# Canonical description of incompressible fluid – Dirac brackets approach

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**Abstract.-** We present a novel canonical description of the incompressible fluid dynamics. This description uses the dynamical constraints, in our case reflecting incompressibility assumption, and leads to replacement of usual hydrodynamical Poisson brackets for density and velocity fields with Dirac brackets. The resulting equations are then known nonlinear, and nonlocal in space, equations for incompressible fluid velocity.

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## 1 Introduction

Canonical description of classical, compressible, isothermal fluid has been developed in the past [1, 2, 3] for various purposes, for example description of superfluid $^4$He [1] or in kinetics of the first order phase transformations [2, 3]. An attempt to extend this formulation for the adiabatic flows has been proposed [4] and used to analyze dynamical properties of thermally driven flows. The isothermal flow canonical description can be generalize for the case of viscous fluids [5] within the framework of the metriplectic dynamics [6]. The Madelung representation for the wave function results in hydrodynamic like picture of quantum mechanics, where the "only" differences from Euler equations are hidden in the quantum pressure term, which is proportional to $\hbar^2$, and in the quantization of the circulation, $\Gamma = n(\hbar/m)$. Apart of that the canonical description of the quantum fluid is then identical to that of the classical one. The dissipative generalization of the Schrödinger equation [7] also allows for metriplectic interpretation, which differs from the classical one [8].

The fundamental point in all of the above listed formulations of fluid dynamics is that the fluid density $\rho$ is one out of the pair of canonically conjugated variables. For potential flows the other canonical variable is the velocity potential $\phi$. In case of general flow the two additional Clebsh potentials $\lambda, \mu$ [9] in the velocity field representation $\mathbf{v} = -\nabla\phi - \lambda\nabla\mu$ are canonically conjugated to each other. None of these descriptions can be applied to the case of incompressible fluids.

The canonical description of incompressible flow is of considerable importance, for example in formulation of statistical mechanics of turbulent flow [5]. Many attempts to provide such a canonical formalism [10] failed to do so. The other important point is that real turbulent flow are hardly

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incompressible, and the issues like compressibility corrections to scaling laws for turbulent flows are still open [11].

In this paper we propose a novel formulation of a canonical description of the incompressible fluid based on the concept of the Dirac brackets [12]. The mathematical introduction to this formalism valid for general dynamic system subject to some set of constraints $\{\Theta_a = 0, a = 1 \ldots N\}$ can be found in [13]. Dirac bracket approach to description of incompressible membranes, within Lagrangian coordinates formulation of continuous mechanics of membranes, was given in [14]. We are unaware of any other application of that formalism to continuum mechanics problems. In separate publication we shall present other application of the Dirac constraints formalism in classical mechanics [15].

The Dirac brackets, for incompressible fluid, are presented, in what follows, within the Poisson bracket formulation of fluid mechanics [2, 3, 5], which avoids cumbersome introduction of the Clebsh potentials. Thus the state of fluid is described by specifying its density and velocity fields.

## 2 Compressible fluid

Consider infinite 3-dimensional volume of the isothermal fluid with density $\varrho(\mathbf{r}, t)$ and velocity $\mathbf{v}(\mathbf{r}, t)$. The Hamiltonian for such a system is given as:

$$\mathcal{H}\{\varrho, \mathbf{v}\} = \int \left[\varrho \mathbf{v}^2/2 + f(\varrho)\right] d^3r \;, \tag{1}$$

where $f(\varrho)$ is the fluid Helmholtz free energy per unit volume, related to the fluid pressure by

$$p = \varrho \frac{\partial f}{\partial \varrho} - f(\varrho). \tag{2}$$

The Poisson bracket relations between fields $\varrho(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ are [5]:

$$\begin{aligned} \{\varrho(\mathbf{x}, t), \varrho(\mathbf{y}, t)\} &= 0 \;, \\ \left\{\varrho(\mathbf{x}, t), v^i(\mathbf{y}, t)\right\} &= -\frac{\partial}{\partial \mathbf{x}^i}\delta(\mathbf{x} - \mathbf{y}) \;, \\ \left\{v^i(\mathbf{x}, t), v^j(\mathbf{y}, t)\right\} &= \delta(\mathbf{x} - \mathbf{y})\frac{1}{\varrho(\mathbf{x}, t)}\epsilon^{ijk}(\nabla \times \mathbf{v})_k(\mathbf{x}, t) \;. \end{aligned} \tag{3}$$

