+ \int_{\Gamma^*} dz \int_\Gamma dz' e^{\frac{i}{\hbar}(z-z')t}(\rho_R(0)|\Phi_{zz'})(\widetilde{\Phi_{zz'}}|O_R) \tag{18}$$

where $z_0 = \omega_0 - \frac{i}{2}\gamma_0$, $\gamma_0 > 0$ and where $|\Phi_z)$, $(\widetilde{\Phi_z}|$, $|\Phi_{zz'})$, and $(\widetilde{\Phi_{zz'}}|$ are the analytical continuation in the lower half-plane of (see ([18] eq. (54))

$$|\Phi_\omega) = |\omega^+\rangle\langle\omega^+|, \quad (\widetilde{\Phi_\omega}| = \widetilde{(\omega|}, \quad |\Phi_{\omega\omega'}) = |\omega^+\rangle\langle\omega'^+|, \tag{19}$$

and

$$(\widetilde{\Phi_{\omega\omega'}}| = \int d\varepsilon[\langle\omega^+|\varepsilon\rangle\langle\varepsilon|\omega'^+\rangle - \delta(\omega-\varepsilon)\delta(\omega'-\varepsilon)](\widetilde{\varepsilon}| + \int d\varepsilon \int d\varepsilon'\langle\omega^+|\varepsilon\rangle\langle\varepsilon'|\omega'^+\rangle\widetilde{(\varepsilon,\varepsilon'}| \tag{20}$$

and where z_0 is the simple pole of Figure 1 in the lower half-plane. $|\Phi_z)$, $(\widetilde{\Phi_z}|$, $|\Phi_{zz'})$, and $(\widetilde{\Phi_{zz'}}|$ can be defined as in the case of eq. (12) and (13). The $|\Phi_z)$, $(\widetilde{\Phi_z}|$, $|\Phi_{zz'})$, and $(\widetilde{\Phi_{zz'}}|$ can also be defined as a simple generalization of the vectors $|f_0\rangle$, $\langle\widetilde{f_0}|$, $|f_z\rangle$, and $\langle\widetilde{f_z}|$ ([18]. eq. (42)). Then the eqs. (19) and (20) allow us to compute the limits (2) and (3) for any $\rho_R(0)$.

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Therefore we can conclude than the last four terms of equation (18) vanish with characteristic times

$$\frac{\hbar}{\gamma_0}; \frac{2\hbar}{\gamma_0}; \frac{2\hbar}{\gamma_0}; \infty \qquad (21)$$

respectively. Let us observe that

i-. The vanishing of the second, third, and fourth therms of eq. (18) are *exponential decaying.* This will also be the case in more complicated models with many poles.

ii.- The ∞ means that the evolution of the last term of this equation corresponds to a polynomial in t^{-1}, i. e. to the *Khalfin evolution.* This is a very weak effect detected in 2006 [27]. If there is a finite number of poles and the curve Γ that yields below these poles the contribution of the integral along Γ corresponds to the Khalfin effect [6]. A closed system model for Khalfin effect can be found in [29], section 6, and an EID-like model in [30], section 5.

Now for times $t > t_D = \frac{\hbar}{\gamma_0}$, eq. (18) reads

$$(\rho_R(t)|O_R) = \int d\omega (\rho(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O) + \int_{\Gamma^*} dz \int_\Gamma dz' e^{\frac{i}{\hbar}(z-z')t}(\rho(0)|\Phi_{zz'})(\widetilde{\Phi_{zz'}}|O) \qquad (22)$$

since for $t > t_D = \frac{\hbar}{\gamma_0}$ the poles term has vanished[7].

Let us diagonalize $\rho_R(t)$ of eq. (18) as [8]

$$\rho_R(t) = \sum_i \rho_i(t)|i(t)\rangle\langle i(t)| \qquad (23)$$

where $\{|i(t)\rangle\}$ is the moving eigenbasis of $\rho_R(t)$.

Then let us define a state $(\rho_P(t)|$, the *preferred state,* such that, *for all times,* it would be

$$(\rho_P(t)|O_R) = \int d\omega (\rho(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O) + \int_{\Gamma^*} dz \int_\Gamma dz' e^{\frac{i}{\hbar}(z-z')t}(\rho(0)|\Phi_{zz'})(\widetilde{\Phi_{zz'}}|O) \qquad (24)$$

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[6] If the there is an infinite set of poles at z_i, with imaginary part $-\frac{1}{2}\gamma_i$ such that $lim_{i \to \infty}\gamma_i = \infty$, then we can choose a curve Γ_j below the poles a $\gamma_1, \gamma_2, ...\gamma_j$. Then the integral along the curve Γ_j contains the effect of the poles $\gamma_{j+1}, \gamma_{j+2}, ...$and the Khalfin effect. Thus we can choose the curve Γ_j in such a way that the decaying times corresponding to these poles, $t_{j+n} = \hbar/\gamma_{j+n}$ would be so small that could be neglected.

[7] Since t_D is just an order of magnitude we consider that the three first imaginary parts of eqs. (21) and (29) are essentially equivalent.

[8] Here, for the sake of simplicity, we will use sum instead of integral, as we will do below in all cases of diagonalization. Moreover, in many cases, the O_R or the initial conditions may just be expanded in a discrete basis of Hilbert space (see below).
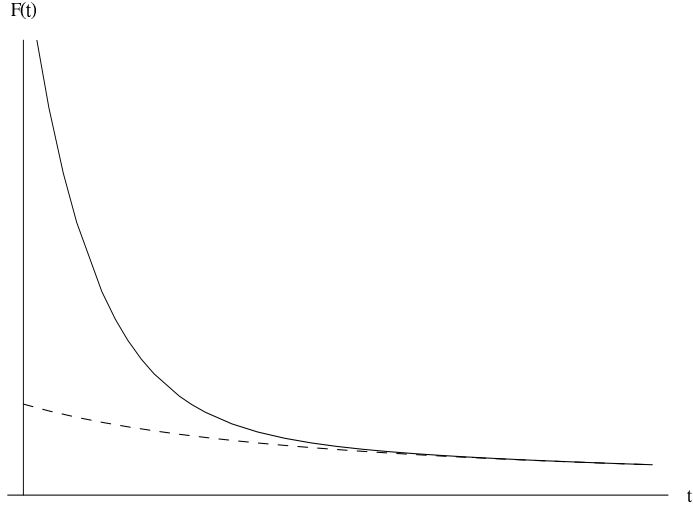
FIG. 2: Evolution of $F(t)$ (solid line), $F_{Khalfin}(t)$ (dashed line) and their coincidence limit at t_D.

So $\rho_P(t)$ is a state that evolves in a model with no poles and with only the Khalfin term . These evolutions exist and can be found using an adequate interaction [9].

It is quite clear that for $t > t_D$ $\rho_R(t) \neq \rho_P(t)$ while for $t < t_D$ $\rho_R(t) = \rho_P(t)$ and that for $t \to t_D$ $\rho_R(t) \to \rho_P(t)$ and also all their derivatives.

The eigen states of the $\rho_P(t)$ are those that we will choose for the moving decoherence basis. In fact, diagonalizing $\rho_P(t)$ we have

$$\rho_P(t) = \sum_j \rho_j(t) |\widetilde{j(t)}\rangle \langle \widetilde{j(t)}| \tag{25}$$

and when $t \to t_D = \frac{\hbar}{\gamma_0}$ we have that $\rho(t) \to \rho_P(t)$ so from eqs. (23) and (25) we see that the eigenbasis of $\rho(t)$ and $\rho_P(t)$ also converge

$$\{|i(t)\rangle\} \to \{|\widetilde{j(t)}\rangle\} \tag{26}$$

Namely the basis $\{|i(t)\rangle\}$ converge to $\{|\widetilde{j(t)}\rangle\}$ and therefore $\rho_R(t)$ becomes diagonal in $\{|\widetilde{j(t)}\rangle\}$. Thus $\{|\widetilde{j(t)}\rangle\}$ is our definition for the *moving preferred basis* for this case. Since $\rho_R(t)$ becomes diagonal in the just defined preferred basis $\{|\widetilde{j(t)}\rangle\}$ when $t \to t_D$ and $t_D = \frac{\hbar}{\gamma_0}$ is really the definition of the decoherence time. In this model the relaxation time t_R is the corresponding to the Khalfin term, i.e. an extremely long time so

$$t_D \ll t_R \tag{27}$$

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[9] All these formulas are confirmed by the coincidence of results with other methods: e.g. those used to study a $^{208}Pb(2d_{5/2})$ proton state in a Woods-Saxon potential (see [18] Figure 3).

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E. Model 2: Two poles are considered and the Khalfin term is neglected.

