SMALL-TIME ASYMPTOTICS FOR FAST MEAN-REVERTING STOCHASTIC VOLATILITY MODELS*

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In this paper, we study stochastic volatility models in regimes where the maturity is small but large compared to the mean-reversion time of the stochastic volatility factor. The problem falls in the class of averaging/homogenization problems for nonlinear HJB type equations where the "fast variable" lives in a non-compact space. We develop a general argument based on viscosity solutions which we apply to the two regimes studied in the paper. We derive a large deviation principle and we deduce asymptotic prices for Out-of-The-Money call and put options, and their corresponding implied volatilities. The results of this paper generalize the ones obtained in [11] (J. Feng, M. Forde and J.-P. Fouque, *Short maturity asymptotic for a fast mean reverting Heston stochastic volatility model*, SIAM Journal on Financial Mathematics, Vol. 1, 2010) by a moment generating function computation in the particular case of the Heston model.

1. Introduction. On one hand, the theory of large deviations has been recently applied to local and stochastic volatility models [1–4, 18], and has given very interesting results on the behavior of implied volatilities near maturity (an implied volatility is the volatility parameter needed in the Black-Scholes formula in order to match a call option price; it is common practice to quote prices in volatility through this transformation). In the context of stochastic volatility models, the rate function involved in the large deviation estimates is given in terms of a distance function, which in general cannot be calculated in closed-form. For particular models, such as the SABR model [17, 19], approximations obtained by expansion techniques have been proposed (see also [12, 16, 20, 26]).

On the other hand, multi-factor stochastic volatility models have been studied during the last ten years by many authors (see for instance [7, 13, 16, 25, 27]). They are quite efficient in capturing the main features of implied

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volatilities known as smiles and skews, but they are usually not simple to calibrate. In the presence of separated time scales, an asymptotic theory has been proposed in [13, 14]. It has the advantage of capturing the main effects of stochastic volatility through a small number of group parameters arising in the asymptotic. The fast time scale expansion is related to the ergodic property of the corresponding fast mean reverting stochastic volatility factor.

It is natural to try to combine these two modeling aspects and limiting results, by considering short maturity options computed with fast mean-reverting stochastic volatility models, in such a way that maturity is of order $\epsilon \ll 1$ and the mean-reversion time, δ, of volatility is even smaller of order $\delta = \epsilon^2$ (fast mean-reversion) or $\delta = \epsilon^4$ (ultra-fast mean-reversion).

In [11], the authors studied the particular case of the Heston model in the regime $\delta = \epsilon^2$ by an explicit computation of the moment generating function of the stock price and its asymptotic analysis.

In this paper, we establish a large deviation principle for general stochastic volatility models in the two regimes of fast and ulta-fast mean-reversion, and we derive asymptotic smiles/skews. For such general dynamics, a moment generating function approach is no longer available. Our problem falls in the class of homogenization/averaging problems for nonlinear HJB type equations where the "fast variable" lives in a non-compact space. We develop a general argument based on viscosity solutions which we apply to the two regimes studied in the paper.

We start by considering the following stochastic differential equations modeling the evolution of the stock price (S_t) under a risk-neutral pricing probability measure, and with a stochastic volatility determined by a process (Y_t):

$$dS_t = rS_t dt + S_t \sigma(Y_t) dW_t^{(1)}, \tag{1.1a}$$

$$dY_t = \frac{1}{\delta}(m - Y_t)dt + \frac{\nu}{\sqrt{\delta}} Y_t^{\beta} dW_t^{(2)}. \tag{1.1b}$$

where $m \in \mathbb{R}, r, \nu > 0$, $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with $\langle W^{(1)}, W^{(2)} \rangle_t = \rho t$, with $|\rho| < 1$ constant. The process (Y_t) is a fast mean-reverting process with rate of mean reversion $1/\delta$ ($\delta > 0$). The parameter β and $\sigma(y)$ are chosen to satisfy the following.

Assumption 1.1. *We assume that*

1. $\beta \in \{0\} \cup [\frac{1}{2}, 1)$;
2. *in the case of* $\beta = 1/2$, *we require* $m > \nu^2/2$ *and* $Y_0 > 0$ *a.s., in the case of* $1/2 < \beta < 1$, *we require* $m > 0$ *and* $Y_0 > 0$ *a.s.;*

3. $\sigma(y) \in C(\mathbb{R}; \mathbb{R}_+)$ *satisfies*

$$\sigma(y) \leq C(1 + |y|^\sigma),$$

for some constants $C > 0$ *and* σ *with* $0 \leq \sigma < 1 - \beta$*.*

These assumptions ensure existence and uniqueness of a strong solution of (1.1). This can be seen as a combination of existence of martingale problem solution (e.g. Theorem 5.3.10 in Ethier and Kurtz [8]) and the Yamada-Watanabe theory for 1-D diffusions (e.g. Chapter 5, Karatzas and Shreve [21]). In particular, Assumption 1.1.2 ensures that, in the case $\beta \in [\frac{1}{2}, 1)$, $Y_t > 0$ a.s. for all $t \geq 0$ (see Appendix A). In the case $\beta = 0$, Y is an Ornstein-Uhlenbeck (OU) process with a natural state space $(-\infty, \infty)$. In order to present both model cases using one simple set of notation, we denote state space for Y as E_0 with $E_0 := \mathbb{R}$ if $\beta = 0$ and $E_0 := (0, \infty)$ when $\beta \in [\frac{1}{2}, 1)$.

Note that the Heston model does not satisfy these assumptions, but it has been treated separately in [11].

The infinitesimal generator of the Y process, when $\delta = 1$, can be identified with the following differential operator on the class of smooth test functions vanishing off compact sets:

$$B := (m - y)\partial_y + \frac{1}{2}\nu^2 |y|^{2\beta} \partial^2_{yy}. \tag{1.2}$$

Following the general theory of 1-D diffusion (e.g. page 221, Karlin and Taylor [23]), we introduce the so called scale and speed measure of the (Y_t) process:

$$s(y) := \exp\left\{ -\int_1^y \frac{2(m-z)}{\nu^2 |z|^{2\beta}} dz \right\}, \quad m(y) := \frac{1}{\nu^2 |y|^{2\beta} s(y)}.$$

Denoting $dS(y) := s(y)dy$ and $dM(y) := m(y)dy$, we then have

$$Bf(y) = \frac{1}{2} \frac{d}{dM} \left[\frac{df(y)}{dS} \right]. \tag{1.3}$$

Under Assumption 1.1 there exists a unique probability measure

$$\pi(dy) := Z^{-1} m(y) dy, \quad Z := \int_{E_0} m(y) dy < \infty \tag{1.4}$$

such that $\int Bf d\pi = 0$ for all $f \in C_c^2(E_0)$. See Appendix C.