The continuity equation is obtained by evaluating the Poisson bracket $\{\varrho, \mathcal{H}\}$, and the Euler equation by $\{\mathbf{v}, \mathcal{H}\}$

$$\begin{aligned} \partial_t \varrho(\mathbf{r}, t) &= \{\varrho, \mathcal{H}\} = -\nabla \cdot \varrho \mathbf{v} \;, \\ \partial_t \mathbf{v}(\mathbf{r}, t) &= \{\mathbf{v}, \mathcal{H}\} = -\mathbf{v} \cdot \nabla \mathbf{v} - (1/\varrho)\nabla p(\varrho) \;. \end{aligned} \tag{4}$$

The above formulation of fluid mechanics can be derived from the least action principle provided we choose the proper lagrangian. As shown by Thellung [1] this lagrangian density is the local pressure.

The incompressible fluid, although an obvious simplification, is adequate for all the flows when the local Mach number is small. The incompressibility condition then is that the density $\varrho(\mathbf{r}, t) - \varrho_0 = 0$ what also implies that $\nabla \cdot \mathbf{v} = 0$. Within the canonical formulation framework both these conditions are regarded as Dirac constraints[12] $\Theta_a(\mathbf{r}, \mathbf{v}, t) = 0$, $a = 1, 2$. Next section contains a brief overview of the Dirac brackets theory.

## 3 Dirac Brackets

The definition of the Dirac brackets we shall use in the following is a natural generalization for the original construction proposed by Dirac [12] and discussed in detail in [13]. When the physical system with phase space $\mathcal{P}$ is subject to a set of constraints $\{\Theta_a = 0\}$ then its motion proceeds on a submanifold $\mathcal{P} \supset \mathcal{S} = \bigcup_{a=1}^{N} \{z \epsilon \mathcal{P} | \Theta_a(z) = 0\}$. If the Poisson bracket for two arbitrary (sufficiently smooth etc.,) phase space functions $F$ and $G$ was $\{F, G\}$, then the Dirac bracket $\sqsubseteq F, G \sqsupseteq$ is defined as:

$$\sqsubseteq F, G \sqsupseteq = \{F, G\} = -\sum_{a,b}^{N} \{F, \Theta_a\} M_{ab} \{\Theta_b, G\} \ , \tag{5}$$

where $M_{ab}$ is the inverse of the constraints Poisson bracket matrix $W_{ab} = \{\Theta_a, \Theta_b\}$.

The generalization of the Dirac bracket to the case of continuous variables, like in hydrodynamics, is straightforward. The sum over the indices $a$ is replaced by sum and integration over the space variables and the inverse of the matrix $W_{ab}(\mathbf{r}, \mathbf{r}') = \{\Theta_a(\mathbf{r}), \Theta_b(\mathbf{r}')\}$ is defined as:

$$\sum_c \int d\mathbf{r}' W_{ac}(\mathbf{r}, \mathbf{r}') M_{cb}(\mathbf{r}', \mathbf{r}'') = \delta_{ab} \delta(\mathbf{r} - \mathbf{r}'') \ . \tag{6}$$

Dirac brackets, given by Eq.(5) replace the original Poisson brackets in the equation of motion for the constrained system. Thus for a phase space function $F$ the time evolution on the submanifold $\mathcal{S}$ is governed by:

$$\left( \frac{\partial F}{\partial t} \right)_{\mathcal{S}} = \sqsubseteq F, \mathcal{H} \sqsupseteq \ , \tag{7}$$

where $\mathcal{H}$ is system Hamiltonian. Next section will contain application of the Dirac brackets to the description of the incompressible fluid.

