

# Extending a Hybrid Godunov Method for Radiation Hydrodynamics to Multiple Dimensions

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## Abstract

This paper presents a hybrid Godunov method for three-dimensional radiation hydrodynamics. The multidimensional technique outlined in this paper is an extension of the one-dimensional method that was developed by Sekora & Stone 2009, 2010. The earlier one-dimensional technique was shown to preserve certain asymptotic limits and be uniformly well behaved from the photon free streaming (hyperbolic) limit through the weak equilibrium diffusion (parabolic) limit and to the strong equilibrium diffusion (hyperbolic) limit. This paper gives the algorithmic details for constructing a multidimensional method. A future paper will present numerical tests that demonstrate the robustness of the computational technique across a wide-range of parameter space.

## 1 Radiation Hydrodynamics

The purpose of this paper is to extend the ideas of Sekora & Stone 2010 to radiation hydrodynamical problems in multiple dimensions. Therefore, this paper is a continuation of that earlier work, where the system of equations for radiation hydrodynamics were non-dimensionalized with respect to the material flow scale [10, 11]. This scaling gives two important parameters:  $\mathbb{C} = c/a_\infty$  which measures relativistic effects and  $\mathbb{P} = a_r T_\infty^4 / \rho_\infty a_\infty^2$  which measures how radiation affects material dynamics. Additionally,  $a_r = 8\pi^5 k^4 / 15c^3 h^3 = 7.57 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$  is a radiation constant,  $T_\infty$  is a reference material temperature in the absence of radiation, and  $\rho_\infty$  is a reference material density in the absence of radiation.

The full system of equations for radiation hydrodynamics in the mixed frame that is correct to  $\mathcal{O}(1/\mathbb{C})$  is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{m}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = -\mathbb{P} \left[ -\sigma_t \left( \mathbf{F}_r - \frac{\mathbf{u} E_r + \mathbf{u} \cdot \mathbf{P}_r}{\mathbb{C}} \right) + \sigma_a \frac{\mathbf{u}}{\mathbb{C}} (T^4 - E_r) \right], \quad (2)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot \left( (E + p) \frac{\mathbf{m}}{\rho} \right) = -\mathbb{P} \mathbb{C} \left[ \sigma_a (T^4 - E_r) + (\sigma_a - \sigma_s) \frac{\mathbf{u}}{\mathbb{C}} \cdot \left( \mathbf{F}_r - \frac{\mathbf{u} E_r + \mathbf{u} \cdot \mathbf{P}_r}{\mathbb{C}} \right) \right], \quad (3)$$

$$\frac{\partial E_r}{\partial t} + \mathbb{C} \nabla \cdot \mathbf{F}_r = \mathbb{C} \left[ \sigma_a (T^4 - E_r) + (\sigma_a - \sigma_s) \frac{\mathbf{u}}{\mathbb{C}} \cdot \left( \mathbf{F}_r - \frac{\mathbf{u} E_r + \mathbf{u} \cdot \mathbf{P}_r}{\mathbb{C}} \right) \right], \quad (4)$$

$$\frac{\partial \mathbf{F}_r}{\partial t} + \mathbb{C} \nabla \cdot \mathbf{P}_r = \mathbb{C} \left[ -\sigma_t \left( \mathbf{F}_r - \frac{\mathbf{u} E_r + \mathbf{u} \cdot \mathbf{P}_r}{\mathbb{C}} \right) + \sigma_a \frac{\mathbf{u}}{\mathbb{C}} (T^4 - E_r) \right], \quad (5)$$

$$\mathbf{P}_r = \mathbf{f} E_r \quad (\text{closure relation}), \quad \mathbf{f} = \frac{1 - \chi}{2} I + \frac{3\chi - 1}{2} \mathbf{n} \otimes \mathbf{n}. \quad (6)$$

For the material quantities,  $\rho$  is density,  $\mathbf{m}$  is momentum density,  $p = (\gamma - 1)e$  is pressure,  $E$  is energy density, and  $T$  is temperature. For the radiation quantities,  $E_r$  is energy density,  $\mathbf{F}_r$  is flux,  $\mathbf{P}_r$  is pressure,  $\mathbf{f}$  is the variable tensor Eddington factor that is used to close the hierarchy of radiation transport moment equations,  $\chi$  is a parameter, and  $\mathbf{n}$  is a unit normal vector aligned with the radiative flux [6, 9]. In the free streaming limit (optically thin regime),  $\chi \rightarrow 1$  such that  $\mathbf{f} \rightarrow \mathbf{n} \otimes \mathbf{n}$ . Yet, in the equilibrium diffusion limit (optically thick regime),  $\chi \rightarrow 1/3$  such that  $\mathbf{f} \rightarrow \frac{1}{3} I$ , where  $I$  is the identity matrix. In the above modified Mihalas-Klein source terms,  $\sigma_a$  is the absorption cross section,  $\sigma_s$  is the scattering cross section, and  $\sigma_t = \sigma_a + \sigma_s$  is the total cross section [10–12].

The numerical approach of the hybrid Godunov method rests on understanding the balance law form of the above system of equations, where:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} + \frac{\partial H(U)}{\partial z} = S(U), \quad (7)$$

$$U = \begin{pmatrix} \rho \\ m_x \\ m_y \\ m_z \\ E \\ E_r \\ F_{r,x} \\ F_{r,y} \\ F_{r,z} \end{pmatrix}, \quad F(U) = \begin{pmatrix} m_x \\ \frac{m_x^2}{\rho} + p \\ \frac{m_x m_y}{\rho} \\ \frac{m_x m_z}{\rho} \\ (E + p) \frac{m_x}{\rho} \\ \mathbb{C} F_{r,x} \\ \mathbb{C} f_{xx} E_r \\ \mathbb{C} f_{yx} E_r \\ \mathbb{C} f_{zx} E_r \end{pmatrix}, \quad G(U) = \begin{pmatrix} m_y \\ \frac{m_y m_x}{\rho} \\ \frac{m_y^2}{\rho} + p \\ \frac{m_y m_z}{\rho} \\ (E + p) \frac{m_y}{\rho} \\ \mathbb{C} F_{r,y} \\ \mathbb{C} f_{xy} E_r \\ \mathbb{C} f_{yy} E_r \\ \mathbb{C} f_{zy} E_r \end{pmatrix}, \quad (8)$$

$$H(U) = \begin{pmatrix} m_z \\ \frac{m_z m_x}{\rho} \\ \frac{m_z m_y}{\rho} \\ \frac{m_z^2}{\rho} + p \\ (E + p) \frac{m_z}{\rho} \\ \mathbb{C} F_{r,z} \\ \mathbb{C} f_{xz} E_r \\ \mathbb{C} f_{yz} E_r \\ \mathbb{C} f_{zz} E_r \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ -\mathbb{P} S^{F_x} \\ -\mathbb{P} S^{F_y} \\ -\mathbb{P} S^{F_z} \\ -\mathbb{P} C S^E \\ \mathbb{C} S^E \\ \mathbb{C} S^{F_x} \\ \mathbb{C} S^{F_y} \\ \mathbb{C} S^{F_z} \end{pmatrix}, \quad (9)$$

where:

$$S^{F_x} = -\sigma_t \left( F_{r,x} - \frac{E_r}{\rho \mathbb{C}} (m_x + m_x f_{xx} + m_y f_{yx} + m_z f_{zx}) \right) + \sigma_a \frac{m_x}{\rho \mathbb{C}} (T^4 - E_r), \quad (10)$$

$$S^{F_y} = -\sigma_t \left( F_{r,y} - \frac{E_r}{\rho \mathbb{C}} (m_y + m_x f_{xy} + m_y f_{yy} + m_z f_{zy}) \right) + \sigma_a \frac{m_y}{\rho \mathbb{C}} (T^4 - E_r), \quad (11)$$

$$S^{F_z} = -\sigma_t \left( F_{r,z} - \frac{E_r}{\rho \mathbb{C}} (m_z + m_x f_{xz} + m_y f_{yz} + m_z f_{zz}) \right) + \sigma_a \frac{m_z}{\rho \mathbb{C}} (T^4 - E_r), \quad (12)$$

$$\begin{aligned} S^E &= \sigma_a (T^4 - E_r) + (\sigma_a - \sigma_s) \frac{m_x}{\rho \mathbb{C}} \left( F_{r,x} - \frac{E_r}{\rho \mathbb{C}} (m_x + m_x f_{xx} + m_y f_{yx} + m_z f_{zx}) \right) \\ &\quad + (\sigma_a - \sigma_s) \frac{m_y}{\rho \mathbb{C}} \left( F_{r,y} - \frac{E_r}{\rho \mathbb{C}} (m_y + m_x f_{xy} + m_y f_{yy} + m_z f_{zy}) \right) \\ &\quad + (\sigma_a - \sigma_s) \frac{m_z}{\rho \mathbb{C}} \left( F_{r,z} - \frac{E_r}{\rho \mathbb{C}} (m_z + m_x f_{xz} + m_y f_{yz} + m_z f_{zz}) \right). \end{aligned} \quad (13)$$