The Khalfin term is so small (see [27]) that can be neglected in most of the experimental cases. So let us consider the case of two poles z_0 and z_1 (and no relevant Khalfin term) where eq. (18) reads:

$$(\rho_R(t)|O_R) = \int d\omega (\rho_R(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O_R) + e^{\frac{i}{\hbar}(z_0^*-z_0)t}(\rho_R(0)|\Phi_{00})(\widetilde{\Phi_{00}}|O_R)+$$

$$e^{\frac{i}{\hbar}(z_1^*-z_0)t}(\rho_R(0)|\Phi_{10})(\widetilde{\Phi_{10}}|O_R)+e^{\frac{i}{\hbar}(z_0^*-z_1)t}(\rho_R(0)|\Phi_{01})(\widetilde{\Phi_{01}}|O_R)+e^{\frac{i}{\hbar}(z_1^*-z_0)t}(\rho_R(0)|\Phi_{11})(\widetilde{\Phi_{11}}|O_R)$$

$$(28)$$

where $z_0 = \omega_0 - \frac{i}{2}\gamma_0$, $\gamma_0 > 0$, $z_1 = \omega_1 - \frac{i}{2}\gamma_1$, $\gamma_1 > 0$, and we will also consider that $\gamma_0 \ll \gamma_1$ (see [31] section 3, for details). Then the four characteristic times (21) now read.

$$\frac{\hbar}{\gamma_0}; \frac{\hbar}{\gamma_1+\gamma_0}; \frac{\hbar}{\gamma_1+\gamma_0} \approx \frac{\hbar}{\gamma_1} \tag{29}$$

Now for times $t > t_D = \frac{\hbar}{\gamma_1}$, eq. (22) reads

$$(\rho_R(t)|O_R) = \int d\omega(\rho_R(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O_R) + e^{\frac{i}{\hbar}(z_0^*-z_0)t}(\rho_R(0)|\Phi_{00})(\widetilde{\Phi_{00}}|O_R)$$

and we can define a state $(\rho_P(t)|$ such that, for *all times*, it would be

$$(\rho_P(t)|O_R) = \int d\omega(\rho_R(0)|\Phi_\omega)(\widetilde{\Phi_\omega}|O_R) + e^{\frac{i}{\hbar}(z_0^*-z_0)t}(\rho_R(0)|\Phi_{00})(\widetilde{\Phi_{00}}|O_R) \tag{30}$$

Repeating the reasoning of eqs. (22) to (26) we can see that, diagonalizing this last equation, as in eq. (??), we obtain the moving preferred basis. Then in this case we see that the relaxation is obtained by an exponential dumping (not a Khalfin term) and

$$t_R = \frac{\hbar}{\gamma_0} \gg t_D = \frac{\hbar}{\gamma_1} \tag{31}$$

Again, in this case when $t \to t_D = \frac{\hbar}{\gamma_0}$ we have that $\rho_R(t) \to \rho_P(t)$ so once more we reach eq. (26). Namely $\rho(t)$ becomes diagonal in the moving preferred basis in a time t_D.

1. Remark

Before considering the many poles case let us make some general remarks.

i.- Let us observe that some $(\widetilde{\Phi_\omega}|O_R)$, $(\widetilde{\Phi_{0z'}}|O_R)$, $(\widetilde{\Phi_{0z}}|O_R)$ and $(\widetilde{\Phi_{zz'}}|O_R)$ may be zero, depending in the observable O_R, so, in the case of many poles, may be some poles can

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be detected by O_R and others may not be detected and disappear from the formulae (see Appendix).

This also is the cases for the initial conditions: $(\rho_R(0)|\Phi_\omega)$, $(\rho_R(0)|\Phi_{0z'})$, $(\rho_R(0)|\Phi_{0z})$, and $(\rho_R(0)|\Phi_{zz'})$ may be zero. But also the O_R or the $\rho_R(0)$ may create some poles. So some poles may be eliminated or created by the observables or the initial conditions while others may be retained. But in general we will choose O_R and $\rho_R(0)$ in such a way that they would neither create or eliminate poles.

ii.- From what we have learned in both models (see eqs. (27) and (31)) we always have

$$t_D < t_R \tag{32}$$

III. THE GENERAL CASE

A. Relaxation, decoherence, and moving preferred basis.

Let us now consider the general case of a system with $N + 1$ poles at $z_i = \omega'_i - i\gamma_i$. These poles are the ones that remain after O_R and $\rho_R(0)$ have eliminated (or created) some poles (see remark above). In this case it is easy to see that eq. (28) (with no Khalfin term) becomes:

$$(\rho_R(t)|O_R) = (\rho_{R*}|O_R) + a_0(t)\exp\left(-\frac{\gamma_0}{\hbar}t\right) + \sum_{i=1}^{N} a_i(t)\exp\left(-\frac{\gamma_i}{\hbar}t\right) = (\rho_{R*}|O_R) + a_0(t)\exp\left(-\frac{\gamma_0}{\hbar}t\right) + f(t) \tag{33}$$

where $(\rho_{R*}|O_R)$ is the final equilibrium value of $(\rho_R(t)|O_R)$ and the $a_i(t)$ are real oscillating functions. In the most general case the z_i will be placed either at random or not. Anyhow in both cases they can be ordered as[10]

$$\gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \dots$$

So we have plotted $F(t) = (\rho_R(t)|O_R) - (\rho_{R*}|O_R)$ in figure 3

Then if $\gamma_0 < \gamma_1$ it is quite clear that the relaxation time is $t_R = \frac{\hbar}{\gamma_0}$. So the relaxation time is defined with no ambiguity

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[10] For simplicity we will consider only the case $\gamma_0 < \gamma_1 < \gamma_2 < \dots$Other special cases will be consider elsewhere.
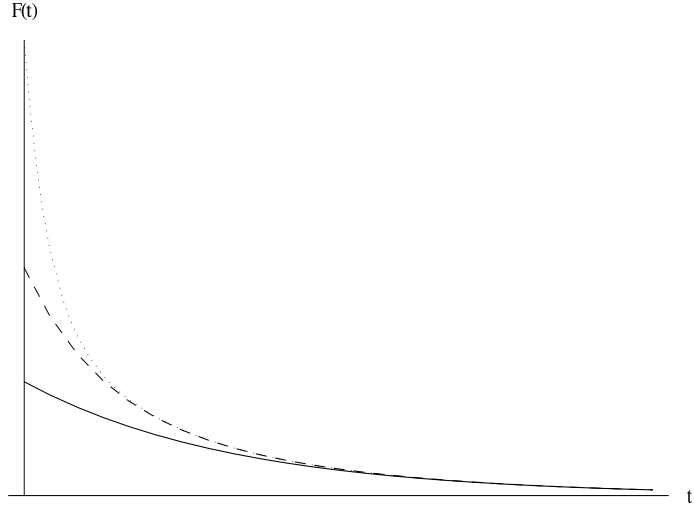
FIG. 3: Evolution of $F(t)$ (solid line), $F_{\gamma_0,\gamma_1}(t)$ (dashed line), $F_{\gamma_0}(t)$ (dot line) and their coincidence limit at t_D. We can see the dominant components in different periods of time.

Let us now consider the decoherence time. Really each pole z_i defines a decaying mode with characteristic time

$$t_i = \frac{\hbar}{\gamma_i}$$

Moreover the poles contain the essence of the decaying phenomenon and the definition of the decoherence time depends on their distribution and other data like the initial condition [11]. In fact our experience, obtained by the study of many models of the literature, is that, t_R seems to be given by the slowest mode irrespectively of the initial conditions. Then we generally define the relaxation time as

$$t_R = \frac{\hbar}{\gamma_0}$$

In doing so we have used the first recipe of section II. to use the pole nearest to the real axis. This t_R will coincide with the one of the model of the next section.

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[11] In a completely random distribution of poles , the best choice seem to be

$$t_R = \frac{\hbar}{\gamma_0}, \quad t_D = \frac{\hbar}{\gamma_1}$$

Then, in this case

$$(\rho_P(t)|O_R) = (\rho_{R*}|O_R) + a_0(t)\exp\left(-\frac{\gamma_0}{\hbar}t\right) \tag{34}$$

and the moving preferred basis would be $\{|\widetilde{j(t)}\rangle\}$, i.e. the basis that diagonalizes $\rho_P(t)$, as before. But, as we will see, we can improve this definition with a more general one that it is valid in a completely general case.

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But the initial conditions seem essential for the definition of t_D. In fact the decoherence time is related to macroscopicity and macroscopicity is defined by the initial conditions.