# 4 Dirac Brackets for Incompressible Fluid

The constraints used in constructing the incompressible fluid dynamics are:

$$\begin{aligned} \Theta_1 &\equiv \varrho(\mathbf{r}) - \varrho_0 = 0 \ , \\ \Theta_2 &\equiv \nabla \cdot \mathbf{v}(\mathbf{r}) = 0 \ . \end{aligned} \tag{8}$$

The constraints Poisson bracket matrix $W_{ab}(\mathbf{r}, \mathbf{r}')$ can be evaluated using Eq.(3), and it reads:

$$W_{ab}(\mathbf{r}, \mathbf{r}') = \nabla_r^i \nabla_r^j \left( \begin{bmatrix} 0 & -\delta^{ij} \\ \delta^{ij}, & \left[ \frac{1}{\varrho(\mathbf{r})} \varepsilon^{ijk} (\nabla \times \mathbf{v}(\mathbf{r}))^k \right] \end{bmatrix} \delta(\mathbf{r} - \mathbf{r}') \right) \ . \tag{9}$$

In the Dirac formalism one needs the inverse of the matrix $W_{ab}(\mathbf{r}, \mathbf{r}')$ defined in(6). The matrix elements $M_{ab}(\mathbf{r}, \mathbf{r}')$ obey the set of partial differential equations, written explicitly in the Appendix A. Solving these equations we find matrix $M_{ab}(\mathbf{r}, \mathbf{r}')$ in the form:

$$M_{ab}(\mathbf{r}, \mathbf{r}') = \begin{bmatrix} \mathcal{M}\{G\}, & -G(\mathbf{r} - \mathbf{r}') \\ G(\mathbf{r} - \mathbf{r}'), & 0 \end{bmatrix} \ , \tag{10}$$

where $G(\mathbf{r} - \mathbf{r}') = \mid \mathbf{r} - \mathbf{r}' \mid / 4\pi$ is the Green function for the Laplace operator in the infinite volume, and

$$\mathcal{M}\{G\} = -\int d\mathbf{x}' G(\mathbf{r} - \mathbf{x}') \nabla_{\mathbf{x}'} \cdot \left[ \frac{1}{\varrho(\mathbf{x}')} \nabla_{\mathbf{x}'} G(\mathbf{x}' - \mathbf{r}') \times (\nabla \times \mathbf{v}(\mathbf{x}')) \right] \ . \tag{11}$$

## 5 Dirac equations of motion for incompressible fluid

Using the definition and the explicit form of the Dirac brackets given in previous section and in the Appendix A, we first calculate the Dirac bracket $\sqsubseteq \varrho, \mathcal{H} \sqsupseteq$. From the definition (5), and (7) we obtain:

$$\frac{\partial \varrho(\mathbf{r},t)}{\partial t} = \sqsubseteq \varrho(\mathbf{r},t), \mathcal{H} \sqsupseteq = \{\varrho(\mathbf{r},t), \mathcal{H}\} - \sum_{ab} \int d\mathbf{z} d\mathbf{z}' \{\varrho(\mathbf{r},t), \Theta_a(\mathbf{z})\} M_{ab}(\mathbf{z},\mathbf{z}') \{\Theta_b(\mathbf{z}'), \mathcal{H}\} \ . \tag{12}$$

Explicit evaluation of the right hand side of Eq.(12) is a bit tedious, but using results from the Appendix A and (23) one finds that it vanishes. Thus the continuity equation for incompressible fluid, within the Dirac formalism reads $\sqsubseteq \varrho(\mathbf{r},t), \mathcal{H} \sqsupseteq = 0$.