By a change of variable $X_t = \log S_t$, we have

$$dX_t = \left(r - \frac{1}{2}\sigma^2(Y_t)\right) dt + \sigma(Y_t) dW_t^{(1)}.$$

In order to study small time behavior of the system, we rescale time $t \mapsto \epsilon t$ for $0 < \epsilon \ll 1$; denoting the rescaled processes by $X_{\epsilon,\delta,t}$ and $Y_{\epsilon,\delta,t}$, we have in distribution:

$$dX_{\epsilon,\delta,t} = \epsilon \left(r - \frac{1}{2}\sigma^2(Y_{\epsilon,\delta,t})\right) dt + \sqrt{\epsilon}\, \sigma(Y_{\epsilon,\delta,t}) dW_t^{(1)} \tag{1.5a}$$

$$dY_{\epsilon,\delta,t} = \frac{\epsilon}{\delta}(m - Y_{\epsilon,\delta,t})dt + \nu\sqrt{\frac{\epsilon}{\delta}}\, Y^{\beta}_{\epsilon,\delta,t} dW_t^{(2)}. \tag{1.5b}$$

We are interested in understanding the two scale $\epsilon, \delta \to 0$ limit behavior of option prices and its implication to implied volatility. In this paper, we restrict our attention to the following two regimes :

$$\delta = \epsilon^4 \quad \text{and} \quad \delta = \epsilon^2.$$

In view of [11], to obtain a large deviation estimate of option prices, it is sufficient to obtain a large deviation principle for $\{X_{\epsilon,\delta,t} : \epsilon > 0\}$. By Bryc's inverse Varadhan lemma [6][Theorem 4.4.2], we know that the key step is proving convergence of the following functionals:

$$u_{\epsilon,\delta}(t, x, y) := \epsilon \log E[e^{\epsilon^{-1} h(X_{\epsilon,\delta,t})} | X_{\epsilon,\delta,0} = x, Y_{\epsilon,\delta,0} = y], \quad h \in C_b(\mathbb{R}), \tag{1.6}$$

to some quantity independent of y.

For each $h \in C_b(\mathbb{R})$, the function $u_{\epsilon,\delta}$ satisfies a nonlinear partial differential equation given in (3.4). In section 3.2, we use heuristic arguments to obtain PDEs that characterize the limit of these $u_{\epsilon,\delta}$. Proving this convergence rigorously however is non-trivial. Intuitively we know that, as Y has a mean reversion rate $1/\delta$ and $\delta \ll \epsilon$, the effect of the Y process should get averaged out. To be exact, the form of nonlinear operator (3.5) indicates that convergence of $u_{\epsilon,\delta}$ is an averaging problem (over the fast y variable) for Hamilton-Jacobi equations. Such problems, in the context of compact state space for the averaging variable, can be handled by extending standard linear equation techniques using viscosity solution language. The Y process in this article lies in E_0, which is $\mathbb{R}$ in the case of $\beta = 0$ and $(0, \infty)$ in other cases. E_0 is a non-compact space, and therein lies an additional difficulty.

We adapt methods developed in Feng and Kurtz [10]. Indeed, an abstract method for large deviation for sequence of Markov processes, based on convergence of HJB equation, is developed fully in [10]. The two schemes treated in this article are of the nature of Example 1.8 and Example 1.9 introduced in Chapter 1 and proved in details in Chapter 11 of [10]. In this article, we not only present a direct proof, but also introduce some argument to further simplify [10] in the setting of multi-scale. This is possible in a large part due to the locally compact state space and mean-reverting nature of the process Y.

In particular, modulo technical subtleties in verification of conditions, the setup of Section 11.6 in [10] corresponds to the large deviation result in our case of $\delta = \epsilon^2$. Since E_0 is locally compact and we only deal with PDEs instead of abstract operator equations, great simplification of [10] can be achieved through the use of a special class of test functions. See Conditions 4.1, 4.2. The techniques we introduce (Lemmas 4.1 and 4.2) are not limited to averaging problems, but are applicable to problems of homogenization also, which we will not delve into in this article. The rigorous justification of convergence of $u_{\epsilon,\delta}$ is shown in section 5.

The main results of the paper are stated in Section 2. Theorem 2.1 is a rare event large deviation type estimate corresponding to short time, out-of-the-money option pricing. Corollary 2.1 and Theorem 2.2 give asymptotics of option price and implied volatility, respectively, for such situation. The proofs are given in the sections that follow, starting with heuristic proofs in section 3.2 and finishing with rigorous justifications in sections 4 and 5. Technical results Lemmas 4.1 and 4.2 may be of independent interest.

2. Main results.

THEOREM 2.1 (Large Deviation). *Suppose that Assumption 1.1 holds. Let*

$$I_4(x; x_0, t) := \frac{|x_0 - x|^2}{2\bar{\sigma}^2 t}, \quad \text{where } \bar{\sigma} \text{ is defined in (3.7) and} \tag{2.1}$$

$$I_2(x; x_0, t) := t\bar{L}_0\left(\frac{x_0 - x}{t}\right) \tag{2.2}$$

where $\bar{L}_0$ is the Legendre transform of $\bar{H}_0$ defined in (5.12). See also equivalent representation (3.15).

Assume $X_{\epsilon,\epsilon^r,0} = x_0$ and $Y_{\epsilon,\epsilon^r,0} = y_0$ where $r = 2, 4$. Then, for each regime $r \in \{2, 4\}$, for every fixed $t > 0$ and $x_0 \in \mathbb{R}, y_0 \in E_0$, a large deviation principle (LDP) holds for $\{X_{\epsilon,\epsilon^r,t} : \epsilon > 0\}$ with speed $1/\epsilon$ and good rate function $I_r(x; x_0, t)$. In particular,

$$\lim_{\eta\to 0}\lim_{\epsilon\to 0} \epsilon \log P\left(|X_{\epsilon,\epsilon^r,t} - x| < \eta \middle| X_{\epsilon,\epsilon^r,0} = x_0, Y_{\epsilon,\epsilon^r,0} = y_0\right) = -I_r(x; x_0, t); \tag{2.3}$$

$x \in \mathbb{R}$, for each regime $r \in \{2, 4\}$.