The quasi-linear form of this system of hyperbolic balance laws is:

$$\frac{\partial U}{\partial t} + A_x \frac{\partial U}{\partial x} + A_y \frac{\partial U}{\partial y} + A_z \frac{\partial U}{\partial z} = S(U), \quad (14)$$

$$A_x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}V^2 - u^2 & -(\gamma-3)u & -(\gamma-1)v & -(\gamma-1)w & (\gamma-1) & 0 & 0 & 0 & 0 \\ -uv & v & u & 0 & 0 & 0 & 0 & 0 & 0 \\ -uw & w & 0 & u & 0 & 0 & 0 & 0 & 0 \\ u\left(\frac{\gamma-1}{2}V^2 - \tilde{H}\right) & \tilde{H} - (\gamma-1)u^2 & -(\gamma-1)uv & -(\gamma-1)uw & \gamma u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{xx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{yx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{zx} & 0 & 0 & 0 \end{pmatrix} \quad (15)$$

$$A_y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -uv & v & u & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}V^2 - v^2 & -(\gamma-1)u & -(\gamma-3)v & -(\gamma-1)w & (\gamma-1) & 0 & 0 & 0 & 0 \\ -vw & 0 & w & v & 0 & 0 & 0 & 0 & 0 \\ v\left(\frac{\gamma-1}{2}V^2 - \tilde{H}\right) & -(\gamma-1)uv & \tilde{H} - (\gamma-1)v^2 & -(\gamma-1)vw & \gamma v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{xy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{yy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{zy} & 0 & 0 & 0 \end{pmatrix} \quad (16)$$

$$A_z = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -uw & w & 0 & u & 0 & 0 & 0 & 0 & 0 \\ -vw & 0 & w & v & 0 & 0 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}V^2 - w^2 & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-3)w & (\gamma-1) & 0 & 0 & 0 & 0 \\ w\left(\frac{\gamma-1}{2}V^2 - \tilde{H}\right) & -(\gamma-1)uw & -(\gamma-1)vw & \tilde{H} - (\gamma-1)w^2 & \gamma w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{C} \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{xz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{yz} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C}f_{zz} & 0 & 0 & 0 \end{pmatrix} \quad (17)$$

Here,  $u = m_x/\rho$ ,  $v = m_y/\rho$ , and  $w = m_z/\rho$  are the velocities in the  $x$ ,  $y$ , and  $z$  directions, respectively.  $V^2 = u^2 + v^2 + w^2$  and  $\tilde{H} = \frac{\gamma E}{\rho} - \frac{\gamma-1}{2}V^2$  is specific enthalpy. The Jacobians  $A_x$ ,  $A_y$ , and  $A_z$  have eigenvalues:  $\lambda_x = \{0, u, u \pm a, \pm f_{xx}^{1/2}\mathbb{C}\}$ ,  $\lambda_y = \{0, v, v \pm a, \pm f_{yy}^{1/2}\mathbb{C}\}$ , and  $\lambda_z = \{0, w, w \pm a, \pm f_{zz}^{1/2}\mathbb{C}\}$ , respectively. However, one must account for how the stiff momentum and energy source terms affect the hyperbolic structure.

## 2 Overview of the Multidimensional Algorithm

In radiation hydrodynamics, there are three important dynamical scales: the speed of sound (material flow), speed of light (radiation flow), and speed at which the source terms interact. Given such variation, one desires a numerical technique that treats the material flow explicitly, radiation flow implicitly, and source terms semi-implicitly.

### 2.1 Effective CFL Condition

Our primary interest when solving radiation hydrodynamical problems is higher-order resolution of material quantities while advancing the overall algorithm according to an effective CFL condition. This temporal advancement is associated with an effective CTU (corner transport upwind) scheme [1]:

$$\Delta t = \frac{\nu}{\max_{i,j,k} \left\{ \frac{|u(i,j,k)| + a_{\text{eff}}(i,j,k)}{\Delta x}, \frac{|v(i,j,k)| + a_{\text{eff}}(i,j,k)}{\Delta y}, \frac{|w(i,j,k)| + a_{\text{eff}}(i,j,k)}{\Delta z} \right\}}, \quad (18)$$

where  $\Delta t$  is the time step and  $\nu \in [0, 1]$  is the CFL number.  $\Delta x = (x_{\text{max}} - x_{\text{min}})/N_{\text{cell}}^x$ ,  $\Delta y = (y_{\text{max}} - y_{\text{min}})/N_{\text{cell}}^y$ , and  $\Delta z = (z_{\text{max}} - z_{\text{min}})/N_{\text{cell}}^z$  are the spatial resolutions for a given number of computational grid cells in the  $x$ ,  $y$ , and  $z$  directions, respectively.  $\max_{i,j,k} \{|u(i,j,k)| + a_{\text{eff}}(i,j,k), |v(i,j,k)| + a_{\text{eff}}(i,j,k), |w(i,j,k)| + a_{\text{eff}}(i,j,k)\}$  is the maximum material wave speed over all grid cells. Furthermore,  $a_{\text{eff}}$  is an estimate of the radiation modified sound speed which is obtained by carrying out an effective eigen-analysis of the material Jacobian. This analysis is presented in a later section. One chooses a CTU-type time step over other alternatives (e.g., donor-cell time step) because of how one couples transport in the corners of the computational grid cells. For the duration of this paper, one assumes that the grid cells are cubic ( $\Delta x = \Delta y = \Delta z$ ).

### 2.2 Algorithmic Steps

After defining  $\Delta t$ , the algorithm loops over the following steps:

1. Backward Euler Upwinding Scheme - implicitly advances the radiation quantities from time  $t_n$  to time  $t_{n+1}$ :

$$(E_r^n, F_{r,x}^n, F_{r,y}^n, F_{r,z}^n) \rightarrow (E_r^{n+1}, F_{r,x}^{n+1}, F_{r,y}^{n+1}, F_{r,z}^{n+1})$$

2. Modified Godunov Predictor Scheme - couples stiff source term effects to the hyperbolic structure of the balance laws for the material quantities and uses effective piecewise

linear extrapolation to spatially reconstruct material quantities at the left/right sides of cell interfaces in the  $x$ ,  $y$ , and  $z$  directions  $\{i \pm 1/2, j \pm 1/2, k \pm 1/2\}$ :

$$\tilde{U}_{L/R,i+1/2,j,k}^{m,n+1/2} \quad \tilde{U}_{L/R,i,j+1/2,k}^{m,n+1/2} \quad \tilde{U}_{L/R,i,j,k+1/2}^{m,n+1/2}$$

3. Flux Function - evaluates the passage of material across cell interfaces using the left/right material states as well as an approximate Riemann solver to obtain:

$$\begin{aligned} \tilde{F}_{i+1/2,j,k}^m &= F(\mathcal{R}(\tilde{U}_{L,i+1/2,j,k}^m, \tilde{U}_{R,i+1/2,j,k}^m)) \\ \tilde{G}_{i,j+1/2,k}^m &= G(\mathcal{R}(\tilde{U}_{L,i,j+1/2,k}^m, \tilde{U}_{R,i,j+1/2,k}^m)) \\ \tilde{H}_{i,j,k+1/2}^m &= H(\mathcal{R}(\tilde{U}_{L,i,j,k+1/2}^m, \tilde{U}_{R,i,j,k+1/2}^m)) \end{aligned}$$

4. CTU Correction - accounts for how the material quantities at the left/right sides of cell interfaces in one spatial direction are affected by the fluxes in the other two spatial directions:

$$\tilde{U}_{L/R}^{m,n+1/2} \rightarrow U_{L/R}^{m,n+1/2}$$

5. Flux Function - evaluates the passage of material across cell interfaces using the corrected left/right material states and the algorithmic machinery of Step 3 above
6. Modified Godunov Corrector Scheme - semi-implicitly advances the material quantities from time  $t_n$  to time  $t_{n+1}$ :

$$(\rho^n, m_x^n, m_y^n, m_z^n, E^n) \rightarrow (\rho^{n+1}, m_x^{n+1}, m_y^{n+1}, m_z^{n+1}, E^{n+1})$$

7. Apply boundary conditions

8. Compute next time step  $\Delta t$

In the above expressions,  $U$ ,  $U^r$ , and  $U^m$  represent all of the conserved quantities, radiation quantities  $(E_r, F_{r,x}, F_{r,y}, F_{r,z})$ , and material quantities  $(\rho, m_x, m_y, m_z, E)$ , respectively.  $F_{i+1/2,j,k}$ ,  $G_{i,j+1/2,k}$ , and  $H_{i,j,k+1/2}$  are fluxes directed across cell faces in the  $x$ ,  $y$ , and  $z$  directions, where  $\mathcal{R}$  represents the solution of a Riemann problem. The tilde that is above some of the left/right material states and fluxes designates quantities that have not yet been adjusted by the CTU correction. Cell centers are defined by three indices  $(i, j, k)$ , such that  $i \pm 1/2$ ,  $j \pm 1/2$ , and  $k \pm 1/2$  represent the location of a cell interface to the right/left of  $i$ ,  $j$ , and  $k$ , respectively.  $n$  designates the time discretization. Details about each step are explained in later sections. Lastly, the one-dimensional hybrid Godunov method of [18] was shown to

be consistent, stable, and accurate as well as coarsely gridded, well-balanced, and having some asymptotic preserving properties. For reasons cited in [18], these numerical properties should be able to be extended to the multidimensional algorithm. However, future tests will justify these claims.

### 3 Backward Euler Upwinding Scheme

This section presents the implicit scheme that advances the radiation quantities  $U^r$  according to the material flow scale. The stability of explicit schemes (e.g., Godunov-type methods) is governed by the CFL condition which restricts the allowable time step according to the fastest characteristic speed. However, if a hyperbolic system consists of multiscale waves (e.g., radiation hydrodynamics where  $c/a_\infty \sim 10^6$ ), then explicit schemes can become inefficient. For these types of problems, implicit methods are useful. Following [4, 7], one can construct implicit flux functions to approximate integrals at cell interfaces. In this context, a HLLE framework was implemented.