The algebra needed to derive equation of motion for the velocity field $\mathbf{v}$ is slightly more complex than these leading to the continuity equation.

Following Dirac procedure we obtain:

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = \sqsubseteq \mathbf{v}(\mathbf{r},t), \mathcal{H} \sqsupseteq = \{\mathbf{v}(\mathbf{r},t), \mathcal{H}\} - \sum_{ab} \int d\mathbf{z} d\mathbf{z}' \{\mathbf{v}(\mathbf{r},t), \Theta_a(\mathbf{z})\} M_{ab}(\mathbf{z},\mathbf{z}') \{\Theta_b(\mathbf{z}'), \mathcal{H}\} \ . \tag{13}$$

Evaluation of the right hand side of (13), with use of expressions (23) gives: :

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = \mathbf{v}(\mathbf{r},t) \times (\nabla \times \mathbf{v}(\mathbf{r},t)) - \nabla_r \left[ \int d\mathbf{z} G(\mathbf{r}-\mathbf{z}) \nabla_z \cdot \{\mathbf{v}(\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{r},t))\} \right] \ . \tag{14}$$

Thus we have obtained nonlinear, nonlocal equation for the velocity field known from previous work [3, 4].

The above exercise in the Dirac brackets calculation provides a novel formulation of the Euler incompressible fluid. The viscous fluid equations can now easily be derived by replacing the Dirac brackets by the metriplectic brackets discussed in [5]. We can also use the Dirac brackets as starting point in the perturbation theory in which compressibility corrections are calculated. To do so one formally associates small parameter $\kappa$ to the matrix elements $M_{ab}$ and expresses the Poisson brackets by the Dirac one. To the first order in $\kappa$ the expression is identical to that in (5) with reversed role of the Poisson and Dirac brackets.

In conclusion we have shown in the above that the Poisson brackets formulation of the fluid dynamics can be used to derive the canonical theory of the incompressible fluid following the Dirac prescription. The application of this theory will be discussed in following publication.

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### Appendix A

**Matrix M:**
The matrix elements $M_{ab}$ satisfy the following system of partial differential equations:

$$\nabla_{\mathbf{z}} \cdot \left[ \nabla_{\mathbf{z}} M_{21}(\mathbf{x},\mathbf{z}) + \frac{1}{\rho(\mathbf{z})} \nabla_{\mathbf{z}} M_{22}(\mathbf{x},\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{z})) \right] = \delta(\mathbf{x}-\mathbf{z}), \tag{15}$$

$$-\nabla_{\mathbf{x}} \cdot \left[ \nabla_{\mathbf{x}} M_{12}(\mathbf{x},\mathbf{z}) + \frac{1}{\rho(\mathbf{x})} \nabla_{\mathbf{x}} M_{22}(\mathbf{x},\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{x})) \right] = \delta(\mathbf{x}-\mathbf{z}), \tag{16}$$

$$\triangle_{\mathbf{z}} M_{12}(\mathbf{x},\mathbf{z}) = -\delta(\mathbf{x}-\mathbf{z}), \tag{17}$$
$$\triangle_{\mathbf{x}} M_{21}(\mathbf{x},\mathbf{z}) = \delta(\mathbf{x}-\mathbf{z}), \tag{18}$$

$$\nabla_{\mathbf{z}} \cdot \left[\nabla_{\mathbf{z}} M_{11}(\mathbf{x},\mathbf{z}) + \frac{1}{\rho(\mathbf{z})}\nabla_{\mathbf{z}} M_{12}(\mathbf{x},\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{z}))\right] = 0, \tag{19}$$

$$\nabla_{\mathbf{x}} \cdot \left[\nabla_{\mathbf{x}} M_{11}(\mathbf{x},\mathbf{z}) + \frac{1}{\rho(\mathbf{x})}\nabla_{\mathbf{x}} M_{21}(\mathbf{x},\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{x}))\right] = 0, \tag{20}$$

$$\triangle_{\mathbf{z}} M_{22}(\mathbf{x},\mathbf{z}) = 0, \tag{21}$$
$$\triangle_{\mathbf{x}} M_{22}(\mathbf{x},\mathbf{z}) = 0. \tag{22}$$