Now, to introduce the initial condition, let us define:

$$f(t) = \sum_{i=1}^{N} a_i(t) \exp\left(-\frac{\gamma_i}{\hbar}t\right), \quad f'(t) = -\frac{i}{\hbar} \sum_{i=1}^{N} a_i(t)\gamma_i \exp\left(-\frac{\gamma_i}{\hbar}t\right)$$

so at $t = 0$ we can write the initial conditions as

$$f(0) = \sum_{i=1}^{N} a_i(0), \qquad f'(0) = -\frac{i}{\hbar} \sum_{i=1}^{N} a_i(0)\gamma_i$$

Let us call $f(t) = const. \exp g(t) \sim \exp g(t)$, and let us make a Taylor expansion of $g(t)$ as

$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(0)t^2 + \ldots \tag{35}$$

Let us now adimensionalize this expression introducing the adimensional variable $\eta = \frac{t}{t_R}$ where we have used t_R the only characteristic time we have up to now.

$$g(\eta) = g(0) + g'(0)t_R\eta + \frac{1}{2}g''(0)t_R^2\eta^2 + \ldots \tag{36}$$

where $g(t) = g(\eta)$, $g'(0)t_R$, $g''(0)t_R^2, \ldots$ are adimensional. Now, as we explained in the introduction, the essential challenge of the second period of the history of decoherence was to obtain a decoherence time clearly smaller than the relaxation time. So let us postulate that the decoherence time is $t_D \ll t_R$. i.e. $\eta \ll 1$. Then with this condition we have the approximations:

$$g(\eta) = g(0) + g'(0)t_R\eta \quad \text{or} \quad g(t) = g(0) + g'(0)t \tag{37}$$

where

$$g(0) = \log f(0) = \log \sum_{i=0}^{N} a_i(0), \quad ..g\prime(0) = \frac{f'(0)}{f(0)} = -\frac{1}{\hbar}\frac{\sum_{i=0}^{N} a_i\gamma_i}{\sum_{i=0}^{N} a_i}$$

These equations contain the initial conditions. Then in this approximation:

$$f(t) = \exp g(0) \exp t g'(0) = f(0) \exp\left(-\frac{t}{\hbar}\frac{\sum_{i=0}^{N} a_i\gamma_i}{\sum_{i=0}^{N} a_i}\right) = f(0)\exp\left(-\frac{\gamma_{eff}}{\hbar}t\right) \tag{38}$$

where

$$\gamma_{eff} = \frac{\sum_{i=0}^{N} a_i\gamma_i}{\sum_{i=0}^{N} a_i} \quad \text{and} \quad f(t) = f(0)\exp\left(-\gamma_{eff}\frac{t}{\hbar}\right) \tag{39}$$

Then as $\gamma_{eff} > \gamma_0$ for the period in study $t_D < t_R$ we have obtain a typical decaying very fast evolution that we consider the one that produce the decoherence phenomenon. So we define

$$t_D = \frac{\hbar}{\gamma_{eff}} \tag{40}$$

Then γ_{eff} and t_D are both functions of the initial conditions. We will see that this t_D coincide with the one of the Omnès example in the next subsection.

Let us now go to the definition of the moving preferred basis. It is clear that, for the time $t > t_D$, the modes with characteristic times $t_i < t_D$ (i.e. $\gamma_i > \gamma_{eff}$), that we will call the *fast modes*, have become negligible in eq. (33) and therefore in this case $\rho_P(t)$ must defined as

$$(\rho_P(t)|O_R) = (\rho_{R*}|O_R) + \sum_{i=0}^{M} a_i(t) \exp\left(-\frac{\gamma_i}{\hbar}t\right) \tag{41}$$

where the sum in this equation contains only the $M < N$ poles such that $\gamma_i < \gamma_{eff}$, that correspond to the *slow modes* [12]. This is our *adiabatic* choice since we have chose the slow modes of decaying to define $\rho_P(t)$ and rejected the fast modes[13]. Moreover, when $t > t_D$ the motions produce by the fast modes, such that $\gamma_i > \gamma_{eff}$, namely those with motions faster than the one of the evolution of eq. (39_2), are ceased to be relevant for $\rho_R(t)$ and $\rho_P(t) \to \rho_R(t)$. Then we call diagonalize $\rho_P(t)$ and we obtain the moving preferred basis $\{|\widetilde{j(t)}\rangle\}$, the eigen basis of $\rho_P(t)$, that evolves only influenced by the poles such that $\gamma_i < \gamma_{eff}$, and such that, when $t \to t_D$, $\{|\widetilde{j(t)}\rangle\} \to$ the eiegenbasis of $\rho_R(t)$. This $\{|\widetilde{j(t)}\rangle\}$ is our candidate for a general definition of moving preferred basis. Finally if $\gamma_0 < \gamma_1 < \gamma_2 < ...$we have that

$$t_R < t_D$$

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[12] E. g. in the case of the eq. (30) only the z_0 appears in the r.h.s. of the equation. This is the result of eq. (41), in the case $a_0 = a_1$, and $\gamma_0 < \gamma_1$.

[13] The requirement of macroscopicity introduced the initial conditions in the play. In turn these initial conditions define the γ_{eff} and then define the fast and slow modes. Our adiabatic choice corresponds to keep the slow modes and disregard the fast ones. Thus for us the *robust* modes are the slow ones, since they are the less affected by the interaction with the environment, that creates the poles, if compared with the fast ones. This is our notion of *robustness*. Analogously, if we compute the *linear entropy* we will have slower variation of this entropy if we consider only the slow modes that if we consider all the modes (including the fast ones). This would be our minimization of the linear entropy: the moving preferred basis evolution contains only the slow modes.

B. The Omnès or Lee-Friedrich model..

Our more complete and simplest example of decoherence in open systems is the Omnès "pendulum" (i. e. oscillator) in a bath of oscillators, that we will compare with the poles theory in the following subsections. In fact the Omnès model could be considered a poles model if we retain the poles and neglect the Khalfin term. Moreover in the Omnès philosophy the moving preferred basis must be related to some "collective variables" in such a way that they would be experimentally accessible. In this case this variable is the center of mass of the pendulum, i. e. the mean value of the position of a coherent state. In [32] page 285 a one dimensional "pendulum" (the system) in a bath of oscillators (the environment) is considered. The Hamiltonian reads [14]

$$H = \hbar\omega a^\dagger a + \sum_k \hbar\omega_k b_k^\dagger b_k + \hbar\sum_k (\lambda_k a^\dagger b_k + \lambda_k^* a b_k^\dagger) \tag{42}$$

where $a^\dagger(a)$ is the creation (annihilation) operator for the system, $b_k^\dagger(b_k)$ are the creation (annihilation) operator for each mode of the environment, ω and ω_k are the energies of the system and each mode of the environment and λ_k are the interaction coefficients.

Then let consider a state

$$|\psi(t)\rangle = a|\alpha_1(t)\rangle \prod_k |\beta_{k1}(t)\rangle + b||\alpha_2(t)\rangle \prod_k |\beta_{k2}(t)\rangle$$

where $|\alpha_1(0)\rangle, |\alpha_2(0)\rangle$ are *coherent* states for the "system" corresponding to the operator $a^\dagger$ and $|\beta_{k1}(0)\rangle, |\beta_{k2}(0)\rangle$ are a coherent state for the environment corresponding to the operator $b_k^\dagger$. Let the initial condition be

$$|\psi(0)\rangle = a|\alpha_1(0) \{\beta_{k1}(0) = 0\}\rangle + b|\alpha_2(t), \{\beta_{k2}(0) = 0\rangle \tag{43}$$

Then

$$\rho_R(0) = Tr_E|\psi(0)\rangle\langle\psi(0)| \text{ and } \rho_R(t) = Tr_E|\psi(t)\rangle\langle\psi(t)| \tag{44}$$

Moreover Omnès shows that, under reasonable hypotheses and approximations (that correspond to the elimination of the Khalfin terms, see below), the evolution of the $|\alpha_1(t)\rangle, |\alpha_2(t)\rangle$

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[14] This Hamiltonian is similar to the one of equation (10) and equation (49). In fact, in some stages of the treatment Omnès is forced to go to the continuos spectrum. A complete treatment of this continuous model can be found in [24]. The present of the factor $\hbar$ in the Hamiltonian will change some formulas (as $t_R = \hbar/\gamma \to t_R = 1/\gamma$) but we prefer to follow the Omnès formalism in this section

is given by

$$\alpha(t) = \alpha(0) \exp[-i(\omega + \delta\omega)t - \gamma t] + \text{small fluctuations} \qquad (45)$$

where $\delta\omega$ is a shift and γ a dumping coefficient that produces that the system would arrive at a state of equilibrium at $t_R = 1/\gamma$[15], the *relaxation time* of the system, (the small fluctuations are usually neglected)

In the next subsections using the concepts of the previous sections we will prove that the Omnès model is a particular case of our general scheme of the section III.A. Let us now consider the condition of *experimentally accessibility.* In fact, in the model under consideration, the initial states corresponds to the linear combination of two coherent, macroscopically different states $|\alpha_1(0)\rangle, |\alpha_2(0)\rangle$ that evolve to $|\alpha_1(t)\rangle, |\alpha_2(t)\rangle$.