#### 3.1 HLLE Framework

The HLLE scheme is based on estimating the minimum and maximum wave speeds ( $s_{\min}, s_{\max}$ ). These quantities are uniquely defined for each Riemann problem that is associated with each interface of a computational grid cell. The numerical flux in one direction is calculated using the following formula:

$$F^{\text{HLLE}}(\mathcal{R}(U_L, U_R)) = \frac{1}{2} ((1 + C^s) (F(U_L) - s_{\min} U_L) + (1 - C^s) (F(U_R) - s_{\max} U_R)). \quad (19)$$

where  $C^s = (s_{\max} + s_{\min}) / (s_{\max} - s_{\min})$ . Defining the left/right states of the Riemann problem according to a first-order accurate (piecewise constant) reconstruction, forces the HLLE flux function to take the following form in each of the spatial directions:

$$F_{i+1/2,j,k}(\mathcal{R}(U_{L,i+1/2,j,k}, U_{R,i+1/2,j,k})) \rightarrow F_{i+1/2,j,k}^{\text{HLLE}}(\mathcal{R}(U_{i,j,k}, U_{i+1,j,k})), \quad (20)$$

$$G_{i,j+1/2,k}(\mathcal{R}(U_{L,i,j+1/2,k}, U_{R,i,j+1/2,k})) \rightarrow G_{i,j+1/2,k}^{\text{HLLE}}(\mathcal{R}(U_{i,j,k}, U_{i,j+1,k})), \quad (21)$$

$$H_{i,j,k+1/2}(\mathcal{R}(U_{L,i,j,k+1/2}, U_{R,i,j,k+1/2})) \rightarrow H_{i,j,k+1/2}^{\text{HLLE}}(\mathcal{R}(U_{i,j,k}, U_{i,j,k+1})), \quad (22)$$

A first-order accurate, backward Euler-type algorithm was used because of total variation diminishing (TVD) conditions. These issues were explored by [4, 18]. One makes the above explicit HLLE scheme implicit by defining the variables in the flux and source terms to be

at time  $t_{n+1}$  such that the exact integral formulation of the conservative differencing is:

$$\begin{aligned}
U_{i,j,k}^{n+1} = U_{i,j,k}^n & - \frac{\Delta t}{\Delta x} (F_{i+1/2,j,k}^{\text{HLL E}}(\mathcal{R}(U_{i,j,k}^{n+1}, U_{i+1,j,k}^{n+1})) - F_{i-1/2,j,k}^{\text{HLL E}}(\mathcal{R}(U_{i-1,j,k}^{n+1}, U_{i,j,k}^{n+1}))) \quad (23) \\
& - \frac{\Delta t}{\Delta y} (G_{i,j+1/2,k}^{\text{HLL E}}(\mathcal{R}(U_{i,j,k}^{n+1}, U_{i,j+1,k}^{n+1})) - G_{i,j-1/2,k}^{\text{HLL E}}(\mathcal{R}(U_{i,j-1,k}^{n+1}, U_{i,j,k}^{n+1}))) \\
& - \frac{\Delta t}{\Delta z} (H_{i,j,k+1/2}^{\text{HLL E}}(\mathcal{R}(U_{i,j,k}^{n+1}, U_{i,j,k+1}^{n+1})) - H_{i,j,k-1/2}^{\text{HLL E}}(\mathcal{R}(U_{i,j,k-1}^{n+1}, U_{i,j,k}^{n+1}))) \\
& + \Delta t S(U_{i,j,k}^{n+1}),
\end{aligned}$$

$$\begin{aligned}
F_{i+1/2,j,k}^{\text{HLL E}} = & \frac{1}{2} \left( 1 + C_{i+1/2,j,k}^{s,x} \right) (F(U_{i,j,k}^{n+1}) - s_{\min}^x U_{i,j,k}^{n+1}) \quad (24) \\
& + \frac{1}{2} \left( 1 - C_{i+1/2,j,k}^{s,x} \right) (F(U_{i+1,j,k}^{n+1}) - s_{\max}^x U_{i+1,j,k}^{n+1}),
\end{aligned}$$

$$\begin{aligned}
G_{i,j+1/2,k}^{\text{HLL E}} = & \frac{1}{2} \left( 1 + C_{i,j+1/2,k}^{s,y} \right) (G(U_{i,j,k}^{n+1}) - s_{\min}^y U_{i,j,k}^{n+1}) \quad (25) \\
& + \frac{1}{2} \left( 1 - C_{i,j+1/2,k}^{s,y} \right) (G(U_{i,j+1,k}^{n+1}) - s_{\max}^y U_{i,j+1,k}^{n+1}),
\end{aligned}$$

$$\begin{aligned}
H_{i,j,k+1/2}^{\text{HLL E}} = & \frac{1}{2} \left( 1 + C_{i,j,k+1/2}^{s,z} \right) (H(U_{i,j,k}^{n+1}) - s_{\min}^z U_{i,j,k}^{n+1}) \quad (26) \\
& + \frac{1}{2} \left( 1 - C_{i,j,k+1/2}^{s,z} \right) (H(U_{i,j,k+1}^{n+1}) - s_{\max}^z U_{i,j,k+1}^{n+1}).
\end{aligned}$$

### 3.2 Applying the Backward Euler HLL E Scheme

If one considers only the radiation part of the equations for radiation hydrodynamics (Equations 4 and 5) [17, 18], then the variables, fluxes, and source terms are:

$$U^r = \begin{pmatrix} E_r \\ F_{r,x} \\ F_{r,y} \\ F_{r,z} \end{pmatrix}, \quad F^r(U) = \begin{pmatrix} \mathbb{C} F_{r,x} \\ \mathbb{C} f_{xx} E_r \\ \mathbb{C} f_{yx} E_r \\ \mathbb{C} f_{zx} E_r \end{pmatrix}, \quad G^r(U) = \begin{pmatrix} \mathbb{C} F_{r,y} \\ \mathbb{C} f_{xy} E_r \\ \mathbb{C} f_{yy} E_r \\ \mathbb{C} f_{zy} E_r \end{pmatrix}, \quad (27)$$

$$H^r(U) = \begin{pmatrix} \mathbb{C} F_{r,z} \\ \mathbb{C} f_{xz} E_r \\ \mathbb{C} f_{yz} E_r \\ \mathbb{C} f_{zz} E_r \end{pmatrix}, \quad S^r(U) = \begin{pmatrix} \mathbb{C} S^E \\ \mathbb{C} S^{F_x} \\ \mathbb{C} S^{F_y} \\ \mathbb{C} S^{F_z} \end{pmatrix}, \quad (28)$$

where the eigenvalues of the radiation subsystem in the free streaming limit ( $\sigma_a, \sigma_t \sim \mathcal{O}(\epsilon)$ ) are  $\lambda_x = \{0, \pm f_{xx}^{1/2} \mathbb{C}\}$ ,  $\lambda_y = \{0, \pm f_{yy}^{1/2} \mathbb{C}\}$ , and  $\lambda_z = \{0, \pm f_{zz}^{1/2} \mathbb{C}\}$  for each of the spatial

directions. Given that the HLLE scheme uses minimum and maximum wave speeds to compute fluxes at cell interfaces, one defines the following equations:

$$\begin{aligned}
 s_{\min}^x &= \lambda_{x,L}^-(i, j, k) = -f_{xx}(i, j, k)^{1/2}\mathbb{C}, & s_{\max}^x &= \lambda_{x,R}^+(i + 1, j, k) = f_{xx}(i + 1, j, k)^{1/2}\mathbb{C}, \\
 C_{i+1/2,j,k}^{s,x} &= \frac{f_{xx}(i + 1, j, k)^{1/2} - f_{xx}(i, j, k)^{1/2}}{f_{xx}(i + 1, j, k)^{1/2} + f_{xx}(i, j, k)^{1/2}}, \\
 s_{\min}^y &= \lambda_{y,L}^-(i, j, k) = -f_{yy}(i, j, k)^{1/2}\mathbb{C}, & s_{\max}^y &= \lambda_{y,R}^+(i, j + 1, k) = f_{yy}(i, j + 1, k)^{1/2}\mathbb{C}, \\
 C_{i,j+1/2,k}^{s,y} &= \frac{f_{yy}(i, j + 1, k)^{1/2} - f_{yy}(i, j, k)^{1/2}}{f_{yy}(i, j + 1, k)^{1/2} + f_{yy}(i, j, k)^{1/2}}, \\
 s_{\min}^z &= \lambda_{z,L}^-(i, j, k) = -f_{zz}(i, j, k)^{1/2}\mathbb{C}, & s_{\max}^z &= \lambda_{z,R}^+(i, j, k + 1) = f_{zz}(i, j, k + 1)^{1/2}\mathbb{C}, \\
 C_{i,j,k+1/2}^{s,z} &= \frac{f_{zz}(i, j, k + 1)^{1/2} - f_{zz}(i, j, k)^{1/2}}{f_{zz}(i, j, k + 1)^{1/2} + f_{zz}(i, j, k)^{1/2}}.
 \end{aligned}$$

Here,  $f_{xx}$ ,  $f_{yy}$ , and  $f_{zz}$  arise from the closure relation  $\mathbb{P}_r = fE_r$  and is either a user defined quantity or obtained by solving the radiation transport equation. If  $f$  varies spatially, then  $C^{s,x}$ ,  $C^{s,y}$ , and  $C^{s,z}$  are non-zero. Defining or computing  $f(\mathbf{x}, t)$  precedes the backward Euler update of  $U^r$ . However, if  $f$  is assumed to be spatially and temporally constant, then  $C^{s,x}, C^{s,y}, C^{s,z} = 0$ . Future work will update  $f(\mathbf{x}, t)$  at each iteration by solving the radiation transport equation.