It is easy to check that these equations are satisfied by matrix elements given below:

$$\begin{aligned}
M_{11}(\mathbf{x},\mathbf{z}) &= -\int d\mathbf{x}' G(\mathbf{x}-\mathbf{x}')\nabla_{\mathbf{x}'} \cdot \left[\frac{1}{\varrho(\mathbf{x}')}\nabla_{\mathbf{x}'} G(\mathbf{x}'-\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{x}'))\right] \equiv \mathcal{M}\{G\}\ , \\
M_{12}(\mathbf{x},\mathbf{z}) &= -M_{21}(\mathbf{x},\mathbf{z}) = -G(\mathbf{x}-\mathbf{z})\ , \\
M_{22}(\mathbf{x},\mathbf{z}) &= 0\ ,
\end{aligned} \tag{23}$$

**Details of the Dirac brackets evaluation for the ideal fluid:**

Consider the Hamiltonian (1), the Dirac bracket $\sqsubseteq \rho(\mathbf{x}), H \sqsupseteq$ reads:

$$\begin{aligned}
\sqsubseteq \rho(\mathbf{x}), H \sqsupseteq &= \{\rho(\mathbf{x}), H\} - \sum_{i,j}\int d\mathbf{z}_1 d\mathbf{z}_2\ \{\rho(\mathbf{x}), \Theta_i(\mathbf{z}_1)\}\, M_{ij}(\mathbf{z}_1,\mathbf{z}_2)\, \{\Theta_j(\mathbf{z}_2), H\} \\
&= \nabla_{\mathbf{x}} \cdot \vec{J}(\mathbf{x}) - \int d\mathbf{z}\ \triangle_{\mathbf{x}} M_{21}(\mathbf{x},\mathbf{z}) \left[\nabla_{\mathbf{z}} \cdot \vec{J}(\mathbf{z})\right] \\
&+ \triangle_{\mathbf{x}} M_{22}(\mathbf{x},\mathbf{z}) \left[\nabla_{\mathbf{z}} \cdot (\mathbf{v}(\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{z}))) - \triangle_{\mathbf{z}} (\mu(\mathbf{v},\varrho))\right] = 0\ .
\end{aligned} \tag{24}$$

One sees immediately that the right hand side of (24) vanishes due to (18,22,23). Here $\mathbf{J} = \varrho\mathbf{v}$ denotes the fluid particle current and $\mu(\mathbf{v},\varrho) = |\mathbf{v}|^2/2 + \partial f(\rho(\mathbf{z}))/\partial\varrho$ is the moving fluid chemical potential.

The continuity equation is then

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) = \sqsubseteq \rho(\mathbf{x},t), H \sqsupseteq = 0\ , \tag{25}$$

as expected.

Evaluating Dirac bracket $\sqsubseteq v^i(\mathbf{x}), H \sqsupseteq$ we obtain:

$$\begin{aligned}
\sqsubseteq v^i(\mathbf{x}), H \sqsupseteq &= \left\{v^i(\mathbf{x}), H\right\} - \sum_{a,b}\int d\mathbf{z}_1 d\mathbf{z}_2\ \left\{v^i(\mathbf{x}), \Theta_a(\mathbf{z}_1)\right\} M_{ab}(\mathbf{z}_1,\mathbf{z}_2)\, \{\Theta_b(\mathbf{z}_2), H\} \\
&= A^i_0 - \sum_{a,b} A^i_{ab}.
\end{aligned} \tag{26}$$