Now the diagonal part of $\rho_R(t)$ reads

$$\rho_R^{(D)}(t) = |a|^2 |\alpha_1(t)\rangle\langle\alpha_1(t)| + |b|^2 |\alpha_2(t)\rangle\langle\alpha_2(t)|$$

and, it can easily be shown [32] that, with the choice of initial conditions of eqs. (87) and (88), that the non diagonal part of $\rho_R(t)$ is

$$\rho_R^{(ND)}(t) = (ab^* |\alpha_1(0)\rangle\langle\alpha_2(0)| + ba^* |\alpha_2(0)\rangle\langle\alpha_1(0)|) \exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}(x_1(0) - x_2(0))^2 (1 - e^{-\gamma t})\right]$$

$$(46)$$

Then if $t \ll t_R = \frac{1}{\gamma}$ (that will be the case if $L_0 = |x_2(0) - x_1(0)|$ is very big) we have

$$\rho_R^{(ND)}(t) \sim (ab^* |\alpha_1(0)\rangle\langle\alpha_2(0)| + ba^* |\alpha_2(0)\rangle\langle\alpha_1(0)|) \exp\left[-\frac{1}{2}\frac{m\omega}{\hbar}(x_1(0) - x_2(0))^2 (1 - 1 + \frac{t}{t_R} + ...)\right]$$

$$(47)$$

where $x_1(0), x_2(0)$ are the initial mean value of the position of the Wigner transform of the two coherent states $|\alpha_1(t)\rangle\langle\alpha_1(t)|, |\alpha_2(t)\rangle\langle\alpha_2(t)|$. This decaying structure is obviously produced by the combination of the initial states and the particular evolution of the system according to the discussion in the introduction of the this section. Then, since $\rho_R^{(ND)}(t) \to 0$ when $t \to \infty$, $\rho_R(t)$ decoheres in the decoherence basis $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\}$, which is the moving preferred basis, and the decoherence time of the system is

$$t_D(L_0) \sim [m\omega(x_1(0) - x_2(0))^2]^{-1} t_R \qquad (48)$$

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[15] So Omnès γ corresponds to our γ_0.

where $L_0 = |x_1(0) - x_2(0)|$.

In the next subsection we will see that we are dealing with a many poles model where the effect of decoherence is produced by these poles and the particular coherent states initial conditions, which produce a "new collective pole mode" with $\gamma_{eff} = \frac{1}{2}\frac{m\omega^2}{\hbar}(x_1(0) - x_2(0))^2$

In the case of the "pendulum" the moving preferred basis $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\}$ is clear experimentally accessible since, in principle, the mean value of the position $x_1(t), x_2(t)$, of the two coherent states $|\alpha_1(t)\rangle, |\alpha_2(t)\rangle$ can be measured and the $x_1(0)$ and $x_2(0)$ turn out to be two "collective variables" (since they are mean values). In fact, in this formalism, the main characteristic of the moving preferred basis is to be related to the "collective variables". Moreover the decoherence time t_D depends on the initial distance $L_0 = |x_1(0) - x_2(0)|$ so we can have different decoherence times depending on the initial conditions.

Let us now consider that

$$\langle \alpha_1(t)|\alpha_2(t)\rangle = \exp\left[-\frac{|\alpha_1 - \alpha_2|^2}{2} + i\frac{\Phi}{2}\right], \quad \Phi = \text{Im}(\alpha_1\alpha_2^* - \alpha_1^*\alpha_2)$$

where

$$|\alpha_1 - \alpha_2| = (2m\hbar\omega)^{-\frac{1}{2}}[m^2\omega^2(x_1(t) - x_2(t))^2 + (p_1(t) - p_2(t))^2]^{\frac{1}{2}}$$

So:

i.- $\langle \alpha_1(t)|\alpha_1(t)\rangle = 1$ even if in general $\langle \alpha_1(t)|\alpha_2(t)\rangle \neq 0$.

ii.- When $(x_1(0) - x_2(0))^2 \to \infty$ or $(p_1(t) - p_2(t))^2 \to \infty$ we have $\langle \alpha_1(t)|\alpha_2(t)\rangle \to 0$.

Thus when the distance between the two centers of the coherent states is very big we have a small t_D and the basis $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\}$would be almost orthonormal. These are the main characteristics of the experimental accessible decoherence basis of Omnès.

But it is important to insist that, generally, $\{|\alpha_i(t)\rangle\}$ is only a *non-orthonormal moving preferred basis*, that we can approximately suppose orthonormal only in the macroscopic case, that is to say, when $x_1(0)$ and $x_2(0)$ are far apart.

In conclusion, in this macroscopic case $\{|\alpha_i(t)\rangle\}$ becomes a orthonormal *moving preferred basis* where $\rho_R(t)$ becomes diagonal in a very small time. This will be the case of the decoherence basis in [17], chapter 17, and in many examples that we can find in the bibliography ([33], [34], [35]). Without this macroscopic property it is difficult to find any trace of a Boolean logic in the moving decoherence basis context of the general case or in this section. In fact, Omnès obtains the Boolean logic by a complete different way (see chapter 6 of [32]).

Anyhow in this particular model the moving preferred basis has a perfect example for the macroscopic case. Let us now present the relation of this formalism with the poles theory.

C. The Lee-Friedrich model. The relaxation time.

Particular important models can be studied, like the one in [24], with Hamiltonian[16]

$$H = \hbar\omega_0 a^\dagger a + \int \hbar\omega_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}} d\mathbf{k} + \hbar \int \lambda_k (a^\dagger b_{\mathbf{k}} + a b^\dagger_{\mathbf{k}}) d\mathbf{k} \tag{49}$$

i.e. a continuous version of (42). In this continuous version we are forced to endow the scalar $(\rho_R(t)|O_R)$ with the some analyticity conditions. Precisely function λ_k (where $k = \omega_k = |\mathbf{k}|$) is chosen in such a way that

$$\eta_\pm(\omega_k) = \omega_k - \omega_0 - \int \frac{d\mathbf{k}\lambda_k^2}{\omega_k - \omega_{k'} \pm i0} \tag{50}$$

which does not vanish for $k \in \mathbb{R}^+$, and its analytic extension $\eta_+(z)$ to the lower half plane only has a simple pole at z_0. This fact will have influence on the poles of $(\rho_R(t)|O_R)$ as in section II and we know that the study of $(\rho_R(t)|O_R)$ is the essential way to understand the whole problem (see section I A).

The Hamiltonian (49) is sometimes called the Lee-Friedrich Hamiltonian and it is characterized by the fact that it contains different *number of modes sector* (number of particle sectors in QFT). In fact, $a^\dagger$ and $b^\dagger_{\mathbf{k}}$ are creation operators that allow to define these numbers of mode sectors. e. g. the one mode sector will contain states like $a^\dagger |0\rangle$ and $b^\dagger_{\mathbf{k}} |0\rangle$ (where $a|0\rangle = b_{\mathbf{k}}|0\rangle = 0$). Then the action of $\exp\left(-\frac{1}{\hbar}Ht\right)$ (or simple the one of H) will conserve the number of modes of this sector in just one mode, since in (49) all the destruction operators are preceded by a creation operator. This also is the case for the $n-$mode sector. The Hamiltonian of the one mode sector, is just the one of the so called Friedrich model i. e.

$$H_F = \hbar\omega_0 |1\rangle \langle 1| + \int \hbar\omega_k |\omega\rangle \langle\omega| d\omega + \hbar \int (\lambda(\omega) |\omega\rangle \langle 1| + \lambda^*(\omega) |1\rangle \langle\omega|) d\omega \tag{51}$$

(expressed just in variable ω, the one that will be analytically continued). As a consequence of the analyticity condition above this simple Friedrich model shows just one resonance. In

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[16] The introduction of $\hbar$ in the equation below produces some changes in the dimensions of some variables, but we prefer to use the Omnès convention in this sub-section also to facilitate the comparison.

fact, this resonance is produced in z_0. Let H_F be the Hamiltonian of the complex extended Friedrich model, then[17] :

$$H_F|z_0\rangle = z_0|z_0\rangle, \quad H_F|z\rangle = z|z\rangle \tag{52}$$

where $z_0 = \omega_0 + \delta\omega_o - i\gamma_0 = \omega_0' - i\gamma_0$ is the only pole and $z \in \Gamma$.

The Lee-Friedrich model, describing the interaction between a quantum oscillator and a scalar field, is extensively analyzed in the literature. Generally, this model is studied by analyzing first the one excited mode sector, i.e. the Friedrich model. Then, if we compute the pole, of this last model, up to the second order in λ_k we obtain that

$$z_0 = \omega_0 + \int \frac{d\mathbf{k}'\lambda_{k'}^2}{\omega_0 - \omega_k + i0} \tag{53}$$

So the pole (that will corresponds to the pole closest to the real axis in the Lee-Friedrich model) can be calculated (see [36] eq. (42)). These results coincide (mutatis mutandis) with the one of Omnès book [32] page 288, for the relaxation time. In fact:

$$\frac{1}{\omega_0 - \omega' + i0} = P\left(\frac{1}{\omega_0 - \omega'}\right) - i\pi\delta(\omega_0 - \omega') \tag{54}$$

where P symbolizes the "principal part", so

$$z_0 = \omega_0 + P\int \frac{d\mathbf{k}'\lambda_{\mathbf{k}'}^2}{\omega_0 - \omega_k} - i\pi\int d\mathbf{k}'\lambda_{\mathbf{k}'}^2\delta(\omega_0 - \omega_k) \tag{55}$$

Then if $d\mathbf{k} = n(\omega)d\omega$ we have

$$\delta\omega_0 = P\int \frac{n(\omega')d\omega'\lambda_{\omega'}^2}{\omega_0 - \omega'}, \quad \gamma_0 = \pi\int n(\omega')d\omega'\lambda_{\omega'}^2\delta(\omega_0 - \omega') \tag{56}$$

namely the results of [32] page 288, and the one contained in eq. (45).:

$$z_0 = (\omega_0 + \delta\omega_0) - i\gamma_0 = \omega_0' - i\gamma_0 \tag{57}$$

So the Omnès result for the relaxation time *coincides*, as we have already said, with the one obtained by the pole theory, precisely

$$t_R = \frac{1}{\gamma_0}$$

in both frameworks.