### 3.3 Matrix Equation for the Radiation Components

Inputting  $U^{r,n+1}$ ,  $F^r(U^{r,n+1})$ ,  $G^r(U^{r,n+1})$ ,  $H^r(U^{r,n+1})$  and  $S^r(U^{m,n}, U^{r,n+1})$  into Equations 23-26 gives the following four implicit difference equations:

$$\begin{aligned}
& E_r^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k-1)^{1/2} \right] \\
+ & F_{r,z}^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) \right] \\
+ & E_r^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j-1, k)^{1/2} \right] \\
+ & F_{r,y}^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) \right] \\
+ & E_r^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i-1, j, k)^{1/2} \right] \\
+ & F_{r,x}^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) \right] \\
+ & E_r^{n+1}(i, j, k) \left[ 1 + d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} \right. \\
& \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} \\
& \quad + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_2 \sigma_a \\
& \quad + \frac{d_2(\sigma_a - \sigma_s)m_x}{\rho^2 C^2} (m_x + m_x f_{xx}(i, j, k) + m_y f_{yx}(i, j, k) + m_z f_{zx}(i, j, k)) \\
& \quad + \frac{d_2(\sigma_a - \sigma_s)m_y}{\rho^2 C^2} (m_y + m_x f_{xy}(i, j, k) + m_y f_{yy}(i, j, k) + m_z f_{zy}(i, j, k)) \\
& \quad \left. + \frac{d_2(\sigma_a - \sigma_s)m_z}{\rho^2 C^2} (m_z + m_x f_{xz}(i, j, k) + m_y f_{yz}(i, j, k) + m_z f_{zz}(i, j, k)) \right] \\
+ & F_{r,x}^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) - d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) - \frac{d_2(\sigma_a - \sigma_s)m_x}{\rho C} \right] \\
+ & F_{r,y}^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) - d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) - \frac{d_2(\sigma_a - \sigma_s)m_y}{\rho C} \right] \\
+ & F_{r,z}^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) - d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) - \frac{d_2(\sigma_a - \sigma_s)m_z}{\rho C} \right] \\
+ & E_r^{n+1}(i+1, j, k) \left[ -d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i+1, j, k)^{1/2} \right] \\
+ & F_{r,x}^{n+1}(i+1, j, k) \left[ d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) \right] \\
+ & E_r^{n+1}(i, j+1, k) \left[ -d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j+1, k)^{1/2} \right] \\
+ & F_{r,y}^{n+1}(i, j+1, k) \left[ d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) \right] \\
+ & E_r^{n+1}(i, j, k+1) \left[ -d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k+1)^{1/2} \right] \\
+ & F_{r,z}^{n+1}(i, j, k+1) \left[ d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) \right] = E_r^n(i, j, k) + d_2 \sigma_a T^4,
\end{aligned} \tag{29}$$

$$\begin{aligned}
 & E_r^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{xz}(i, j, k-1) \right] \\
 + & F_{r,x}^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k-1)^{1/2} \right] \\
 + & E_r^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{xy}(i, j-1, k) \right] \\
 + & F_{r,x}^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j-1, k)^{1/2} \right] \\
 + & E_r^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i-1, j, k) \right] \\
 + & F_{r,x}^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i-1, j, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i, j, k) - d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i, j, k) \right. \\
 & \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{xy}(i, j, k) - d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{xy}(i, j, k) \\
 & \quad + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{xz}(i, j, k) - d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{xz}(i, j, k) \\
 & \quad \left. - \frac{d_2 \sigma_t}{\rho C} (m_x + m_x f_{xx}(i, j, k) + m_y f_{yx}(i, j, k) + m_z f_{zx}(i, j, k)) + \frac{d_2 \sigma_a m_x}{\rho^n C} \right] \\
 + & F_{r,x}^{n+1}(i, j, k) \left[ 1 + d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} \right. \\
 & \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} \\
 & \quad \left. + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_2 \sigma_t \right] \\
 + & E_r^{n+1}(i+1, j, k) \left[ d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i+1, j, k) \right] \\
 + & F_{r,x}^{n+1}(i+1, j, k) \left[ -d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i+1, j, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j+1, k) \left[ d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{xy}(i, j+1, k) \right] \\
 + & F_{r,x}^{n+1}(i, j+1, k) \left[ -d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j+1, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j, k+1) \left[ d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{xz}(i, j, k+1) \right] \\
 + & F_{r,x}^{n+1}(i, j, k+1) \left[ -d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k+1)^{1/2} \right] = F_{r,x}^n(i, j, k) + \frac{d_2 \sigma_a m_x T^4}{\rho C},
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 & E_r^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{yz}(i, j, k-1) \right] \\
 + & F_{r,y}^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k-1)^{1/2} \right] \\
 + & E_r^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j-1, k) \right] \\
 + & F_{r,y}^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j-1, k)^{1/2} \right] \\
 + & E_r^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{yx}(i-1, j, k) \right] \\
 + & F_{r,y}^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i-1, j, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{yx}(i, j, k) - d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{yx}(i, j, k) \right. \\
 & \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j, k) - d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j, k) \\
 & \quad + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{yz}(i, j, k) - d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{yz}(i, j, k) \\
 & \quad \left. - \frac{d_2 \sigma_t}{\rho C} (m_y + m_x f_{xy}(i, j, k) + m_y f_{yy}(i, j, k) + m_z f_{zy}(i, j, k)) + \frac{d_2 \sigma_a m_y}{\rho C} \right] \\
 + & F_{r,y}^{n+1}(i, j, k) \left[ 1 + d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} \right. \\
 & \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} \\
 & \quad \left. + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_2 \sigma_t \right] \\
 + & E_r^{n+1}(i+1, j, k) \left[ d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{yx}(i+1, j, k) \right] \\
 + & F_{r,y}^{n+1}(i+1, j, k) \left[ -d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i+1, j, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j+1, k) \left[ d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j+1, k) \right] \\
 + & F_{r,y}^{n+1}(i, j+1, k) \left[ -d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j+1, k)^{1/2} \right] \\
 + & E_r^{n+1}(i, j, k+1) \left[ d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{yz}(i, j, k+1) \right] \\
 + & F_{r,y}^{n+1}(i, j, k+1) \left[ -d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k+1)^{1/2} \right] = F_{r,y}^n(i, j, k) + \frac{d_2 \sigma_a m_y T^4}{\rho C},
 \end{aligned} \tag{31}$$

$$\begin{aligned}
& E_r^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k-1) \right] \\
& + F_{r,z}^{n+1}(i, j, k-1) \left[ -d_1 \left( 1 + C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k-1)^{1/2} \right] \\
& + E_r^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{zy}(i, j-1, k) \right] \\
& + F_{r,z}^{n+1}(i, j-1, k) \left[ -d_1 \left( 1 + C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j-1, k)^{1/2} \right] \\
& + E_r^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{zx}(i-1, j, k) \right] \\
& + F_{r,z}^{n+1}(i-1, j, k) \left[ -d_1 \left( 1 + C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i-1, j, k)^{1/2} \right] \\
& + E_r^{n+1}(i, j, k) \left[ d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{zx}(i, j, k) - d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{zx}(i, j, k) \right. \\
& \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{zy}(i, j, k) - d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{zy}(i, j, k) \\
& \quad + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k) - d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k) \\
& \quad \left. - \frac{d_2 \sigma_t}{\rho C} (m_z + m_x f_{xz}(i, j, k) + m_y f_{yz}(i, j, k) + m_z f_{zz}(i, j, k)) + \frac{d_2 \sigma_a m_z}{\rho C} \right] \\
& + F_{r,z}^{n+1}(i, j, k) \left[ 1 + d_1 \left( 1 + C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i-1/2,j,k}^{s,x} \right) f_{xx}(i, j, k)^{1/2} \right. \\
& \quad + d_1 \left( 1 + C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j-1/2,k}^{s,y} \right) f_{yy}(i, j, k)^{1/2} \\
& \quad \left. + d_1 \left( 1 + C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_1 \left( 1 - C_{i,j,k-1/2}^{s,z} \right) f_{zz}(i, j, k)^{1/2} + d_2 \sigma_t \right] \\
& + E_r^{n+1}(i+1, j, k) \left[ d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{zx}(i+1, j, k) \right] \\
& + F_{r,z}^{n+1}(i+1, j, k) \left[ -d_1 \left( 1 - C_{i+1/2,j,k}^{s,x} \right) f_{xx}(i+1, j, k)^{1/2} \right] \\
& + E_r^{n+1}(i, j+1, k) \left[ d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{zy}(i, j+1, k) \right] \\
& + F_{r,z}^{n+1}(i, j+1, k) \left[ -d_1 \left( 1 - C_{i,j+1/2,k}^{s,y} \right) f_{yy}(i, j+1, k)^{1/2} \right] \\
& + E_r^{n+1}(i, j, k+1) \left[ d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k+1) \right] \\
& + F_{r,z}^{n+1}(i, j, k+1) \left[ -d_1 \left( 1 - C_{i,j,k+1/2}^{s,z} \right) f_{zz}(i, j, k+1)^{1/2} \right] = F_{r,z}^n(i, j, k) + \frac{d_2 \sigma_a m_z T^4}{\rho C},
\end{aligned} \tag{32}$$

where  $d_1 = \Delta t C / 2 \Delta x$ ,  $d_2 = \Delta t C$ . All of the material quantities ( $\rho, m_x, m_y, m_z, E$ ) that appear in the above equations have the spatial index  $(i, j, k)$  and temporal index  $n$ . Additionally, the material temperature enters the above equations via the following relation:  $T_{i,j,k}^n = p_{i,j,k}^n / \rho_{i,j,k}^n = (\gamma - 1) \left( \frac{E_{i,j,k}^n}{\rho_{i,j,k}^n} - \frac{1}{2} \left( \frac{M_{i,j,k}^n}{\rho_{i,j,k}^n} \right)^2 \right)$ , where  $M^2 = m_x^2 + m_y^2 + m_z^2$ . To understand how the multidimensional equations fit into the linear algebraic description  $Ax = b$ , it is important to understand that the algorithm cycles through the indices  $(i, j, k)$  in the following manner: `for(k=1;N;k++){ for(j=1;N;j++){ for(i=1;N;i++){ ... }}`. Furthermore, the algorithm cycles through the radiation quantities in the following order:  $(E_r, F_{r,x}, F_{r,y}, F_{r,z})$ . Assuming that  $N_{cell}^x = N_{cell}^y = N_{cell}^z$ , one expects the matrix  $A$  to have dimensions  $\dim(A) = 4N^3 \times 4N^3$ . Moreover, the solution vector  $x$  will contain the following sequence of entries:

$$\begin{aligned}
x = & \left[ E_r^{n+1}(i, j, k), F_{r,x}^{n+1}(i, j, k), F_{r,y}^{n+1}(i, j, k), F_{r,z}^{n+1}(i, j, k), E_r^{n+1}(i+1, j, k), \dots, \right. \\
& E_r^{n+1}(i+2, j, k), \dots, F_{r,z}^{n+1}(N, j, k), E_r^{n+1}(1, j+1, k), \dots, \\
& \left. F_{r,z}^{n+1}(N, N, k), E_r^{n+1}(1, 1, k+1), \dots \right]^T.
\end{aligned}$$