After straightforward but lengthy calculations we obtain:

$$
\begin{aligned}
A_0^i &= \left\{v^i(\mathbf{x}), H\right\} = [\mathbf{v}(\mathbf{x}) \times (\nabla \times \mathbf{v}(\mathbf{x}))]^i - \nabla^i \left[\mu(\mathbf{v}, \varrho)\right] \ , \\
A_{11}^i &= -\nabla_{\mathbf{x}}^i \int d\mathbf{z}\, M_{11}(\mathbf{x}, \mathbf{z}) \left[\nabla_{\mathbf{z}} \cdot \vec{J}(\mathbf{z})\right] \ , \\
A_{22}^i &= 0 \ , \\
A_{12}^i &= \nabla_{\mathbf{x}}^i \int d\mathbf{z}\, G(\mathbf{x} - \mathbf{z}) \left\{\nabla_{\mathbf{z}} \cdot [\mathbf{v}(\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{z}))] - \triangle_{\mathbf{z}} \left[\mu(\mathbf{v}, \varrho)\right]\right\} \ , \\
A_{21}^i &= -\int d\mathbf{z}\, \frac{1}{\rho(\mathbf{x})} \left[\nabla_{\mathbf{x}} G(\mathbf{x} - \mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{x}))\right]^i \left[\nabla_{\mathbf{z}} \cdot \vec{J}(\mathbf{z})\right] \ ,
\end{aligned}
\tag{27}
$$

Using above, together with equations (17) and (21) we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial t} \mathbf{v}(\mathbf{x}, t) &= \mathbf{v}(\mathbf{x}) \times (\nabla \times \mathbf{v}(\mathbf{x})) \\
&+ \int d\mathbf{z} \left[\nabla_{\mathbf{x}} M_{11}(\mathbf{x}, \mathbf{z}) + \frac{1}{\rho(\mathbf{x})} \nabla_{\mathbf{x}} G(\mathbf{x} - \mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{x}))\right] \left[\nabla_{\mathbf{z}} \cdot \vec{J}(\mathbf{z})\right] \\
&- \int d\mathbf{z}\, \left[\nabla_{\mathbf{x}} G(\mathbf{x} - \mathbf{z})\right] \left\{\nabla_{\mathbf{z}} \cdot [\mathbf{v}(\mathbf{z}) \times (\nabla \times \mathbf{v}(\mathbf{z}))]\right\} \ .
\end{aligned}
\tag{28}
$$

Acting on both sides of equation (28) with operator $\mathbf{div} = \nabla_{\mathbf{x}} \cdot$, and using equations (16,20) one gets:

$$
\frac{\partial}{\partial t} \left[\nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t)\right] = \sqsubseteq \nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t), H \sqsupseteq = \nabla_{\mathbf{x}} \cdot \sqsubseteq \mathbf{v}(\mathbf{x}, t), H \sqsupseteq = \nabla_{\mathbf{x}} \cdot \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t) = 0. \tag{29}
$$

Thus $\Theta_2(\mathbf{x}) = \nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t)$ is a constant of motions, as expected. Condition $\rho = \rho_0$ implies that $\nabla_{\mathbf{x}} \cdot \vec{J}(\mathbf{x}, t)$ is also a constant of motions, and $\nabla_{\mathbf{x}} \cdot \vec{J}(\mathbf{x}, t) = 0$.

Now, using equations (23,29) one easily sees that equation (28) reduces to

$$
\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} = \mathbf{v}(\mathbf{x}, t) \times (\nabla \times \mathbf{v}(\mathbf{x}, t)) - \nabla_{\mathbf{x}} \left[\int d\mathbf{z}\, G(\mathbf{x} - \mathbf{z}) \ \nabla_{\mathbf{z}} \cdot [\mathbf{v}(\mathbf{z}, t) \times (\nabla \times \mathbf{v}(\mathbf{z}, t))]\right] \ , \tag{30}
$$

which is exactly the Euler equation for an ideal, incompressible fluid in its integral form.

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