REMARK 2.1. *The rate function $I_r(x; x_0, t)$, in both regimes, are convex, continuous functions of x and $I_r(x_0; x_0, t) = 0$.*

REMARK 2.2. *In the case $\delta = \epsilon^4$, observe that the rate function I_4, in (2.1), is the same as the rate function for the Black-Scholes model with constant volatility $\bar{\sigma}$. In other words, in the ultra fast regime, to the leading order, it is the same as averaging first and then take the short maturity limit.*

REMARK 2.3. *In the case $\delta = \epsilon^2$, no explicit formula for the rate function is obtained. However, an explicit formula of the rate function is obtained for the Heston model in [11] which corroborates the formula in (2.2). The Heston model per se does not fall in the category of stochastic volatility models covered in this paper, but direct computation of $\bar{H}_0$, given by (3.15), and $\bar{L}_0$, its Legendre transform, is possible for this model.*

Let $S_0 > 0$ be the initial value of stock price and let $X_{\epsilon,\epsilon^r,0} = x_0 = \log S_0$. The asymptotic behavior of the price of out-of-the-money European call option with strike price K and short maturity time $T = \epsilon t$ is given in the following corollary. Since we are only considering out-of-the-money call options,

$$S_0 < K \quad \text{or} \quad x_0 < \log K. \tag{2.4}$$

COROLLARY 2.1 (Option Price). *For fixed $t > 0$,*

$$\lim_{\epsilon \to 0^+} \epsilon \log E\left[e^{-r\epsilon t}\left(S_{\epsilon,\epsilon^r,t} - K\right)^+\right] = -I_r(\log K; x_0, t)$$

for $r = 2, 4$.

Denote the Black-Scholes implied volatility for out-of the-money European call option, with strike price K, by $\sigma_{r,\epsilon}(t, \log K, x_0)$, where $r = 2, 4$ correspond to the two regimes. By the same argument used in [11], we get an asymptotic formula for implied volatility:

THEOREM 2.2 (Implied Volatilities).

$$\lim_{\epsilon \to 0^+} \sigma_{r,\epsilon}(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2 I_r(\log K; x_0, t) t}.$$

Remark 2.4. *In the case $\delta = \epsilon^4$, the implied volatility is $\bar{\sigma}$, which is obtained by averaging the volatility term $\sigma^2(y)$ with respect to the equilibrium measure for Y. It is likely that more features of the Y process, beyond its equilibrium, will be manifested in higher order terms of implied volatility. Studying the next order term of implied volatility is a topic for future research.*

3. Preliminaries. The process $(X_{\epsilon,\delta}, Y_{\epsilon,\delta})$ is Markovian, and can be identified through a martingale problem given by generator

$$A_{\epsilon,\delta} f(x,y) = \epsilon\Big((r - \frac{1}{2}\sigma^2(y))\partial_x f(x,y) + \frac{1}{2}\sigma^2(y)\partial^2_{xx} f(x,y)\Big) + \frac{\epsilon}{\delta} B f(x,y) + \frac{\epsilon}{\sqrt{\delta}}\rho\sigma(y)\nu y^\beta \partial^2_{xy} f(x,y), \tag{3.1}$$

where $f \in C^2_c(\mathbb{R} \times E_0)$. Recall that B is given by (1.2). Let $g \in C_b(\mathbb{R})$ and define

$$v_{\epsilon,\delta}(t,x,y) := E\left[g(X_{\epsilon,\delta,t}) | X_{\epsilon,\delta,0} = x, Y_{\epsilon,\delta,0} = y\right]. \tag{3.2}$$

In general, $v_{\epsilon,\delta} \in C_b([0,T] \times \mathbb{R} \times E_0)$. If moreover $v_{\epsilon,\delta} \in C^{1,2}([0,T] \times \mathbb{R} \times \mathbb{R})$, then it solves the following Cauchy problem in classical sense:

$$\partial_t v = A_{\epsilon,\delta} v, \quad \text{in } (0,T] \times \mathbb{R} \times E_0; \tag{3.3a}$$

$$v(0,x,y) = g(x), \quad (x,y) \in \mathbb{R} \times E_0. \tag{3.3b}$$

3.1. *Logarithmic transformation method.* Recall the definition of $u_{\epsilon,\delta}$ in (1.6). That is, $u_{\epsilon,\delta} := \epsilon \log v_{\epsilon,\delta}$ when $g(x) = e^{\epsilon^{-1}h(x)}, h \in C_b(\mathbb{R})$ in (3.2). By (3.3) and some calculus, at least informally, (3.4) is satisfied. This is the logarithmic transform method by Fleming and Sheu. See Chapters VI and VII in [15]. In general, in the absence of knowledge on smoothness of $v_{\epsilon,\delta}$, we can only conclude that $u_{\epsilon,\delta}$ solves the Cauchy problem (3.4) in the sense of viscosity solution (Definition 4.1). In additon to Fleming and Soner [15], such arguments can also be found in Section 5 of Feng [9].

Lemma 3.1. *For $h \in C_b(\mathbb{R})$, $u_{\epsilon,\delta}$ defined as in (1.6), is a bounded continuous function satisfying the following nonlinear Cauchy problem in viscosity solution sense:*

$$\partial_t u = H_{\epsilon,\delta} u, \quad \textit{in } (0,T] \times \mathbb{R} \times E_0; \tag{3.4a}$$

$$u(0,x,y) = h(x), \quad (x,y) \in \mathbb{R} \times E_0. \tag{3.4b}$$

In the above,

(3.5)
$$\begin{aligned} H_{\epsilon,\delta}u(t,x,y) &= \epsilon e^{-\epsilon^{-1}u}A_{\epsilon,\delta}e^{\epsilon^{-1}u}(t,x,y) \\ &= \epsilon\Big((r-\frac{1}{2}\sigma^2)\partial_x u + \frac{1}{2}\sigma^2\partial^2_{xx}u\Big) + \frac{1}{2}|\sigma\partial_x u|^2 \\ &\quad + \frac{\epsilon^2}{\delta}e^{-\epsilon^{-1}u}Be^{\epsilon^{-1}u} + \rho\sigma(y)\nu y^\beta(\frac{\epsilon}{\sqrt{\delta}}\partial^2_{xy}u + \frac{1}{\sqrt{\delta}}\partial_x u\partial_y u) \end{aligned}$$

where,

$$\frac{\epsilon^2}{\delta}e^{-\epsilon^{-1}u}Be^{\epsilon^{-1}u} = \frac{\epsilon}{\delta}Bu + \delta^{-1}\frac{1}{2}|\nu y^\beta\partial_y u|^2.$$

Note that $H_{\epsilon,\delta}$ only operates on the spatial variables x and y.

3.2. *Heuristic expansion.* By Bryc's inverse Varadhan lemma (e.g. Theorem 4.4.2 of [6]), we know that convergence of $u_{\epsilon,\delta}$ is a necessary condition to obtain the LDP for $\{X_{\epsilon,\delta,t} : \epsilon > 0\}$. In this section, we describe heuristically PDEs characterizing $u_{\epsilon,\delta}$ in the limit and the nature of convergence itself.