## 4.1 Applying the Modified Godunov Predictor Scheme

If one considers only the material component in Equations 1-3, then the variables, fluxes, and source terms are:

$$U = \begin{pmatrix} \rho \\ m_x \\ m_y \\ m_z \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} m_x \\ \frac{m_x^2}{\rho} + p \\ \frac{m_x m_y}{\rho} \\ \frac{m_x m_z}{\rho} \\ (E + p) \frac{m_x}{\rho} \end{pmatrix}, \quad G(U) = \begin{pmatrix} m_y \\ \frac{m_y m_x}{\rho} \\ \frac{m_y^2}{\rho} + p \\ \frac{m_y m_z}{\rho} \\ (E + p) \frac{m_y}{\rho} \end{pmatrix},$$

$$H(U) = \begin{pmatrix} m_z \\ \frac{m_z m_x}{\rho} \\ \frac{m_z m_y}{\rho} \\ \frac{m_z^2}{\rho} + p \\ (E + p) \frac{m_z}{\rho} \end{pmatrix}, \quad S^m(U) = \begin{pmatrix} 0 \\ -\mathbb{P}S^{F_x} \\ -\mathbb{P}S^{F_y} \\ -\mathbb{P}S^{F_z} \\ -\mathbb{P}CS^E \end{pmatrix}.$$

In order to compute  $\mathcal{I}$ , one first computes  $\nabla_{U^m} S^m(U)$ . Therefore:

$$\nabla_{U^m} S^m(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\mathbb{P}S_{\rho}^{F_x} & -\mathbb{P}S_{m_x}^{F_x} & -\mathbb{P}S_{m_y}^{F_x} & -\mathbb{P}S_{m_z}^{F_x} & -\mathbb{P}S_E^{F_x} \\ -\mathbb{P}S_{\rho}^{F_y} & -\mathbb{P}S_{m_x}^{F_y} & -\mathbb{P}S_{m_y}^{F_y} & -\mathbb{P}S_{m_z}^{F_y} & -\mathbb{P}S_E^{F_y} \\ -\mathbb{P}S_{\rho}^{F_z} & -\mathbb{P}S_{m_x}^{F_z} & -\mathbb{P}S_{m_y}^{F_z} & -\mathbb{P}S_{m_z}^{F_z} & -\mathbb{P}S_E^{F_z} \\ -\mathbb{P}CS_{\rho}^E & -\mathbb{P}CS_{m_x}^E & -\mathbb{P}CS_{m_y}^E & -\mathbb{P}CS_{m_z}^E & -\mathbb{P}CS_E^E \end{pmatrix}, \quad (35)$$

where the partial derivatives are presented in Appendix 1.

### 4.1.1 Simplifying $\nabla_{U^m} S^m(U)$

In its current form,  $\nabla_{U^m} S^m(U)$  in Equation 35 leads to a propagation operator  $\mathcal{I}$  that is difficult to work with algebraically. By inspection, the material momentum source terms are not the dominant factors defining the stiffness associated with the matter-radiation coupling, such that  $-\mathbb{P}\mathbf{S}^{\mathbf{F}} < \mathcal{O}(1)$  even in the strong equilibrium diffusion limit. Additionally, one finds that the derivative of the material momentum source term with respect to the variables  $U^m$  has the following magnitude  $-\mathbb{P}\mathbf{S}_{U^m}^{\mathbf{F}} < \mathcal{O}(1)$ . Therefore,  $-\mathbb{P}\mathbf{S}^{\mathbf{F}}$  can be included as a body force (e.g., gravity).

It is the material energy source term  $-\mathbb{P}CS^E$  that defines the stiffness associated with the problem. By inspection of the contributing terms,  $-\mathbb{P}CS^E < \mathcal{O}(C)$  in the strong equilibrium

diffusion limit. Additionally, one finds that the derivative of the material energy source term with respect to the variables  $U^m$  has the following magnitude  $-\mathbb{P}CS_{U^m}^E < \mathcal{O}(\mathbb{C}^2)$ . Therefore, one only needs to use  $-\mathbb{P}CS^E$  to define  $\nabla_{U^m} S^m(U)$ , such that:

$$\nabla_{U^m} S^m(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\mathbb{P}CS_\rho^E & -\mathbb{P}CS_{m_x}^E & -\mathbb{P}CS_{m_y}^E & -\mathbb{P}CS_{m_z}^E & -\mathbb{P}CS_E^E \end{pmatrix}. \quad (36)$$

$\nabla_{U^m} S^m(U)$  is further simplified by examining  $S_\rho^E$ ,  $S_{m_x}^E$ ,  $S_{m_y}^E$ ,  $S_{m_z}^E$ , and  $S_E^E$  in the equilibrium diffusion limit and neglecting terms that have magnitudes of or less than  $\mathcal{O}(\mathbb{C})$ . Therefore:

$$S_\rho^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-E}{\rho^2} + \frac{M^2}{\rho^3} \right) \quad (37)$$

$$S_{m_x}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_x}{\rho^2} \right) \quad (38)$$

$$S_{m_y}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_y}{\rho^2} \right) \quad (39)$$

$$S_{m_z}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_z}{\rho^2} \right) \quad (40)$$

$$S_E^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{1}{\rho} \right). \quad (41)$$

It is important to note that these partial derivatives have the same stiff magnitude  $4\sigma_a T^3 (\gamma - 1)$ . This insight simplifies algebraic manipulation.

If  $\nabla_{U^m} S^m(U)$  is diagonalizable, then  $\nabla_{U^m} S^m(U) = RDR^{-1}$ . Here,  $D = \text{diag}(0, 0, 0, 0, -\mathbb{P}CS_E^E)$  and  $R$  is a matrix whose columns are the right eigenvectors. Below, one sees how the stiff magnitudes cancel out:

$$R = \begin{pmatrix} \frac{-S_E^E}{S_\rho^E} & \frac{-S_{m_z}^E}{S_\rho^E} & \frac{-S_{m_y}^E}{S_\rho^E} & \frac{-S_{m_x}^E}{S_\rho^E} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} \frac{-S_\rho^E}{S_E^E} & \frac{-S_{m_x}^E}{S_E^E} & \frac{-S_{m_y}^E}{S_E^E} & \frac{-S_{m_z}^E}{S_E^E} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{S_\rho^E}{S_E^E} & \frac{S_{m_x}^E}{S_E^E} & \frac{S_{m_y}^E}{S_E^E} & \frac{S_{m_z}^E}{S_E^E} & 1 \end{pmatrix}. \quad (42)$$

### 4.1.2 Propagation Operator $\mathcal{I}$

Since one is considering a modified Godunov scheme with a predictor step of  $\Delta t/2$  [13]:

$$\mathcal{I}\left(\frac{\Delta t}{2}\right) = \frac{1}{\Delta t/2} \int_0^{\Delta t/2} e^{\tau \nabla_{U^m} S^m(U)} d\tau \quad (43)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (\alpha - 1) \frac{S_\rho^E}{S_E^E} & (\alpha - 1) \frac{S_{m_x}^E}{S_E^E} & (\alpha - 1) \frac{S_{m_y}^E}{S_E^E} & (\alpha - 1) \frac{S_{m_z}^E}{S_E^E} & \alpha \end{pmatrix} \quad (44)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (1 - \alpha) \left(\frac{E}{\rho} - \frac{M^2}{\rho^2}\right) & (1 - \alpha) \frac{m_x}{\rho} & (1 - \alpha) \frac{m_y}{\rho} & (1 - \alpha) \frac{m_z}{\rho} & \alpha \end{pmatrix}, \quad (45)$$

where  $\alpha = (1 - \exp(-\text{PCS}_E^E \Delta t/2)) / (\text{PCS}_E^E \Delta t/2)$ . Since  $S_E^E \geq 0$  across all relevant parameter space,  $0 \leq \alpha \leq 1$ . This property is important when considering stability and the subcharacteristic condition which is discussed later in this paper.