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[17] Only symbolically, since the poles belong to the scalar $(\rho(t)|O)$, as in section II.

D. Other poles of the Lee-Friedrich model.

Let us now consider the Lee-Friedrich Hamiltonian (49) for the many modes sectors, e. g., as an example, for the three mode sector. Then we have that[18]:

$$H|z_1, z_2, z_3\rangle = (z_1 + z_2 + z_3)|z_1, z_2, z_3\rangle \tag{58}$$

$$H|z_1, z_2, z_0\rangle = (z_1 + z_2 + z_0)|z_1, z_2, z_0\rangle \tag{59}$$

$$H|z_1, z_0, z_3\rangle = (z_1 + z_0 + z_3)|z_1, z_0, z_3\rangle \tag{60}$$

$$H|z_0, z_2, z_3\rangle = (z_0 + z_2 + z_3)|z_0, z_2, z_3\rangle \tag{61}$$

$$H|z_1, z_0, z_0\rangle = (z_1 + 2z_0)|z_1, z_0, z_0\rangle \tag{62}$$

$$H|z_0, z_2, z_0\rangle = (z_2 + 2z_0)|z_0, z_2, z_0\rangle \tag{63}$$

$$H|z_0, z_0, z_3\rangle = (z_3 + 2z_0)|z_0, z_0, z_3\rangle \tag{64}$$

$$H|z_0, z_0, z_0\rangle = 3z_0|z_0, z_0, z_0\rangle \tag{65}$$

where $z_1, z_2, z_3 \epsilon \Gamma$. So in the real complex plane the spectrum of H is

1.- From the eigenvalue $(z_1 + z_2 + z_3)$ three points of the curve Γ

2.- From the eigenvalue $(z_1 + z_2 + z_0)$, $(z_1 + z_0 + z_3)$, $(z_0 + z_2 + z_3)$, a pole at z_0 and two points of the curve Γ

3.- From the eigenvalue $(z_1 + 2z_0)$, $(z_2 - 2z_0)$, $(z_3 + 2z_0)$ a pole at $2z_0$, and one point of the curve Γ

4.- From the eigenvalue a pole at $3z_0$

See figure 3 (**verificar**):

Of course in the general case $3 \to n$ and, as a consequence, the spectrum is nz_0+ the curve Γ, where $n = 0, 1, 2.\ 3,...$ in fact

$$z_n = nz_0 + \bigcup_j z_j, \ z_j \epsilon \Gamma \tag{66}$$

Then if we neglect the Khalfin term, since it corresponds to extremely long times, the Γ disappears and we simply have

$$z_n = nz_0 = n(\omega'_0 - i\gamma_0) \tag{67}$$

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[18] Only symbolically as we have already explained in a previous footnote.
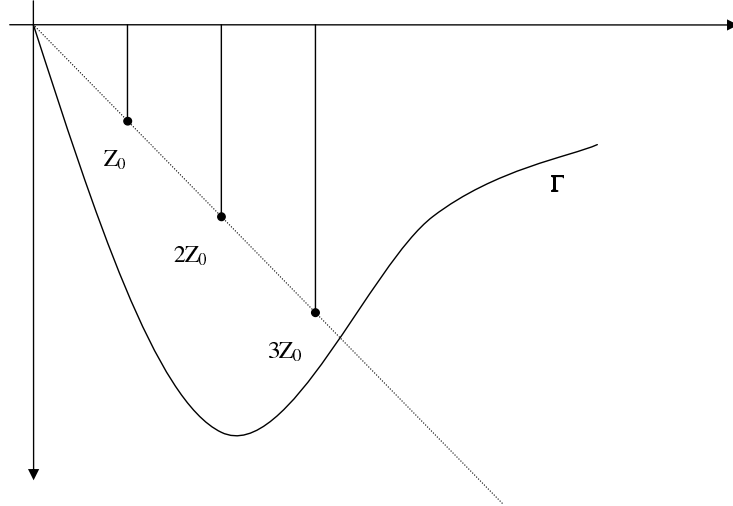
FIG. 4: Complex contour on the lower complex energy plane for the three modes model. The energy poles z_0, $2z_0$ $3z_0$ are assumed to be simple.

Then under this approximation the system has an effective (non Hermitian) Hamiltonian

$$H_{eff} = \hbar z_0 a_0^\dagger a_0 = \hbar N z_0$$

where $a_0^\dagger, a_0$ are the creation and annihilation operators for the mode corresponding to the pole z_0 and N is the corresponding number of poles operator. Now the Hamiltonian of the harmonic oscillator is

$$H_o = \left(N + \frac{1}{2} \right) \hbar\omega \tag{68}$$

Thus we see that in the no Khalfin terms approximation, and taking $\omega_0' = \omega$ (of the last equation) and if n is very large[19]

$$H_{eff} = H_o - i\frac{\gamma_0}{\omega} H_o \tag{69}$$

So, in this approximation, the effective Lee-Friedrich Hamiltonian H_{eff} simply is a (non Hermitian) version of H_o with a dumping term $\frac{\gamma_0}{\hbar\omega} H_o$. Moreover the basis of H_{eff} and H_o are the same one , i. e.$\{|n\rangle\}$, the basis of the H_o of eq. (68).

---

[19] Or, in the general case, since $\frac{\hbar\omega}{2}$ only affects the real part of the pole and not the imaginary namely the one that produces the time scales.

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E. Poles dependency on the initial conditions

1. The amplitude of probability

The probability amplitude that a pure state $|\varphi\rangle$ would be in the pure state $|\psi\rangle$ at time t is:

$$A(t) = \langle \psi | \varphi(t) \rangle \tag{70}$$

The most general linear superposition of the eigenvectors of H_{eff}, in basis $\{|n\rangle\}$ is:

$$|\psi\rangle = \sum_{n=0}^{N} a_n |n\rangle \tag{71}$$

and the time evolution for $|\varphi\rangle$ must be:

$$|\varphi(t)\rangle = \sum_{n=0}^{N} b_n |n(t)\rangle \tag{72}$$

Then

$$A(t) = \sum_{n,n'=0}^{N} b_n a_n^* \langle n | n'(t) \rangle = \sum_{n,n'=0}^{N} b_n a_{n'}^* A_{nn'} \tag{73}$$

We can compute $A_{nn'} = \langle n | n'(t) \rangle = \langle n | \exp\left(-\frac{i}{\hbar} H t\right) |n'\rangle = \langle n | e^{-i z_n t} |n'\rangle = e^{-i z_n t} \delta_{nn'}$, then

$$A(t) = \sum_{n=0}^{N} b_n a_n^* e^{-i z_n t} \tag{74}$$

where from eqs. (45) and (56), or eq. 4.47 of [31] we have:

$$z_n = \omega_n' + i \gamma_n \tag{75}$$

Then, as we know, if we neglect the Khalfin term the "energy" levels are multiples of the fundamental "energy" i. e.

$$z_n = n z_0 \tag{76}$$

where $z_0 = \omega_0' - i \gamma_0$ and the coefficients a_n and b_n depend in the initial conditions (according to eq. 4.26 of [31]).

With the expression (67) eq. (74) becomes

$$A(t) = \sum_{n=0}^{N} b_n a_n^* e^{-i n z_0 t} = \sum_{n=0}^{N} b_n a_n^* \left(e^{-i z_0 t}\right)^n \tag{77}$$

The same recipe could be used in the fundamental scalar $(\rho_R(t)|O_R)$ instead of $\langle\psi|\varphi(t)\rangle$ with similar results but with more difficult calculations.