Henceforth, for notational simplicity, we will drop the subscript δ and write u_ϵ and H_ϵ for $u_{\epsilon,\delta}$ and $H_{\epsilon,\delta}$ respectively. We begin by the following heuristic expansion of u_ϵ in integer powers of ϵ

$$u_\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \epsilon^4 u_4 + \dots \tag{3.6}$$

in both regimes. The $u_i, i = 0, 1, \dots$ are functions of t, x, y. In this heuristic section, we make reasonable choices of u_i which a posteriori, following a rigorous proof of the convergence of u_ϵ in section 5, are shown to be the right choice.

3.2.1. *The case of $\delta = \epsilon^4$.* Computation of $H_\epsilon u_\epsilon$ (see (3.5)) reveals that, in this scale, the fast process Y oscillates so fast that averaging occurs up to terms of order ϵ^2. Viz., $u_0 = u_0(t,x)$, $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ will not depend on y. To see this, we equate coefficients of powers of ϵ in $\partial_t u_\epsilon = H_\epsilon u_\epsilon$.

$O\left(\frac{1}{\epsilon^4}\right)$: $\quad 0 = \frac{1}{2}\nu^2 y^{2\beta}(\partial_y u_0)^2$,

$O\left(\frac{1}{\epsilon^2}\right)$: $\quad 0 = Bu_1 + \frac{1}{2}\nu^2 y^{2\beta}(\partial_y u_1)^2$, and

$O\left(\frac{1}{\epsilon}\right)$: $\quad 0 = Bu_2.$

By choosing u_0, u_1 and u_2 independent of y, the above three equations are satisfied. Then, it follows that

$$\begin{aligned} H_\epsilon u_\epsilon(x,y) &= \frac{1}{2}|\sigma\partial_x u_0|^2 + Bu_3 \\ &\quad + \epsilon(\sigma^2\partial_x u_0 \partial_x u_1 + \frac{1}{2}\sigma^2\partial_{xx}u_0 + (r - \frac{1}{2}\sigma^2)\partial_x u_0 + \nu\rho\sigma y^\beta \partial_x u_0 \partial_y u_3 \\ &\qquad + Bu_4) \\ &\qquad\quad + o(\epsilon) \\ &= \bar{H}_0 u_0 + \epsilon \bar{H}_1(u_0, u_1) + o(\epsilon) \end{aligned}$$

The ϵ^0 order term has to be y independent, therefore u_3 should be chosen so that

$$\frac{1}{2}|\partial_x u_0(x)|^2\sigma^2(y) + Bu_3(x,y) = \bar{H}_0 u_0(x)$$

i.e.

$$Bu_3(x,y) = \bar{H}_0 u_0(x) - \frac{1}{2}|\partial_x u_0(x)|^2\sigma^2(y).$$

The above is a Poisson equation for u_3 with respect to the operator B in the y variable. We impose the condition that the right hand side is centered with respect to the invariant distribution π (given in (1.4)). This ensures a solution to the Poisson equation, which is unique up to a constant in y. That is, we take

$$\bar{H}_0 u_0(x) = \frac{1}{2}|\bar{\sigma}\partial_x u_0(x)|^2;$$

where

$$\overline{\sigma}^2 = \int \sigma^2(y)\pi(dy). \tag{3.7}$$

Thus the leading order term in the heuristic expansion satisfies

$$\partial_t u_0 = \frac{1}{2}|\bar{\sigma}\partial_x u_0(x)|^2. \tag{3.8}$$

3.2.2. *The case of $\delta = \epsilon^2$.* When δ goes to zero at a slower rate ϵ^2, limits become very different and more features in the Y process (rather than just its equilibrium) is retained. We observe that while u_0 is independent of y as in the faster scaling regime, u_1 may now depend on y. Equating coefficients of ϵ^{-2} in $\partial_t u_\epsilon = H_\epsilon u_\epsilon$, we get

$$O\left(\frac{1}{\epsilon^2}\right): \quad 0 = \frac{1}{2}\nu^2 y^{2\beta}(\partial_y u_0)^2,$$

and so we choose $u_0 = u_0(t,x)$ independent of y. Then $H_\epsilon u_\epsilon$ reduces to

$$\begin{aligned} H_\epsilon u_\epsilon(t,x,y) &= \frac{1}{2}|\sigma\partial_x u_0|^2 + \rho\sigma\nu y^\beta \partial_x u_0 \partial_y u_1 + e^{-u_1} B e^{u_1} \\ &\quad +\epsilon\Big(\sigma^2 \partial_x u_0 \partial_x u_1 + \frac{1}{2}\sigma^2 \partial_{xx} u_0 + (r - \frac{1}{2}\sigma^2)\partial_x u_0 \\ &\qquad + B u_2 + \nu y^{2\beta} \partial_y u_1 \partial_y u_2 + \rho\sigma\nu y^\beta \partial_{xy} u_1 + \rho\sigma\nu y^\beta \partial_x u_1 \partial_y u_1 \\ &\qquad + \rho\sigma\nu y^\beta \partial_x u_0 \partial_y u_2\Big) \\ &\qquad\quad + o(\epsilon) \\ &= \bar{H}_0 u_0 + \epsilon \bar{H}_1(u_0, u_1) + o(\epsilon). \end{aligned}$$

Observe that the leading order term should satisfy

$$\partial_t u_0 = \bar{H}_0 u_0,$$

where

$$\begin{aligned} \bar{H}_0 u_0(t,x) = \tfrac{1}{2}|\partial_x u_0(t,x)|^2 \sigma^2(y) + \rho\nu\sigma(y) y^\beta \partial_x u_0(t,x) \partial_y u_1(t,x,y) \\ + e^{-u_1} B e^{u_1}(t,x,y). \end{aligned} \tag{3.9}$$

The ϵ^0 order term has to be y independent, therefore u_1 should be chosen so that

$$\frac{1}{2}|\partial_x u_0|^2 \sigma^2(y) + \rho\nu\sigma(y) y^\beta \partial_x u_0 \partial_y u_1 + e^{-u_1} B e^{u_1} = \lambda(t,x), \tag{3.10}$$

where λ is a function of t and x and is a constant with respect to y. We will rewrite the above equation as an eigenvalue problem.