## 4.2 Effective Material Jacobians - $A_{x,\text{eff}}^m$ , $A_{y,\text{eff}}^m$ , $A_{z,\text{eff}}^m$

The effects of the stiff source terms on the hyperbolic structure are accounted for by transforming to a moving-mesh (Lagrangian) frame ( $A_{x,L}^m = A_x^m - uI$ ,  $A_{y,L}^m = A_y^m - vI$ ,  $A_{z,L}^m = A_z^m - wI$ ), applying the propagation operator  $\mathcal{I}$  to  $A_L^m$ , and transforming back to an Eulerian frame, such that the effective material Jacobians ( $A_{x,\text{eff}}^m = \mathcal{I}A_{x,L}^m + uI$ ,  $A_{y,\text{eff}}^m = \mathcal{I}A_{y,L}^m + vI$ ,  $A_{z,\text{eff}}^m = \mathcal{I}A_{z,L}^m + wI$ ) are given by [13]:

$$A_{x,\text{eff}}^m = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{\gamma-1}{2}V^2 - u^2 & -(\gamma-3)u & -(\gamma-1)v & -(\gamma-1)w & (\gamma-1) \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ u \left[ \frac{\gamma-1}{2}V^2 - \alpha\tilde{H} - (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) \right] & -(\gamma-1)u^2 + \alpha\tilde{H} + (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) & -(\gamma-1)uv & -(\gamma-1)uw & \gamma u \end{pmatrix},$$

$$A_{y,\text{eff}}^m = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -uv & v & u & 0 & 0 \\ \frac{\gamma-1}{2}V^2 - v^2 & -(\gamma-1)u & -(\gamma-3)v & -(\gamma-1)w & (\gamma-1) \\ -vw & 0 & w & v & 0 \\ v \left[ \frac{\gamma-1}{2}V^2 - \alpha\tilde{H} - (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) \right] & -(\gamma-1)uv & -(\gamma-1)v^2 + \alpha\tilde{H} + (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) & -(\gamma-1)vw & \gamma v \end{pmatrix},$$

$$A_{z,\text{eff}}^m = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -uw & w & 0 & u & 0 \\ -vw & 0 & w & v & 0 \\ \frac{\gamma-1}{2}V^2 - w^2 & -(\gamma-1)u & -(\gamma-1)v & -(\gamma-3)w & (\gamma-1) \\ w \left[ \frac{\gamma-1}{2}V^2 - \alpha\tilde{H} - (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) \right] & -(\gamma-1)uw & -(\gamma-1)vw & -(\gamma-1)w^2 + \alpha\tilde{H} + (1-\alpha) \left( \frac{T}{\gamma-1} + \frac{1}{2}V^2 \right) & \gamma w \end{pmatrix}.$$

These Jacobians have the following eigenvalues:  $\lambda_{x,\text{eff},\{-,0,+\}}^m = \{u - a_{\text{eff}}, u, u + a_{\text{eff}}\}$ ,  $\lambda_{y,\text{eff},\{-,0,+\}}^m = \{v - a_{\text{eff}}, v, v + a_{\text{eff}}\}$ , and  $\lambda_{z,\text{eff},\{-,0,+\}}^m = \{w - a_{\text{eff}}, w, w + a_{\text{eff}}\}$ , respectively. Here, the effective sound speed  $a_{\text{eff}}$  (i.e., the radiation modified sound speed) is:

$$a_{\text{eff}}^2 = -\frac{\gamma-1}{2}V^2 + \alpha(\gamma-1)\tilde{H} + (1-\alpha) \left( T + \frac{\gamma-1}{2}V^2 \right) \quad (46)$$

$$= \alpha \frac{\gamma p}{\rho} + (1-\alpha)T \quad (47)$$

$$= (\alpha(\gamma-1) + 1) \frac{p}{\rho}, \quad (48)$$

where  $T = p/\rho$  because of the choice of non-dimensionalization. Here, one notices that  $\tilde{H}, (T + (\gamma-1)V^2/2) \geq 0$  across all relevant parameter space such that the effective sound speed  $a_{\text{eff}}$  admits the following limits:

$$-\text{PCSE}_E^E \rightarrow 0 \Rightarrow \alpha \rightarrow 1 \Rightarrow a_{\text{eff}}^2 \rightarrow -\frac{\gamma-1}{2}V^2 + (\gamma-1)\tilde{H} = \frac{\gamma p}{\rho} \quad (\text{adiabatic}) \quad (49)$$

$$-\text{PCSE}_E^E \rightarrow -\infty \Rightarrow \alpha \rightarrow 0 \Rightarrow a_{\text{eff}}^2 \rightarrow T = \frac{p}{\rho} \quad (\text{isothermal}). \quad (50)$$

When examining Equations 49 and 50, one sees that the subcharacteristic condition for material wave speeds is satisfied in each spatial direction, such that [13]:

$$\lambda_{x,-}^m = u - a_{\text{ad}} \leq \lambda_{x,\text{eff},-}^m = u - a_{\text{eff}} \leq \lambda_{x,0}^m = \lambda_{x,\text{eff},0}^m = u \leq \lambda_{x,\text{eff},+}^m = u + a_{\text{eff}} \leq \lambda_{x,\text{eff},+}^m = u + a_{\text{ad}}, \quad (51)$$

$$\lambda_{y,-}^m = v - a_{\text{ad}} \leq \lambda_{y,\text{eff},-}^m = v - a_{\text{eff}} \leq \lambda_{y,0}^m = \lambda_{y,\text{eff},0}^m = v \leq \lambda_{y,\text{eff},+}^m = v + a_{\text{eff}} \leq \lambda_{y,\text{eff},+}^m = v + a_{\text{ad}}, \quad (52)$$

$$\lambda_{z,-}^m = w - a_{\text{ad}} \leq \lambda_{z,\text{eff},-}^m = w - a_{\text{eff}} \leq \lambda_{z,0}^m = \lambda_{z,\text{eff},0}^m = w \leq \lambda_{z,\text{eff},+}^m = w + a_{\text{eff}} \leq \lambda_{z,\text{eff},+}^m = w + a_{\text{ad}}, \quad (53)$$

This condition is necessary for the stability of the system and guarantees that the numerical solution tends to the solution of the equilibrium equation as the relaxation time tends to zero. Additionally, the structure of the equations remains consistent with respect to classical Godunov methods. Therefore, the CFL CTU time-stepping condition applies. Lastly, the right material eigenvectors  $R_{\{x,y,z\},\text{eff}}^m$  (stored as columns) and left material eigenvectors  $L_{\{x,y,z\},\text{eff}}^m$  (stored as rows) are given in Appendix 2.

### 4.3 Computing Left/Right States

In the modified Godunov predictor scheme one uses effective piecewise linear extrapolation to spatially reconstruct material quantities at the left/right sides of cell interfaces. This technique is shown in the following relations for each spatial direction:

$$\begin{aligned} \tilde{U}_{L,i+1/2,j,k}^{m,n+1/2} &= U_{i,j,k}^{m,n} + \frac{1}{2} \left( I - \frac{\Delta t}{\Delta x} A_{x,\text{eff}}^m(U_{i,j,k}^{m,n}) \right) P_{\Delta}^x(\Delta U_{i,j,k}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i,j,k}^{m,n}, U_{i,j,k}^{r,n+1}), \end{aligned} \quad (54)$$

$$\begin{aligned} \tilde{U}_{R,i+1/2,j,k}^{m,n+1/2} &= U_{i+1,j,k}^{m,n} - \frac{1}{2} \left( I + \frac{\Delta t}{\Delta x} A_{x,\text{eff}}^m(U_{i+1,j,k}^{m,n}) \right) P_{\Delta}^x(\Delta U_{i+1,j,k}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i+1,j,k}^{m,n}, U_{i+1,j,k}^{r,n+1}), \end{aligned} \quad (55)$$

$$\begin{aligned} \tilde{U}_{L,i,j+1/2,k}^{m,n+1/2} &= U_{i,j,k}^{m,n} + \frac{1}{2} \left( I - \frac{\Delta t}{\Delta y} A_{y,\text{eff}}^m(U_{i,j,k}^{m,n}) \right) P_{\Delta}^y(\Delta U_{i,j,k}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i,j,k}^{m,n}, U_{i,j,k}^{r,n+1}), \end{aligned} \quad (56)$$

$$\begin{aligned} \tilde{U}_{R,i,j+1/2,k}^{m,n+1/2} &= U_{i,j+1,k}^{m,n} - \frac{1}{2} \left( I + \frac{\Delta t}{\Delta y} A_{y,\text{eff}}^m(U_{i,j+1,k}^{m,n}) \right) P_{\Delta}^y(\Delta U_{i,j+1,k}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i,j+1,k}^{m,n}, U_{i,j+1,k}^{r,n+1}), \end{aligned} \quad (57)$$

$$\begin{aligned} \tilde{U}_{L,i,j,k+1/2}^{m,n+1/2} &= U_{i,j,k}^{m,n} + \frac{1}{2} \left( I - \frac{\Delta t}{\Delta z} A_{z,\text{eff}}^m(U_{i,j,k}^{m,n}) \right) P_{\Delta}^z(\Delta U_{i,j,k}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i,j,k}^{m,n}, U_{i,j,k}^{r,n+1}), \end{aligned} \quad (58)$$

$$\begin{aligned} \tilde{U}_{R,i,j,k+1/2}^{m,n+1/2} &= U_{i,j,k+1}^{m,n} - \frac{1}{2} \left( I + \frac{\Delta t}{\Delta z} A_{z,\text{eff}}^m(U_{i,j,k+1}^{m,n}) \right) P_{\Delta}^z(\Delta U_{i,j,k+1}^{m,n}) \\ &+ \frac{\Delta t}{2} \mathcal{I} \left( \frac{\Delta t}{2} \right) S^m(U_{i,j,k+1}^{m,n}, U_{i,j,k+1}^{r,n+1}). \end{aligned} \quad (59)$$

where  $P_{\Delta}^{\{x,y,z\}}$  is a slope limiting function used to eliminate spurious oscillations in the  $x$ ,  $y$ , and  $z$  directions,  $re_y$ , and  $and_y$ , and  $z$  directions,  $re_y$ , and  $and_y$ , and  $z$  directions,  $re_y$ , and  $and_y$ , and  $z$  directions, respectively. Slope limiting is performed for each of the material quantities. Although most slope limiters can be used, this algorithm employs the extremum-preserving [2, 16] and traditional van Leer limiters (also referred to as the MUSCL limiter).

These techniques can be implemented either componentwise or across characteristic fields. After reconstructing the material quantities in each of the spatial directions, an approximate Riemann solver evaluates the passage of material into and out of each computational cell by using the material states that are to the left/right of the cell interfaces [15]. It is important to emphasize that these flux functions do not directly account for the influence of radiation on the material quantities. Rather, the radiation effects are included via the source terms, propagation operator, and effective material Jacobian.