2. Initial conditions and evolution

As initial conditions, $|\alpha_1(0)\rangle$, $|\alpha_2(0)\rangle$, it is possible to choose any linear combination of the elements $\{|n\rangle\}$ with $n = 0, 1..., \infty$. So we can choose the coherent states

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \tag{78}$$

But we can also choose as the initial conditions an approximated version where the number modes is N and we take $n = 0, 1..., N$. namely an approximated quasi-coherent states or quasi-Gaussian (that becomes a coherent state when $N \to \infty$ as we will consider below). Thus

$$|\lambda\rangle = \left(\sum_{k=0}^{N} \frac{|\lambda|^{2k}}{k!}\right)^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{\lambda^n}{\sqrt{n!}} |n\rangle \tag{79}$$

Then let us choose the initial conditions as the sum of two quasi-Gaussian functions, namely:

$$|\Phi(0)\rangle = a |\alpha_1(0)\rangle + b |\alpha_2(0)\rangle \tag{80}$$

where $|\alpha_1(0)\rangle$ and $|\alpha_2(0)\rangle$ are quasi-coherent states, precisely

$$|\alpha_1(0)\rangle = \left(\sum_{k=0}^{N} \frac{|\alpha_1(0)|^{2k}}{k!}\right)^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{(\alpha_1(0))^n}{\sqrt{n!}} |n\rangle \tag{81}$$

and

$$|\alpha_2(0)\rangle = \left(\sum_{k=0}^{N} \frac{|\alpha_2(0)|^{2k}}{k!}\right)^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{(\alpha_2(0))^n}{\sqrt{n!}} |n\rangle \tag{82}$$

Thus the initial state is:

$$\rho_0 = |\Phi(0)\rangle \langle\Phi(0)| = |a|^2 |\alpha_1(0)\rangle \langle\alpha_1(0)| + ab^* |\alpha_1(0)\rangle \langle\alpha_2(0)|$$
$$+ a^*b |\alpha_2(0)\rangle \langle\alpha_1(0)| + |b|^2 |\alpha_2(0)\rangle \langle\alpha_2(0)| \tag{83}$$

Therefore the time evolved state is

$$\rho(t) = |\Phi(t)\rangle \langle\Phi(t)| = \rho_D(t) + \rho_{ND}(t) \tag{84}$$

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where $\rho_D(t)$ is the diagonal part (in the basis $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$) of $\rho(t)$

$$\rho^{(D)}(t) = |a|^2 |\alpha_1(t)\rangle \langle \alpha_1(t)| + |b|^2 |\alpha_2(t)\rangle \langle \alpha_2(t)| \tag{85}$$

and $\rho^{(ND)}$ is the non-diagonal part of $\rho(t)$

$$\rho^{(ND)}(t) = ab^* |\alpha_1(t)\rangle \langle \alpha_2(t)| + a^*b |\alpha_2(t)\rangle \langle \alpha_1(t)| \tag{86}$$

We choose the two quasi-Gaussian (81) and (82) with center at $p_{1,2}(0) = 0$, (see [32] eq. (7.15) page 284) and

$$\alpha_1(0) = \frac{m\omega}{\sqrt{2m\hbar\omega}} x_1(0) \tag{87}$$

$$\alpha_2(0) = \frac{m\omega}{\sqrt{2m\hbar\omega}} x_2(0) \tag{88}$$

So $\alpha_1(0)$ and $\alpha_2(0)$ are real numbers.

Without loss of generality (since with a change of coordinates we can shift $x_1(0)$ and $x_2(0)$) we can consider that the $\alpha_1(0)$ and $\alpha_2(0)$ are both positive. For this reason we will interchange $\alpha_i(0)$ and $|\alpha_i(0)|$ below.

3. Components of the non-diagonal part of the state and the macroscopic case

Let us not consider $\rho^{(ND)}(t)$ in the basis of the initial condition $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$. Then we have

$$\rho^{(ND)}(t) = \rho_{11}^{(ND)}(t) |\alpha_1(0)\rangle \langle \alpha_1(0)| + \rho_{12}^{(ND)}(t) |\alpha_1(0)\rangle \langle \alpha_2(0)|$$
$$+ \rho_{21}^{(ND)}(t) |\alpha_2(0)\rangle \langle \alpha_1(0)| + \rho_{22}^{(ND)}(t) |\alpha_2(0)\rangle \langle \alpha_2(0)| \tag{89}$$

We will prove that for macroscopic initial conditions, i.e. when the peaks of the two Gaussians are far from each other, the states $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$ are quasi-orthogonal basis, i.e.

$$\langle \alpha_1(0)|\alpha_2(0)\rangle \cong \langle \alpha_2(0)|\alpha_1(0)\rangle \cong 0 \tag{90}$$

and indeed this is the macroscopicity condition. In fact

$$\langle \alpha_1(0)|\alpha_2(0)\rangle = \left(\sum_{k=0}^{N} \frac{|\alpha_1(0)|^{2k}}{k!} \right)^{-\frac{1}{2}} \left(\sum_{k=0}^{N} \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{(\alpha_1(0)\alpha_2(0))^n}{n!} \tag{91}$$

So using the Cauchy product and the binomial theorem we have

$$\langle \alpha_1(0)|\alpha_2(0)\rangle = \left(\sum_{k=0}^{N} \frac{\left(|\alpha_1(0)|^2 + |\alpha_2(0)|^2\right)^k}{k!} \right)^{-\frac{1}{2}} \sum_{n=0}^{N} \frac{(\alpha_1(0)\alpha_2(0))^n}{n!} \tag{92}$$

and again using the Cauchy product and the binomial theorem we have

$$\langle \alpha_1(0)|\alpha_2(0)\rangle = \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{1}{2} \left(|\alpha_1(0)|^2 + |\alpha_2(0)|^2 - 2\alpha_1(0)\alpha_2(0) \right) \right)^n \tag{93}$$

then

$$\langle \alpha_1(0)|\alpha_2(0)\rangle = \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{(\alpha_1(0) - \alpha_2(0))^2}{2} \right)^n \tag{94}$$

so for $|\alpha_1(0) - \alpha_2(0)| \to \infty$ we have orthogonality as we have promised to demonstrate.

In fact, we can consider the limit $N \to \infty$. Thus the last scalar product is equal to the truncated Taylor series of exponential function. Then we may introduce R_{N+1}, the difference with the complete Taylor series, and we obtain

$$\langle \alpha_1(0)|\alpha_2(0)\rangle = e^{-\frac{(\alpha_1(0) - \alpha_2(0))^2}{2}} - R_{N+1} \tag{95}$$

where R_{N+1} is a correction of order $N + 1$

$$R_{N+1} = \frac{e^{\xi}}{(N+1)!} \left(-\frac{(\alpha_1(0) - \alpha_2(0))^2}{2} \right)^{N+1} \tag{96}$$

with $\xi \in \left(-\frac{(\alpha_1(0)-\alpha_2(0))^2}{2}, 0 \right)$, then we have

$$R_{N+1} \leq \frac{1}{(N+1)!} \left(-\frac{(\alpha_1(0) - \alpha_2(0))^2}{2} \right)^{N+1} \tag{97}$$

Thus we can prove the orthogonality conditions:

1. To eliminate the first term of (95) we make

$$e^{-\frac{(\alpha_1(0)-\alpha_2(0))^2}{2}} \ll 1 \tag{98}$$

   i.e.

$$|\alpha_1(0) - \alpha_2(0)| \gg 1 \tag{99}$$

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2. To eliminate the second term of (95), $|R_{N+1}| \ll 1$, we make

$$\left| \frac{1}{(N+1)!} \left(-\frac{(\alpha_1(0) - \alpha_2(0))^2}{2} \right)^{N+1} \right| \ll 1 \tag{100}$$

i.e.

$$|\alpha_1(0) - \alpha_2(0)| \ll [2(N+1)!]^{\frac{1}{2(N+1)}} \tag{CI-21l}$$

This expression can be simplified by a huge N using the Stirling's approximation

$$|\alpha_1(0) - \alpha_2(0)| \ll \sqrt{2(N+1)} \tag{101}$$

Then the orthogonality is proved if eqs. (99) and (101) are satisfy.

In fact these are the two *macroscopicity* condition: then $|\alpha_1(0) - \alpha_2(0)|$ and N should be large.

We will consider that $\alpha_1(0) - \alpha_2(0)$ and $N + 1$ always satisfy these macroscopicity conditions. Then from (95) the basis $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$ is quasi-orthogonal and we have

$$\rho_{11}^{(ND)}(t) = \langle \alpha_1(0)| \rho^{(ND)}(t) |\alpha_1(0)\rangle$$
$$\rho_{12}^{(ND)}(t) = \langle \alpha_1(0)| \rho^{(ND)}(t) |\alpha_2(0)\rangle$$
$$\rho_{21}^{(ND)}(t) = \langle \alpha_2(0)| \rho^{(ND)}(t) |\alpha_1(0)\rangle$$
$$\rho_{22}^{(ND)}(t) = \langle \alpha_2(0)| \rho^{(ND)}(t) |\alpha_2(0)\rangle \tag{102}$$

then from eq. (86) we have

$$\rho_{11}^{(ND)}(t) = ab^* \langle \alpha_1(0)|\alpha_1(t)\rangle \langle \alpha_2(t)|\alpha_1(0)\rangle + a^*b \langle \alpha_1(0)|\alpha_2(t)\rangle \langle \alpha_1(t)|\alpha_1(0)\rangle$$
$$\rho_{12}^{(ND)}(t) = ab^* \langle \alpha_1(0)|\alpha_1(t)\rangle \langle \alpha_2(t)|\alpha_2(0)\rangle + a^*b \langle \alpha_1(0)|\alpha_2(t)\rangle \langle \alpha_1(t)|\alpha_2(0)\rangle$$
$$\rho_{21}^{(ND)}(t) = ab^* \langle \alpha_2(0)|\alpha_1(t)\rangle \langle \alpha_2(t)|\alpha_1(0)\rangle + a^*b \langle \alpha_2(0)|\alpha_2(t)\rangle \langle \alpha_1(t)|\alpha_1(0)\rangle$$
$$\rho_{22}^{(ND)}(t) = ab^* \langle \alpha_2(0)|\alpha_1(t)\rangle \langle \alpha_2(t)|\alpha_2(0)\rangle + a^*b \langle \alpha_2(0)|\alpha_2(t)\rangle \langle \alpha_1(t)|\alpha_2(0)\rangle \tag{103}$$

We can compute these products with eqs. (67) and (77).