Define the perturbed generator

$$B^p g(y) = B g(y) + \rho\sigma\nu y^\beta p \partial_y g(y). \tag{3.11}$$

Then

$$e^{-u_1} B e^{u_1} + \rho\sigma\nu y^\beta \partial_x u_0 \partial_y u_1 = e^{-u_1} B^{\partial_x u_0(t,x)} e^{u_1}. \tag{3.12}$$

Fix t and x, and rewrite (3.10) in terms of the perturbed generator (3.12):

$$e^{-u_1} B^{\partial_x u_0(t,x)} e^{u_1}(t,x,y) + \frac{1}{2}|\partial_x u_0(t,x)|^2 \sigma^2(y) = \lambda(t,x).$$

Multiplying the above equation by e^{u_1}, we get the eigenvalue problem:

$$\left(B^{\partial_x u_0} + V\right) g(y) = \lambda g(y), \tag{3.13}$$

where $V(\cdot) = \frac{1}{2}|\partial_x u_0(t,x)|^2\sigma^2(\cdot)$ and $g(\cdot) = e^{u_1(t,x,\cdot)}$. Choose u_1 such that (λ, g) is the solution to the principal (positive) eigenvalue problem (3.13). Since the dependence of λ on t and x is only through $\partial_x u_0$, we can define

$$\bar{H}_0(\partial_x u_0) := \lambda(t,x) = \bar{H}_0 u_0(t,x). \tag{3.14}$$

Constructing classical solution for (3.13) is a considerably hard problem even in the 1-D situation. We give an alternate definition for $\bar{H}_0$ in (3.15) below. If (3.13) can be solved with a nice g, then (3.15) always holds with the $\bar{H}_0$ given by (3.14). The converse is not always true. Especially, (3.15) says nothing about the g. We only need the definition in (3.15) in rigorous treatment of the problem.

Let Y^p be the process corresponding to generator B^p and define

$$\bar{H}_0(p) := \limsup_{T\to+\infty} \sup_{y\in E_0} T^{-1} \log E[e^{\frac{1}{2}|p|^2\int_0^T \sigma^2(Y_s^p)ds} | Y_0^p = y]. \tag{3.15}$$

Y^p has strong enough ergodic properties that the left hand above does not depend upon y even if we omitted the $\sup_{y\in E_0}$ on the right hand side; and, in fact, the $\limsup_{T\to\infty}$ can be replaced with $\lim_{T\to\infty}$ in the above definition. We will justify this fact in the rigorous derivations. By Girsanov transformation

$$\bar{H}_0(p) = \limsup_{T\to+\infty} \sup_{y\in E_0} T^{-1} \log E[e^{\int_0^T \rho p\sigma(Y_s)dW_2(s) + \frac{(1-\rho^2)}{2}|p|^2\int_0^T \sigma^2(Y_s)ds}] \tag{3.16}$$

where Y is the process with generator B. From this expression, we see that $\bar{H}$ is convex in p.

To summarize,

$$\partial_t u_0(t,x) = \bar{H}_0(\partial_x u_0(t,x)) \tag{3.17}$$

where $\bar{H}_0$ is given by (3.15) or (3.16).

4. Convergence of HJB equations. The results of this section can be independently read from the rest of the article.

We reformulate and simplify some techniques, regarding multi-scale convergence of HJB equations, introduced in [10]. Compared with [10], the simplification makes ideas more transparent and readily applicable. These are made possible because we are dealing with Euclidean state spaces which is local compact. All these results are generalizations of Barles-Perthame's half-relaxed limit argument first introduced in single scale, compact state space setting.

Let $E \subset \mathbb{R}^m$, $E_0 \subset \mathbb{R}^n$ and $E' := E \times E_0 \subset \mathbb{R}^d$ where $d = m + n$. A typical element in E is denoted as x and a typical element in E' is denoted as $z = (x, y)$ with $x \in E$ and $y \in E_0$. We denote a class of compact sets in E'

$$Q := \{K \times \tilde{K} : \text{compact } K \subset\subset E, \text{ compact } \tilde{K} \subset\subset E_0\}.$$

We specify a family of differential operators next. Let Λ be an index set and

$$H_i(x, p, P; \alpha) : E \times \mathbb{R}^m \times M_{m\times m} \times \Lambda \mapsto \mathbb{R}, \quad i = 0, 1;$$
$$H_\epsilon(z, p, P) : E' \times \mathbb{R}^d \times M_{d\times d} \mapsto \mathbb{R}$$

be continuous. For each $f \in C^2(\mathbb{R}^d)$, let $\nabla f(x) \in \mathbb{R}^d$ and $D^2 f(x) \in M_{d\times d}$ respectively denote gradient and Hessian matrix evaluated at x. We consider sequence of differential operators

$$H_\epsilon f(z) := H_\epsilon(z, \nabla f(z), D^2 f(z)),$$

for f belongs to the following two domains

$$\begin{aligned} D_{\epsilon,+} &:= \{f : f \in C^2(E'), f \text{ has compact finite level sets}\} \\ D_{\epsilon,-} &:= -D_{\epsilon,+} := \{-f : f \in C^2(E'), f \text{ has compact finite level sets}\}. \end{aligned}$$

We will separately consider these two domains depending on the situation of sub- or super-solution. We also define domains D_+, D_- similarly replacing E' by E.

We will give conditions where $u_\epsilon(t, z) = u_\epsilon(t, x, y)$ solving

$$\partial_t u_\epsilon(t, z) = H_\epsilon(z, \nabla u_\epsilon(t, z), D^2 u_\epsilon(t, z)), \tag{4.1}$$

converges to $u(t, x)$ which is a sub-solution to

$$\partial_t u(t, x) \le \inf_{\alpha\in\Lambda} H_0(x, \nabla u(t, x), D^2 u(t, x); \alpha), \tag{4.2}$$

and a super-solution to

$$\partial_t u(t, x) \ge \sup_{\alpha\in\Lambda} H_1(x, \nabla u(t, x), D^2 u(t, x); \alpha). \tag{4.3}$$

The meaning of sub- super-solutions is defined as follows.

Definition 4.1 (viscosity sub- super- solutions). *We call a bounded measurable function u a viscosity sub-solution to (4.2) (respectively super-solution to (4.3)), if u is upper semicontinuous (respectively lower semicontinuous), and for each*

$$u_0(t, x) = \phi(t) + f_0(x), \quad \phi \in C^1(\mathbb{R}_+), f_0 \in D_+,$$

and each $x_0 \in E$ satisfying $u - u_0$ has a local maximum (respectively each

$$u_1(t,x) = \phi(t) + f_1(x), \quad \phi \in C^1(\mathbb{R}_+), f_1 \in D_-,$$

and each $x_0 \in E$ satisfying $u - u_1$ has a local minimum) at x_0, we have

$$\partial_t u_0(t_0,x_0) - \inf_{\alpha\in\Lambda} H_0(x_0, \nabla u_0(t_0,x_0), D^2 u_0(t_0,x_0); \alpha) \leq 0;$$

(respectively

$$\partial_t u_1(t_0,x_0) - \sup_{\alpha\in\Lambda} H_1(x_0, \nabla u_1(t_0,x_0), D^2 u_1(t_0,x_0); \alpha) \geq 0).$$

If a function is both a sub- as well as super- solution, then it is a solution.

We will assume the following two conditions.