## 5 Corner Transport Upwind Correction

Before advancing the material quantities from time  $t_n$  to time  $t_{n+1}$ , one accounts for how the left/right states ( $\tilde{U}_{L/R,i+1/2,j,k}^m$ ,  $\tilde{U}_{L/R,i,j+1/2,k}^m$ ,  $\tilde{U}_{L/R,i,j,k+1/2}^m$ ) and thus the fluxes ( $\tilde{F}_{i+1/2,j,k}^m$ ,  $\tilde{G}_{i,j+1/2,k}^m$ ,  $\tilde{H}_{i,j,k+1/2}^m$ ) are affected by transport in the other spatial directions. In particular, one corrects for material propagating across the corners of a computational cell in the following manner [1]:

$$U_{L,i+1/2,j,k}^{m,n+1/2} = \tilde{U}_{L,i+1/2,j,k}^{m,n+1/2} - \frac{\Delta t}{2\Delta y} \left( \tilde{G}_{i,j+1/2,k}^m - \tilde{G}_{i,j-1/2,k}^m \right) \quad (60)$$

$$U_{R,i+1/2,j,k}^{m,n+1/2} = \tilde{U}_{R,i+1/2,j,k}^{m,n+1/2} - \frac{\Delta t}{2\Delta z} \left( \tilde{H}_{i,j,k+1/2}^m - \tilde{H}_{i,j,k-1/2}^m \right),$$

$$U_{R,i+1/2,j,k}^{m,n+1/2} = \tilde{U}_{R,i+1/2,j,k}^{m,n+1/2} - \frac{\Delta t}{2\Delta y} \left( \tilde{G}_{i+1,j+1/2,k}^m - \tilde{G}_{i+1,j-1/2,k}^m \right) \quad (61)$$

$$- \frac{\Delta t}{2\Delta z} \left( \tilde{H}_{i+1,j,k+1/2}^m - \tilde{H}_{i+1,j,k-1/2}^m \right),$$

$$U_{L,i,j+1/2,k}^{m,n+1/2} = \tilde{U}_{L,i,j+1/2,k}^{m,n+1/2} - \frac{\Delta t}{2\Delta x} \left( \tilde{F}_{i+1/2,j,k}^m - \tilde{F}_{i-1/2,j,k}^m \right) \quad (62)$$

$$- \frac{\Delta t}{2\Delta z} \left( \tilde{H}_{i,j,k+1/2}^m - \tilde{H}_{i,j,k-1/2}^m \right),$$

$$U_{R,i,j+1/2,k}^{m,n+1/2} = \tilde{U}_{R,i,j+1/2,k}^{m,n+1/2} - \frac{\Delta t}{2\Delta x} \left( \tilde{F}_{i+1/2,j+1,k}^m - \tilde{F}_{i-1/2,j+1,k}^m \right) \quad (63)$$

$$- \frac{\Delta t}{2\Delta z} \left( \tilde{H}_{i,j+1,k+1/2}^m - \tilde{H}_{i,j+1,k-1/2}^m \right),$$

$$U_{L,i,j,k+1/2}^{m,n+1/2} = \tilde{U}_{L,i,j,k+1/2}^{m,n+1/2} - \frac{\Delta t}{2\Delta x} \left( \tilde{F}_{i+1/2,j,k}^m - \tilde{F}_{i-1/2,j,k}^m \right) \quad (64)$$

$$- \frac{\Delta t}{2\Delta y} \left( \tilde{G}_{i,j+1/2,k}^m - \tilde{G}_{i,j-1/2,k}^m \right),$$

$$U_{R,i,j,k+1/2}^{m,n+1/2} = \tilde{U}_{R,i,j,k+1/2}^{m,n+1/2} - \frac{\Delta t}{2\Delta x} \left( \tilde{F}_{i+1/2,j,k+1}^m - \tilde{F}_{i-1/2,j,k+1}^m \right) \quad (65)$$

$$- \frac{\Delta t}{2\Delta y} \left( \tilde{G}_{i,j+1/2,k+1}^m - \tilde{G}_{i,j-1/2,k+1}^m \right).$$

With these corrected left/right material states, one again uses an approximate Riemann solver to evaluate the passage of material.

## 6 Modified Godunov Corrector Scheme

The time discretization for the source term is a single-step, second order accurate scheme based on [5, 13, 14]. Given the material system of balance laws, one aims for a scheme that has an explicit approach for the flux divergence ( $\nabla \cdot F^m + \nabla \cdot G^m + \nabla \cdot H^m$ ) and an implicit approach for the stiff source term  $S^m(U)$ . At each grid point, one solves the following collection of ordinary differential equations:

$$\frac{dU^m}{dt} = S^m(U) - (\nabla \cdot F^m)^{n+1/2} - (\nabla \cdot G^m)^{n+1/2} - (\nabla \cdot H^m)^{n+1/2}, \quad (66)$$

where the time-centered flux divergence terms are inputted from the predictor step and taken to be constant valued. Using Picard iteration and the method of deferred corrections, an initial guess for the solution to the collection of ordinary differential equations is:

$$\hat{U} = U^{m,n} + \Delta t (I - \Delta t \nabla_{U^m} S^m(U)|_{U^{m,n}, U^{r,n+1}})^{-1} \quad (67)$$

$$\left( S^m(U^{m,n}, U^{r,n+1}) - (\nabla \cdot F^m)^{n+1/2} - (\nabla \cdot G^m)^{n+1/2} - (\nabla \cdot H^m)^{n+1/2} \right).$$

Here,  $\nabla_{U^m} S^m(U)$  has the same functional form as that which was used to define the propagation operator  $\mathcal{I}$  in a previous section. Therefore:

$$(I - \Delta t \nabla_{U^m} S^m(U)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \Delta t \text{PCS}_\rho^E & \Delta t \text{PCS}_{m_x}^E & \Delta t \text{PCS}_{m_y}^E & \Delta t \text{PCS}_{m_z}^E & 1 + \Delta t \text{PCS}_E^E \end{pmatrix}. \quad (68)$$

By inverting the above matrix, one finds:

$$(I - \Delta t \nabla_{U^m} S(U))^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{-\Delta t \text{PCS}_\rho^E}{1 + \Delta t \text{PCS}_E^E} & \frac{-\Delta t \text{PCS}_{m_x}^E}{1 + \Delta t \text{PCS}_E^E} & \frac{-\Delta t \text{PCS}_{m_y}^E}{1 + \Delta t \text{PCS}_E^E} & \frac{-\Delta t \text{PCS}_{m_z}^E}{1 + \Delta t \text{PCS}_E^E} & \frac{1}{1 + \Delta t \text{PCS}_E^E} \end{pmatrix}. \quad (69)$$

The error estimate  $\epsilon$  is the difference between the solution obtained from one iteration of the Picard technique where  $\hat{U}$  is used as the starting value and the initial guess  $\hat{U}$ :

$$\begin{aligned} \epsilon(\Delta t) = U^{m,n} &+ \frac{\Delta t}{2} \left( S^m(\hat{U}, U^{r,n+1}) + S^m(U^{m,n}, U^{r,n+1}) \right) \\ &- \Delta t \left( (\nabla \cdot F^m)^{n+1/2} + (\nabla \cdot G^m)^{n+1/2} + (\nabla \cdot H^m)^{n+1/2} \right) - \hat{U}. \end{aligned} \quad (70)$$

Following Miniati & Colella 2007, the correction to the initial guess is given by:

$$\delta(\Delta t) = \left( I - \Delta t \nabla_{U^m} S^m(U)|_{\hat{U}, U^{r,n+1}} \right)^{-1} \epsilon(\Delta t). \quad (71)$$

Therefore, the material quantities at time  $t_{n+1}$  are:

$$U^{m,n+1} = \hat{U} + \delta(\Delta t). \quad (72)$$

## 7 Conclusions and Future Work

This paper presents the algorithmic details for constructing a hybrid Godunov method to solve three-dimensional radiation hydrodynamical problems. Careful consideration was taken when developing this technique such that one can compute numerical solutions for a host of physical phenomena (e.g., free streaming, weak equilibrium diffusion, and strong equilibrium diffusion limits). Additionally, the algorithmic ideas in this paper were cast in such a way so that they can be easily implemented in existing codes, particularly ones that carry out MHD calculations. Future papers will showcase (i) a series of multidimensional radiation hydrodynamical tests which demonstrates the robustness of the hybrid Godunov method and (ii) how to combine a numerical method for radiation hydrodynamics with a technique for updating the variable tensor Eddington factor  $f$ .

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## Appendix 1: Partial Derivatives

$$S_{\rho}^{F_x} = \frac{-\sigma_t E_r}{\rho^2 C} (m_x + m_x f_{xx} + m_y f_{yx} + m_z f_{zx}) - \frac{\sigma_a m_x (T^4 - E_r)}{\rho^2 C} + \frac{4\sigma_a m_x T^3}{\rho C} (\gamma - 1) \left( \frac{-E}{\rho^2} + \frac{M^2}{\rho^3} \right), \quad (73)$$

$$S_{m_x}^{F_x} = \frac{\sigma_t E_r (1 + f_{xx})}{\rho C} + \frac{\sigma_a (T^4 - E_r)}{\rho C} + \frac{4\sigma_a m_x T^3}{\rho C} (\gamma - 1) \left( \frac{-m_x}{\rho^2} \right), \quad (74)$$

$$S_{m_y}^{F_x} = \frac{\sigma_t E_r f_{yx}}{\rho C} + \frac{4\sigma_a m_x T^3}{\rho C} (\gamma - 1) \left( \frac{-m_y}{\rho^2} \right), \quad (75)$$

$$S_{m_z}^{F_x} = \frac{\sigma_t E_r f_{zx}}{\rho C} + \frac{4\sigma_a m_x T^3}{\rho C} (\gamma - 1) \left( \frac{-m_z}{\rho^2} \right), \quad (76)$$

$$S_E^{F_x} = \frac{4\sigma_a m_x T^3}{\rho C} (\gamma - 1) \left( \frac{1}{\rho} \right), \quad (77)$$