- For $\langle \alpha_1(0)|\alpha_1(t)\rangle$ we have that $|\psi\rangle = |\alpha_1(0)\rangle$ and $|\varphi(t)\rangle = |\alpha_1(t)\rangle$, then from (81) and since $\alpha_1(t)$ is a real number

$$a_n^* = e^{-\frac{|\alpha_1(0)|^2}{2}} \frac{(\alpha_1(0))^n}{\sqrt{n!}} \, and \, b_n = e^{-\frac{|\alpha_1(0)|^2}{2}} \frac{(\alpha_1(0))^n}{\sqrt{n!}}$$

then we have

$$\langle \alpha_1(0)|\alpha_1(t)\rangle = e^{-|\alpha_1(0)|^2} \sum_{n=0}^{N} \frac{\left(|\alpha_1(0)|^2\right)^n}{n!} \left(e^{-iz_0 t}\right)^n$$

$$= e^{-|\alpha_1(0)|^2} e^{|\alpha_1(0)|^2 e^{-iz_0 t}} \tag{104}$$

- For $\langle \alpha_1(0)|\alpha_2(t)\rangle$ we have that $|\psi\rangle = |\alpha_1(0)\rangle$ and $|\varphi(t)\rangle = |\alpha_2(t)\rangle$, then from (81), (82) and since $\alpha_1(t)$ and $\alpha_2(t)$ are real numbers we can find the coefficients of eqs. (71) and (72) as

$$a_n^* = e^{-\frac{|\alpha_1(0)|^2}{2}} \frac{|\alpha_1(0)|^n}{\sqrt{n!}} \quad \text{and} \quad b_n = e^{-\frac{|\alpha_2(0)|^2}{2}} \frac{|\alpha_2(0)|^n}{\sqrt{n!}} \tag{105}$$

then, form eq. (74) we have

$$\langle \alpha_1(0)|\alpha_2(t)\rangle = e^{-\frac{|\alpha_1(0)|^2 + |\alpha_2(0)|^2}{2}} \sum_{n=0}^{N} \frac{\left(|\alpha_1(0)| \, |\alpha_2(0)|\right)^n}{n!} \left(e^{-iz_0 t}\right)^n$$

$$= e^{-\frac{|\alpha_1(0)|^2 + |\alpha_2(0)|^2}{2}} e^{|\alpha_1(0)||\alpha_2(0)|e^{-iz_0 t}} \tag{106}$$

- For $\langle \alpha_2(0)|\alpha_1(t)\rangle$ we have that $|\psi\rangle = |\alpha_2(0)\rangle$ and $|\varphi(t)\rangle = |\alpha_1(t)\rangle$, then from (81), (82) and since $\alpha_1(t)$ and $\alpha_2(t)$ are real numbers

$$a_n^* = e^{-\frac{|\alpha_2(0)|^2}{2}} \frac{(\alpha_2(0))^n}{\sqrt{n!}} \text{ and } b_n = e^{-\frac{|\alpha_1(0)|^2}{2}} \frac{(\alpha_1(0))^n}{\sqrt{n!}} \tag{107}$$

then

$$\langle \alpha_2(0)|\alpha_1(t)\rangle = e^{-\frac{|\alpha_1(0)|^2 + |\alpha_2(0)|^2}{2}} \sum_{n=0}^{N} \frac{\left(|\alpha_1(0)| \, |\alpha_2(0)|\right)^n}{n!} \left(e^{-iz_0 t}\right)^n$$

$$= e^{-\frac{|\alpha_1(0)|^2 + |\alpha_2(0)|^2}{2}} e^{|\alpha_1(0)||\alpha_2(0)|e^{-iz_0 t}} \tag{108}$$

- For $\langle \alpha_2(0)|\alpha_2(t)\rangle$ we have that $|\psi\rangle = |\alpha_2(0)\rangle$ and $|\varphi(t)\rangle = |\alpha_2(t)\rangle$, then from (82) and since $\alpha_2(t)$ is a real number

$$a_n^* = e^{-\frac{|\alpha_2(0)|^2}{2}} \frac{(\alpha_2(0))^n}{\sqrt{n!}} \text{ and } b_n = e^{-\frac{|\alpha_2(0)|^2}{2}} \frac{(\alpha_2(0))^n}{\sqrt{n!}} \tag{109}$$

then

$$\langle \alpha_2(0)|\alpha_2(t)\rangle = e^{-|\alpha_2(0)|^2} \sum_{n=0}^{N} \frac{\left(|\alpha_2(0)|^2\right)^n}{n!} \left(e^{-iz_0 t}\right)^n$$

$$= e^{-|\alpha_2(0)|^2} e^{|\alpha_2(0)|^2 e^{-iz_0 t}} \tag{110}$$

Now if we consider eqs. (87) and (88) and remember that the initial centers of the Gaussians are given by eqs. (81) and (82), with no lost of generality we can choose:

$$\alpha_1(0) = 0 \tag{111}$$

and

$$\alpha_2(0) = \frac{m\omega}{\sqrt{2m\hbar\omega}}L_0 \tag{112}$$

Remember that we have imposed a macroscopic condition to the initial conditions, i.e. $|\alpha_1(0) - \alpha_2(0)| \gg 1$ and $|\alpha_1(0) - \alpha_2(0)| \ll [2(N+1)!]^{\frac{1}{2(N+1)}}$. So in the case given by (111) and (112) we have

$$|\alpha_1(0) - \alpha_2(0)| = \alpha_2(0) \gg 1 \quad \text{and} \quad |\alpha_1(0) - \alpha_2(0)| \ll [2(N+1)!]^{\frac{1}{2(N+1)}} \tag{113}$$

i.e.

$$\frac{m\omega}{\sqrt{2m\hbar\omega}}L_0 \gg 1 \quad \text{and} \quad [2(N+1)!]^{\frac{1}{2(N+1)}} \gg \frac{m\omega}{\sqrt{2m\hbar\omega}}L_0 \tag{114}$$

Then if we substitute (111), (112) and (113) in eq. (104), (106), (108) and (110) and we take into account (113)

$$\langle \alpha_1(0)|\alpha_1(t)\rangle = 1 \tag{115}$$

$$\langle \alpha_1(0)|\alpha_2(t)\rangle = e^{-\frac{|\alpha_2(0)|^2}{2}} \cong 0 \tag{116}$$

$$\langle \alpha_2(0)|\alpha_1(t)\rangle = e^{-\frac{|\alpha_2(0)|^2}{2}} \cong 0 \tag{117}$$

$$\langle \alpha_2(0)|\alpha_2(t)\rangle = e^{-|\alpha_2(0)|^2\left(1-e^{-iz_0 t}\right)} \tag{118}$$

Moreover if we substitute (115), (116), (117) and (118) in eq. (103) we have

$$\rho_{11}^{(ND)}(t) \cong 0$$

$$\rho_{12}^{(ND)}(t) \cong ab^* e^{-|\alpha_2(0)|^2\left(1-e^{-iz_0 t}\right)}$$

$$\rho_{21}^{(ND)}(t) \cong a^* b e^{-|\alpha_2(0)|^2\left(1-e^{-iz_0 t}\right)}$$

$$\rho_{22}^{(ND)}(t) \cong 0 \tag{119}$$

We see that in these equations there is an exponential produced by the poles of eq. (76), Then we have from eq. (89) and eq. (112).

$$\rho_{ij}^{(ND)}(t) \propto \exp[-\frac{m\omega}{2\hbar}L_0^2(1 - e^{-\gamma_0 t}) \tag{120}$$

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Then we can make the expansion we have done in eq. (35).

$$\rho_{ij}^{(ND)}(t) \propto \exp[-\frac{m\omega}{2\hbar}L_0^2(1 - 1 + \gamma_0 t - \frac{1}{2}\gamma_0^2 t^2 - ...)]$$

and introduce the recipe of eq. (37) to obtain

$$\rho_{ij}^{(ND)}(t) \propto \exp(-\frac{m\omega}{2\hbar}L_0^2\gamma_0 t)$$

So a largest decaying time $t_R = \frac{1}{\gamma_0}$ is given by the original pole of eq. (55) but, using our poles technique, a new decaying mode appears, as in eq. (38), i.e.

$$f(t) \sim \exp(-\frac{\gamma_{eff}}{\hbar}t)$$

with the coefficient γ_{eff} that, with the convention of this section, reads:

$$\gamma_{eff} = \frac{m\omega}{2\hbar}L_0^2\gamma_0 \tag{121}$$

so, the new characteristic time is $t_D = \frac{1}{\gamma_{eff}}$ or

$$t_D = \frac{2\hbar}{m\omega}\frac{1}{L_0^2}t_R \tag{122}$$

the same time was found by Omnès in [32] or in eq. (48) and corresponding to the definition (40) of section III.A. So in fact, we have recovered the same result. Also in [32] the result for t_D is valid only for small t as in section III.A. In the general case, and considering that $\alpha_1(0) = 0$, from eqs. (89) and (120) we have:

$$\rho^{(ND)}(t) = \{ab^* |\alpha_1(0)\rangle \langle\alpha_2(0)| + ba^* |\alpha_1(0)\rangle \langle\alpha_2(0)|\}$$
$$\exp\left[-\frac{m\omega}{2\hbar}L_0^2 \left(1 - e^{-\gamma_0 t}\right)\right] \tag{123}$$

the same expression that can be found on page 290 of [32] or in eq. (46). So the coincidence of both formalisms is completely proved.