Condition 4.1 (limsup convergence of operators). *For each $f_0 \in D_+$ and each $\alpha \in \Lambda$, there exists $f_{0,\epsilon} \in D_{\epsilon,+}$ (may depend on α) such that*

1. *for each $c > 0$, there exists $K \times \tilde{K} \in \mathcal{Q}$ satisfying*

$$\{(x,y) : H_\epsilon f_{0,\epsilon}(x,y) \geq -c\} \cap \{(x,y) : f_{0,\epsilon}(x,y) \leq c\} \subset K \times \tilde{K};$$

2. *for each $K \times \tilde{K} \in \mathcal{Q}$,*

$$\lim_{\epsilon\to 0} \sup_{(x,y)\in K\times\tilde{K}} |f_{0,\epsilon}(x,y) - f_0(x)| = 0; \tag{4.4}$$

3. *whenever $(x_\epsilon, y_\epsilon) \in K \times \tilde{K} \in \mathcal{Q}$ satisfies $x_\epsilon \to x$,*

$$\limsup_{\epsilon\to 0} H_\epsilon f_{0,\epsilon}(x_\epsilon, y_\epsilon) \leq H_0(x, \nabla f_0(x), D^2 f_0(x); \alpha). \tag{4.5}$$

Condition 4.2 (liminf convergence of operators). *For each $f_1 \in D_-$ and each $\alpha \in \Lambda$, there exists $f_{1,\epsilon} \in D_{\epsilon,-}$ (may depend on α) such that*

1. *for each $c > 0$, there exists $K \times \tilde{K} \in \mathcal{Q}$ satisfying*

$$\{(x,y) : H_\epsilon f_{1,\epsilon}(x,y) \leq c\} \cap \{(x,y) : f_{1,\epsilon}(x,y) \geq -c\} \subset K \times \tilde{K};$$

2. *for each $K \times \tilde{K} \in \mathcal{Q}$,*

$$\lim_{\epsilon\to 0} \sup_{(x,y)\in K\times\tilde{K}} |f_1(x) - f_{1,\epsilon}(x,y)| = 0;$$

3. whenever $(x_\epsilon, y_\epsilon) \in K \times \tilde{K} \in \mathcal{Q}$*, and* $x_\epsilon \to x$*,*

$$\liminf_{\epsilon \to 0} H_\epsilon f_{1,\epsilon}(x_\epsilon, y_\epsilon) \geq H_1(x, \nabla f_1(x), D^2 f_1(x); \alpha).$$

Let u_ϵ be viscosity solutions to (4.1), we define

$$u_3(t,x) := \sup\{\limsup_{\epsilon \to 0+} u_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) : \exists (t_\epsilon, x_\epsilon, y_\epsilon) \in [0,T] \times K \times \tilde{K}, \\ (t_\epsilon, x_\epsilon) \to (t,x), K \times \tilde{K} \in \mathcal{Q}\},$$

$$u_4(t,x) := \inf\{\liminf_{\epsilon \to 0+} u_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) : \exists (t_\epsilon, x_\epsilon, y_\epsilon) \in [0,T] \times K \times \tilde{K}, \\ (t_\epsilon, x_\epsilon) \to (t,x), K \times \tilde{K} \in \mathcal{Q}\},$$

and $\overline{u} = u_3^*$ the upper semicontinuous regularization of u_3 and $\underline{u} = (u_4)_*$ the lower semicontinuous regularization of u_4.

LEMMA 4.1. *Suppose that* $\sup_{\epsilon > 0} \|u_\epsilon\|_\infty < \infty$*. Then,*

1. *under Condition 4.1,* $\overline{u}$ *is a sub-solution to* (4.2)*;*
2. *under Condition 4.2,* $\underline{u}$ *is a supersolution to* (4.3)*.*

PROOF. Let $u_0(t,x) = \phi(t) + f_0(x)$ for a fixed $\phi \in C^1(\mathbb{R}_+)$ and $f_0 \in D_+$. Let (t_0, x_0) be a local maximum of $\overline{u} - u_0$, $t_0 > 0$. We can modify f_0 and ϕ if necessary so that (t_0, x_0) is a strict global maximum. For instance, by taking $\tilde{f}_0(x) = f_0(x) + k|x - x_0|^4$ and $\tilde{\phi}(t) = \phi(t) + k|t - t_0|^2$ for $k > 0$ large enough. Note that such modification has the property that

$$\lim_{\epsilon \to 0+} \sup_{|x - x_0| < \epsilon} |\nabla \tilde{f}_0(x) - \nabla f_0(x_0)| + |D^2 \tilde{f}_0(x) - D^2 f_0(x_0)| = 0.$$

Let $\tilde{u}_0 = \tilde{\phi} + \tilde{f}_0$.

Let $\alpha \in \Lambda$ be given. We now take $u_{0,\epsilon}(t,z) = \tilde{\phi}(t) + f_{0,\epsilon}(z)$ where $f_{0,\epsilon}$ is the approximate of $\tilde{f}_0$ in Condition 4.1. Since u_ϵ is bounded and $u_{0,\epsilon}$ has compact level sets, there exists $(t_\epsilon, z_\epsilon) \in [0,T] \times E'$ such that

$$(u_\epsilon - u_{0,\epsilon})(t_\epsilon, z_\epsilon) \geq (u_\epsilon - u_{0,\epsilon})(t,z) \quad \text{for } (t,z) \in [0,T] \times E', \tag{4.6}$$

and

$$\partial_t \tilde{\phi}(t_\epsilon) - H_\epsilon f_{0,\epsilon}(z_\epsilon) \leq 0. \tag{4.7}$$

The above implies $\inf_\epsilon H_\epsilon f_{0,\epsilon}(z_\epsilon) > -\infty$. We verify next that $f_{0,\epsilon}(z_\epsilon) < c < \infty$. Then by Condition 4.1.1, there exists $K \times \tilde{K} \in \mathcal{Q}$ such that $z_\epsilon = (x_\epsilon, y_\epsilon) \in K \times \tilde{K}$.

Take a $(\hat{t}, \hat{x})$ such that $\tilde{u}_0(\hat{t}, \hat{x}) < \infty$. Take $\hat{z} = (\hat{x}, \hat{y})$ for some $\hat{y} \in E_0$. Then

$$u_{0,\epsilon}(\hat{t}, \hat{z}) = \tilde{\phi}(\hat{t}) + f_{0,\epsilon}(\hat{z}) \to \tilde{\phi}(\hat{t}) + f_0(\hat{x}) = \tilde{u}_0(\hat{t}, \hat{x}) < \infty.$$

Combined with (4.6),

$$u_{0,\epsilon}(t_\epsilon, z_\epsilon) \le 2 \sup_{\epsilon > 0} \|u_\epsilon\|_\infty + \sup_{\epsilon > 0} u_{0,\epsilon}(\hat{t}, \hat{z}) < \infty,$$

and $\sup_{\epsilon>0} f_{0,\epsilon}(z_\epsilon) < \infty$ follows.