$$S_{\rho}^{F_y} = \frac{-\sigma_t E_r}{\rho^2 C} (m_y + m_x f_{xy} + m_y f_{yy} + m_z f_{zy}) - \frac{\sigma_a m_y (T^4 - E_r)}{\rho^2 C} + \frac{4\sigma_a m_y T^3}{\rho C} (\gamma - 1) \left( \frac{-E}{\rho^2} + \frac{M^2}{\rho^3} \right), \quad (78)$$

$$S_{m_x}^{F_y} = \frac{\sigma_t E_r f_{xy}}{\rho C} + \frac{4\sigma_a m_y T^3}{\rho C} (\gamma - 1) \left( \frac{-m_x}{\rho^2} \right), \quad (79)$$

$$S_{m_y}^{F_y} = \frac{\sigma_t E_r (1 + f_{yy})}{\rho C} + \frac{\sigma_a (T^4 - E_r)}{\rho C} + \frac{4\sigma_a m_y T^3}{\rho C} (\gamma - 1) \left( \frac{-m_y}{\rho^2} \right), \quad (80)$$

$$S_{m_z}^{F_y} = \frac{\sigma_t E_r f_{zy}}{\rho C} + \frac{4\sigma_a m_y T^3}{\rho C} (\gamma - 1) \left( \frac{-m_z}{\rho^2} \right), \quad (81)$$

$$S_E^{F_y} = \frac{4\sigma_a m_y T^3}{\rho C} (\gamma - 1) \left( \frac{1}{\rho} \right), \quad (82)$$

$$S_{\rho}^{F_z} = \frac{-\sigma_t E_r}{\rho^2 C} (m_z + m_x f_{xz} + m_y f_{yz} + m_z f_{zz}) - \frac{\sigma_a m_z (T^4 - E_r)}{\rho^2 C} + \frac{4\sigma_a m_z T^3}{\rho C} (\gamma - 1) \left( \frac{-E}{\rho^2} + \frac{M^2}{\rho^3} \right), \quad (83)$$

$$S_{m_x}^{F_z} = \frac{\sigma_t E_r f_{xz}}{\rho C} + \frac{4\sigma_a m_z T^3}{\rho C} (\gamma - 1) \left( \frac{-m_x}{\rho^2} \right), \quad (84)$$

$$S_{m_y}^{F_z} = \frac{\sigma_t E_r f_{yz}}{\rho C} + \frac{4\sigma_a m_z T^3}{\rho C} (\gamma - 1) \left( \frac{-m_y}{\rho^2} \right), \quad (85)$$

$$S_{m_z}^{F_z} = \frac{\sigma_t E_r (1 + f_{zz})}{\rho C} + \frac{\sigma_a (T^4 - E_r)}{\rho C} + \frac{4\sigma_a m_z T^3}{\rho C} (\gamma - 1) \left( \frac{-m_z}{\rho^2} \right), \quad (86)$$

$$S_E^{F_z} = \frac{4\sigma_a m_z T^3}{\rho C} (\gamma - 1) \left( \frac{1}{\rho} \right), \quad (87)$$

$$S_{\rho}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-E}{\rho^2} + \frac{M^2}{\rho^3} \right) - \frac{(\sigma_a - \sigma_s) m_x F_{r,x}}{\rho^2 C} + \frac{2(\sigma_a - \sigma_s) m_x E_r}{\rho^3 C^2} (m_x + m_x f_{xx} + m_y f_{yx} + m_z f_{zx}) \\ - \frac{(\sigma_a - \sigma_s) m_y F_{r,y}}{\rho^2 C} + \frac{2(\sigma_a - \sigma_s) m_y E_r}{\rho^3 C^2} (m_y + m_x f_{xy} + m_y f_{yy} + m_z f_{zy}) \\ - \frac{(\sigma_a - \sigma_s) m_z F_{r,z}}{\rho^2 C} + \frac{2(\sigma_a - \sigma_s) m_z E_r}{\rho^3 C^2} (m_z + m_x f_{xz} + m_y f_{yz} + m_z f_{zz}), \quad (88)$$

$$S_{m_x}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_x}{\rho^2} \right) + \frac{(\sigma_a - \sigma_s) F_{r,x}}{\rho C} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} (2m_x + 2m_x f_{xx} + m_y f_{yx} + m_z f_{zx}) \\ - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_y f_{xy} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_z f_{xz}, \quad (89)$$

$$S_{m_y}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_y}{\rho^2} \right) + \frac{(\sigma_a - \sigma_s) F_{r,y}}{\rho C} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} (2m_y + m_x f_{xy} + 2m_y f_{yy} + m_z f_{zy}) \\ - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_x f_{yx} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_z f_{yz}, \quad (90)$$

$$S_{m_z}^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{-m_z}{\rho^2} \right) + \frac{(\sigma_a - \sigma_s) F_{r,z}}{\rho C} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} (2m_z + m_x f_{xz} + m_y f_{yz} + 2m_z f_{zz}) \\ - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_x f_{zx} - \frac{(\sigma_a - \sigma_s) E_r}{\rho^2 C^2} m_y f_{zy}, \quad (91)$$

$$S_E^E = 4\sigma_a T^3 (\gamma - 1) \left( \frac{1}{\rho} \right). \quad (92)$$

## Appendix 2: Effective Material Eigenvectors

$$R_{x,\text{eff}}^m = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ u - a_{\text{eff}} & u & 0 & 0 & u + a_{\text{eff}} \\ v & v & 1 & 0 & v \\ w & w & 0 & 1 & w \\ \frac{V^2}{2} - ua_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} & \frac{V^2}{2} & v & w & \frac{V^2}{2} + ua_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} \end{pmatrix}, \quad (93)$$

$$L_{x,\text{eff}}^m = \begin{pmatrix} \frac{1}{2a_{\text{eff}}} \left( u + \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \left( 1 + \frac{(\gamma-1)u}{a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)v}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)w}{a_{\text{eff}}} & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \\ 1 - \frac{(\gamma-1)V^2}{2a_{\text{eff}}^2} & \frac{(\gamma-1)u}{a_{\text{eff}}^2} & \frac{(\gamma-1)v}{a_{\text{eff}}^2} & \frac{(\gamma-1)w}{a_{\text{eff}}^2} & \frac{-(\gamma-1)}{a_{\text{eff}}^2} \\ -v & 0 & 1 & 0 & 0 \\ -w & 0 & 0 & 1 & 0 \\ \frac{-1}{2a_{\text{eff}}} \left( u - \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{1}{2a_{\text{eff}}} \left( 1 - \frac{(\gamma-1)u}{a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)v}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)w}{a_{\text{eff}}} & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \end{pmatrix}, \quad (94)$$

$$R_{y,\text{eff}}^m = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ u & 1 & u & 0 & u \\ v - a_{\text{eff}} & 0 & v & 0 & v + a_{\text{eff}} \\ w & 0 & w & 1 & w \\ \frac{V^2}{2} - va_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} & u & \frac{V^2}{2} & w & \frac{V^2}{2} + va_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} \end{pmatrix}, \quad (95)$$

$$L_{y,\text{eff}}^m = \begin{pmatrix} \frac{1}{2a_{\text{eff}}} \left( v + \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)u}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \left( 1 + \frac{(\gamma-1)v}{a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)w}{a_{\text{eff}}} & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \\ -u & 1 & 0 & 0 & 0 \\ 1 - \frac{(\gamma-1)V^2}{2a_{\text{eff}}^2} & \frac{(\gamma-1)u}{a_{\text{eff}}^2} & \frac{(\gamma-1)v}{a_{\text{eff}}^2} & \frac{(\gamma-1)w}{a_{\text{eff}}^2} & \frac{-(\gamma-1)}{a_{\text{eff}}^2} \\ -w & 0 & 0 & 1 & 0 \\ \frac{-1}{2a_{\text{eff}}} \left( v - \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)u}{a_{\text{eff}}} & \frac{1}{2a_{\text{eff}}} \left( 1 - \frac{(\gamma-1)v}{a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)w}{a_{\text{eff}}} & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \end{pmatrix}, \quad (96)$$

$$R_{z,\text{eff}}^m = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ u & 1 & 0 & u & u \\ v & 0 & 1 & v & v \\ w - a_{\text{eff}} & 0 & 0 & w & w + a_{\text{eff}} \\ \frac{V^2}{2} - wa_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} & u & v & \frac{V^2}{2} & \frac{V^2}{2} + wa_{\text{eff}} + \frac{a_{\text{eff}}^2}{\gamma-1} \end{pmatrix}, \quad (97)$$

$$L_{z,\text{eff}}^m = \begin{pmatrix} \frac{1}{2a_{\text{eff}}} \left( w + \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)u}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)v}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \left( 1 + \frac{(\gamma-1)w}{a_{\text{eff}}} \right) & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \\ -u & 1 & 0 & 0 & 0 \\ -v & 0 & 1 & 0 & 0 \\ 1 - \frac{(\gamma-1)V^2}{2a_{\text{eff}}^2} & \frac{(\gamma-1)u}{a_{\text{eff}}^2} & \frac{(\gamma-1)v}{a_{\text{eff}}^2} & \frac{(\gamma-1)w}{a_{\text{eff}}^2} & \frac{-(\gamma-1)}{a_{\text{eff}}^2} \\ \frac{-1}{2a_{\text{eff}}} \left( w - \frac{(\gamma-1)V^2}{2a_{\text{eff}}} \right) & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)u}{a_{\text{eff}}} & \frac{-1}{2a_{\text{eff}}} \frac{(\gamma-1)v}{a_{\text{eff}}} & \frac{1}{2a_{\text{eff}}} \left( 1 - \frac{(\gamma-1)w}{a_{\text{eff}}} \right) & \frac{(\gamma-1)}{2a_{\text{eff}}^2} \end{pmatrix}. \quad (98)$$