Finally, for $t < t_D$ the evolution of $\rho_R(t)$ contains all the modes (more or less as in figure of 3.315_a of [35]). For times $t > t_D$ the fast modes are not important anymore and the evolution corresponds to eq. (41) where only the slow modes have influence in the evolution of $\rho_R(t)$ (more or less as in figure of 3.315_b of [35]). Then in order to define the moving preferred basis we must consider the definition of $\rho_R(t)$ in eq. (41). For larger times only the "big mountains" of figure 3.315_b of [35] remain while the central interference pattern

has vanished. these mountains in motion are the moving preferred basis in this case. Finally for $t \sim t_R$ the two mountains have reached equilibrium and only the $\rho_D(t)$ remains. This would we the evolution of the moving preferred basis of Omnès (even if he never use this name) explained according to our formalism.

So we have proved that all the characters of the Omnès model: t_R, t_D, and the moving preferred basis, coincide with our definitions of section III.A.

Precisely, as we have said in the macroscopic case the basis $\{|\alpha_1(t)\rangle, |\alpha_2(t)\rangle\}$ is orthogonal and it is the one defined in section III.A. In fact for $t > t_D$ the evolution of $\rho_R(t)$ is produced by the poles such that $\gamma_i < \gamma_{eff}$ while for $t < t_D$ the evolution is produced by all the poles.. Moreover the corresponding $\rho_R(t)$ and $\rho_P(t)$ coincide at $t = t_D$, with all their derivatives.

IV.   CONCLUSIONS

In this paper we have:

i.- Discussed a general scheme for decoherence, that in principle could be used in many examples.

ii.- We have given a quite general definition of moving preferred basis $\{|\widetilde{j(t)}\rangle\}$, and of relaxation and decoherence times for a general systems.

iii.- We have proved that our definitions coincide with those of the Omnès model..

We hope that these general results will produce some light in the general problem of decoherence.

The Omnès formalism, of references [16], [17], and [32] contains the most general definition of moving preferred basis of the literature on the subject. Our basis have another conceptual frame: the catalogue of decaying modes in the non-unitary evolution of a quantum system. But since the Omnès formalism is the best available it would be very important for us to show, in the future, the coincidence of both formalisms, as we have done at least for one model in this paper (see section III.B).

Of course we realize that, to prove our proposal, more examples must be added, as we will do elsewhere. But we also believe that we have a good point of depart. In fact, probably the coincidences that we have found in the Omnès model could be a general feature of the decoherence phenomenon. Essentially because, being the poles catalogue the one that contains *all the possible decaying modes* of the non unitary evolutions, since relaxation

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and decoherence are non-unitary evolutions, necessarily they must be contained within this catalogue, .

V. ACKNOWLEDGMENTS

VI. APPENDIX A.

A. Observables that see some poles.

In this appendix we will introduce a particular example of observables, of the same system, such that some observables would see some poles while other would see other poles. Essentially it is a bi-Friedrich-model.

Let us consider a system $\mathcal{S}$ with Hamiltonian:

$$H = H_0 + H_{Int}$$

where

$$H_0 = \Omega_1 |1\rangle\langle 1| + \Omega_2 |2\rangle\langle 2| + 2\int_0^\infty \omega |\omega\rangle\langle\omega| d\omega$$

and

$$H_{Int} = \int_0^a V_\omega^{(1)} \left[|\omega\rangle\langle 1| + |1\rangle\langle\omega|\right] d\omega + \int_b^\infty V_{\omega'}^{(2)} \left[|\omega'\rangle\langle 2| + |2\rangle\langle\omega'|\right] d\omega$$

where $a < b$ and $\langle 1|2\rangle = \langle\omega|2\rangle = \langle 1|\omega\rangle = 0$. This Hamiltonian can also reads:

$$H = H_1 + H_2$$

where

$$H_1 = \Omega_1 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| d\omega + \int_0^a V_\omega^{(1)} \left[|\omega\rangle\langle 2| + |2\rangle\langle\omega|\right] d\omega$$

and

$$H_2 = \Omega_2 |2\rangle\langle 2| + \int_0^\infty \omega' |\omega'\rangle\langle\omega'| d\omega' + \int_b^\infty V_{\omega'}^{(2)} \left[|\omega'\rangle\langle 1| + |1\rangle\langle\omega'|\right] d\omega'$$

. Then it is easy to prove that

$$[H_1, H_2] = 0$$

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and that

$$\exp(-\frac{i}{\hbar}Ht) = \exp(-\frac{i}{\hbar}H_1 t)\exp(-\frac{i}{\hbar}H_2 t)$$

Let us now decompose the system as $\mathcal{S} = \mathcal{P}_1 \cup \mathcal{P}_2$ where part $\mathcal{P}_1$ is related with Hamiltonian H_1 and part $\mathcal{P}_2$ related with Hamiltonian H_2. Let us observe that these two parts are not independent since they share a common continuous spectrum, i. e. $2\int_0^\infty \omega |\omega\rangle\langle\omega| d\omega$. Moreover let the corresponding relevant observable spaces be $\mathcal{O}_1 \otimes I_{E1}$ for $\mathcal{P}_1$ and $\mathcal{O}_2 \otimes I_{E2}$ for $\mathcal{P}_2$, where $\mathcal{O}_1$ has basis $\{|1\rangle\}$, and $\mathcal{O}_{E1}$ has basis $\{|\omega\rangle\}$ while $\mathcal{O}_2$ has basis $\{|2\rangle\}$, and $\mathcal{O}_{E2}$ basis $\{|\omega'\rangle\}$. Moreover let us consider the two relevant observables of system $\mathcal{S} = \mathcal{P}_1 \cup \mathcal{P}_2$

$$\mathbb{O}_i = O_1 \otimes I_{E1} \otimes \mathcal{I}_2 \otimes I_{E2} \text{ and } \mathbb{O}_2 = I_1 \otimes I_{E1} \otimes O_2 \otimes I_{E2}$$

where the I are the corresponding unit operators. Then

$$(\rho(t)|\mathbb{O}_1) = (\rho(0)|\exp(\frac{i}{\hbar}H_1 t)O_1 \exp(-\frac{i}{\hbar}H_1 t) \otimes I_{E1} \otimes \mathcal{I}_2 \otimes I_{E2})$$

$$(\rho(t)|\mathbb{O}_2) = (\rho(0)|I_1 \otimes I_{E1} \otimes \exp(\frac{i}{\hbar}H_2 t)O_2 \exp(\frac{i}{\hbar}H_2 t) \otimes I_{E2})$$

and therefore $\mathbb{O}_1$ only sees the evolution in part $\mathcal{P}_1$ while $\mathbb{O}_2$ only sees the evolution in part $\mathcal{P}_2$. Then, since the poles of part $\mathcal{P}_1$ correspond to the decaying modes of the evolution of this part (and we know that the Friedrich model of this subsystem generically do have poles) $\mathbb{O}_1$ only sees the poles of part $\mathcal{P}_1$. Respectively $\mathbb{O}_2$ only sees the poles of part $\mathcal{P}_2$. q. e. d.

Now we can consider that the poles of part $\mathcal{P}_1$ define a relaxation time t_{R1} while the poles of part $\mathcal{P}_2$ define a relaxation time t_{R2}. If $t_{R1} \ll t_{R2}$ part $\mathcal{P}_1$ decoheres and becomes classical in a short time $t > t_{R1}$ while part $\mathcal{P}_2$ remains quantum for a large time $t < t_{R2}$. Then for t such that $t_{R1} < t < t_{R2}$ part $\mathcal{P}_1$ behaves classically while part $\mathcal{P}_2$ remains quantum. Precisely: system $\mathcal{S}$ observed by $\mathbb{O}_i = O_1 \otimes I_{E1} \otimes \mathcal{I}_2 \otimes I_{E2}$ seems classical while observed by $\mathbb{O}_2 = I_1 \otimes I_{E1} \otimes O_2 \otimes I_{E2}$ seems quantum. In fact this is the behavior of a generic physical system.

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