Since $K \times \tilde{K}$ is compact in E', there exists a subsequence of $\{(t_\epsilon, z_\epsilon)\}$ (to simplify, we still use the ϵ to index it) and a $(\tilde{t}_0, \tilde{x}_0) \in [0, T] \times E$ such that $t_\epsilon \to \tilde{t}_0$ and $x_\epsilon \to \tilde{x}_0$. Such $(\tilde{t}_0, \tilde{x}_0)$ has to be the unique global maximizer (t_0, x_0) for $\overline{u} - \tilde{u}_0$ that appeared earlier. This is because, by using $x_\epsilon \to \tilde{x}_0$ and $z_\epsilon = (x_\epsilon, y_\epsilon)$, the definition of $\overline{u}$ and (4.4), from (4.6) we have

$$(\overline{u} - u_0)(\tilde{t}_0, \tilde{x}_0) \ge (\overline{u} - u_0)(t, x), \quad \forall (t, x). \tag{4.8}$$

Now, from (4.7) and (4.5), we also have

$$\partial_t u_0(t_0, x_0) \le H_0(x_0, \nabla f_0(x_0), D^2 f_0(x_0); \alpha).$$

Note that t_0, x_0 and u_0 are all chosen prior to, and independent of, α. We can take $\inf_{\alpha \in \Lambda}$ on both sides to get

$$\partial_t u_0(t_0, x_0) - \inf_{\alpha \in \Lambda} H_0(x_0, \nabla u_0(t_0, x_0), D^2 u_0(t_0, x_0); \alpha) \le 0.$$

The proof that $\underline{u}$ is a super-solution of (4.3) under Condition 4.2 follows similarly. □

Lemma 4.2. *Suppose that the conditions in Lemma 4.1 hold and that there exists $h \in C_b(E)$ such that*

$$\lim_{\epsilon \to 0} \sup_{(x,y) \in K \times \tilde{K}} |h(x) - u_\epsilon(0, x, y)| = 0, \quad \forall K \times \tilde{K} \in \mathcal{Q}.$$

Further suppose that for any sub-solution $u_0(t, x)$ of (4.2) with $u_0(0, x) = h(x)$ and super-solution u_1 of (4.3) with $u_1(0, x) = h(x)$, we have

$$u_0(t, x) \le u_1(t, x), \quad (t, x) \in [0, T] \times E.$$

That is, a comparison principle holds for sub-solutions of (4.2) and super-solutions of (4.3) with initial data h.

Then $u = \overline{u} = \underline{u}$ and

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \sup_{(x,y)\in K\times\tilde{K}} |u(t,x) - u_\epsilon(t,x,y)| = 0, \quad \forall K\times\tilde{K} \in \mathcal{Q}.$$

5. Rigorous justification of expansions. To rigorously prove the convergence of operators H_ϵ given by (3.5) to operators $\bar{H}_0$ obtained by heuristic arguments in section 3.2, we rely on, and extend, results developed in [10]. An exposition of the relevant results from [10] was laid out in section 4. In this section we verify conditions 4.1 and 4.2 and prove the comparison principle in Lemma 4.2. We will adhere to notation used in section 4.

Conditions 4.1 and 4.2 require us to carefully choose a class of perturbed test functions with an index set Λ, and a family of operators $\{H_0(\cdot;\alpha), H_1(\cdot;\alpha); \alpha\in\Lambda\}$ to obtain viscosity sub- and super- solution estimates of u_0, the limit of u_ϵ. This technique was first introduced in [10] and illustrated through examples in Chapter 11 of that book. Our presentation simplifies the technique in the context of application here. We will make the sub-solution estimate given by $H_0(\cdot,\alpha)$ tight, by inf-ing over α. Hence introducing yet another operator H_0. Similarly, we sup over α to tighten up the super-solution type estimate provided by $H_1(\cdot,\alpha)$ which introduces operator H_1.

Let

$$\zeta(y) := |y-m|^\zeta, \tag{5.1}$$

where $\zeta > 0$ is any number satisfying $2\sigma < \zeta < 2(1-\beta)$ with σ and β given as in Assumption 1.1. Throughout the two regimes ($\delta = \epsilon^4, \epsilon^2$), we take the index set

$$\Lambda := \{\alpha = (\xi,\theta) : \xi \in C_c^2(E_0), 0<\theta<1\};$$

and define two domains

$$D_+ := \{f : f(x) = \varphi(x) + \gamma\log(1+|x|^2); \varphi\in C_c^2(\mathbb{R}), \gamma>0\},$$

and

$$D_- := \{f : f(x) = \varphi(x) - \gamma\log(1+|x|^2); \varphi\in C_c^2(\mathbb{R}), \gamma>0\}.$$

A collection of compact sets in $\mathbb{R}\times E_0$ is defined by

$$\mathcal{Q} := \{K\times\tilde{K} : \text{ compact } K \subset\subset \mathbb{R}, \tilde{K}\subset\subset E_0\}.$$

5.0.3. *Case* $\delta = \epsilon^4$. For each $f = f(x) \in D_+$, and each $\alpha = (\xi, \theta) \in \Lambda$, we let

$$g(y) := \xi(y) + \theta\zeta(y),$$

and define perturbed test function

$$f_\epsilon(x,y) := f(x) + \epsilon^3 g(y) \quad = \quad f(x) + \epsilon^3\xi(y) + \epsilon^3\theta\zeta(y).$$

Note that $\|\partial_x f\|_\infty + \|\partial^2_{xx} f\|_\infty < \infty$. Then

$$\begin{aligned} H_\epsilon f_\epsilon(x,y) \quad = \quad & \epsilon\left[(r - \frac{1}{2}\sigma^2(y))\partial_x f + \frac{1}{2}\sigma^2(y)\partial^2_{xx} f\right] + \frac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ & + B\xi(y) + \theta B\zeta(y) + \frac{1}{2}\epsilon^2\nu^2 y^{2\beta}|\partial_y\xi(y) + \theta\partial_y\zeta(y)|^2 \\ & + \epsilon\rho\sigma(y)\nu y^\beta \partial_x f(\partial_y\xi(y) + \partial_y\zeta(y)). \end{aligned}$$

The choice of the number ζ in definition of the function $\zeta(y)$ in (5.1) guarantees that $B\zeta(y) \leq -C\zeta(y)$. Moreover, with earlier assumption that $0 \leq \sigma < 1 - \beta$, the growth of $\zeta(y)$ as $|y| \to \infty$ dominates the growth in y of all other terms in $H_\epsilon f_\epsilon$. Therefore, there exist constants $c_0, c_1 > 0$ with

$$H_\epsilon f_\epsilon(x,y) \leq \frac{1}{2}|\sigma(y)\partial_x f(x)|^2 + B\xi(y) - \theta c_0\zeta(y) + \epsilon c_1$$

In addition,

$$f_\epsilon(x,y) = f(x) + \epsilon^3 g(y) \geq f(x) - \epsilon\|\xi\|_\infty.$$

Furthermore, for each $c > 0$, we can find $K \times \tilde{K} \in \mathcal{Q}$, such that

$$\{(x,y) : H_\epsilon f_\epsilon(x,y) \geq -c\} \cap \{(x,y) : f_\epsilon(x,y) \leq c\} \subset K \times \tilde{K}, \tag{5.2}$$

verifying Condition 4.1.1. The rest of Condition 4.1 can be verified by taking

$$H_0(x,p;\xi,\theta) = \sup_{y\in E_0}\Big(\frac{1}{2}|\sigma(y)p|^2 + B\xi(y) - \theta c_0\zeta(y)\Big).$$

We define

$$\begin{aligned} H_0 f(x) : &= \inf_{\alpha\in\Lambda} H_0(x, \partial_x f(x); \alpha) \\ &= \inf_{0<\theta<1}\ \inf_{\xi\in C_c^2(E_0)}\ \sup_{y\in E_0}\Big(\frac{1}{2}|\sigma(y)\partial_x f(x)|^2 + B\xi(y) - \theta c_0\zeta(y)\Big). \end{aligned}$$

Similarly, for $f \in D_-$, $\alpha = (\xi, \theta) \in \Lambda$, we can choose

$$f_\epsilon(x,y) \quad = \quad f(x) + \epsilon^3\xi(y) - \epsilon^3\theta\zeta(y).$$

Then Condition 4.2 holds for the choice of

$$H_1(x,p;\xi,\theta) = \inf_{y\in\mathbb{R}} \Big(\frac{1}{2}|\sigma(y)p|^2 + B\xi(y) + \theta c_0\zeta(y)\Big).$$

We define

$$\begin{aligned} H_1 f(x) :&= \sup_{\alpha\in\Lambda} H_1(x,\partial_x f(x);\alpha) \\ &= \sup_{0<\theta<1} \sup_{\xi\in C_c^2(E_0)} \inf_{y\in E_0} \Big(\frac{1}{2}|\sigma(y)\partial_x f(x)|^2 + B\xi(y) + \theta c_0\zeta(y)\Big). \end{aligned}$$

Next, to verify Lemma 4.2, we estimate $H_0 f$ from above and $H_1 f$ from below using some simple quantity.

Lemma 5.1.

$$H_0 f(x) \le \frac{1}{2}|\bar\sigma\partial_x f(x)|^2, f\in D_+; \quad H_1 f(x) \ge \frac{1}{2}|\bar\sigma\partial_x f(x)|^2, f\in D_-.$$

We note that H_0, H_1 have different domains D_+ and D_- respectively; $D_+\cap D_- = \emptyset$.

Proof. The key to obtaining the estimates in the statement of the Lemma is the Poisson equation

$$B\chi(y) = \frac{1}{2}|p|^2(\bar\sigma^2 - \sigma^2(y)). \tag{5.3}$$

We will need growth estimates for χ. In the case of $\beta = 0$ (i.e. Y is an O-U process), section 5.2.2 of Fouque, Papanicolaou and Sircar [13] contains such estimates. Specifically, if $\sigma(y)$ is bounded, $|\chi(y)| \le C(1+\log(1+|y|))$; if $\sigma(y)$ has polynomial growth, χ has polynomial growth estimates of the same order. The following growth estimates for the situation $\frac{1}{2} \le \beta < 1$ are derived in Appendix B:

$$|\chi'(y)| \le C_1 y^{2\sigma-1} \quad \text{as } y\to\infty, \quad \text{for some positive constant } C_1. \tag{5.4}$$

Therefore $|\chi(y)| \le C(1+\log(1+|y|))$ if $\sigma(y)$ is bounded and $|\chi(y)| \le \tilde C(1+y^{2\sigma})$ when $0<\sigma<1-\beta$.

We will make use of χ as test function in the expressions for $H_0 f$ and $H_1 f$. However, χ does not have compact support. We choose a cut-off function φ to approximate it using localization arguments. Let non-negative $\varphi(y) \in$

$C^\infty(E_0)$ be such that $\varphi(y) = 1$ when $|y| \le 1$ and 0 when $|y| > 2$. We take a sequence of $\xi_n(y) = \varphi(\frac{y}{n})\chi(y)$ which are truncated versions of χ. Then

$$\begin{aligned} B\xi_n(y) &= \varphi\left(\frac{y}{n}\right) B\chi(y) + (m-y)\chi(y)n^{-1}\varphi\left(\frac{y}{n}\right) \\ &\quad + \frac{1}{2}\nu^2 y^{2\beta}\chi(y)n^{-2}\varphi''\left(\frac{y}{n}\right) + \nu^2 y^{2\beta}\chi'(y)n^{-1}\varphi'\left(\frac{y}{n}\right) \end{aligned}$$

Suppose $\sigma > 0$. Noting that $|\varphi(y)|, |\varphi'(y)|$ and $|\varphi''(y)|$ are uniformly bounded and are 0 when $|y| > 2$, and using the growth estimates (5.4) for χ and χ', we get,

$$\begin{aligned} |B\xi_n(y)| &\le cy^{2\sigma}\left(1 + \frac{(m-y)}{n} + \left(\frac{y}{n}\right)^{2\beta} n^{2\beta-2} + y^{\beta-1}\left(\frac{y}{n}\right)^{\beta} n^{\beta-1}\right) 1_{\{\frac{y}{n}\le 2\}} \\ &\le cy^{2\sigma} \quad \text{for all } n. \end{aligned}$$

In the above, we used the fact that $\frac{y}{n} \le 2$ and $\beta - 1 < 0$. Similarly, if $\sigma(y)$ is bounded i.e. $\sigma = 0$, we get $|B\xi_n(y)|$ is uniformly bounded for all n. Therefore, for large y, $\zeta(y)$ dominates $B\xi_n(y)$ uniformly in n in the following sense: there exists a sub-linear function $\psi : \mathbb{R} \mapsto \mathbb{R}_+$ such that

$$\sup_{n=1,2,\ldots} |B\xi_n(y)| \le \psi(\zeta(y)).$$

With the above estimate, we have

$$\begin{aligned} H_0 f(x) &\le \limsup_{n\to\infty} \inf_{0<\theta<1} \sup_{y\in E_0} \left(\frac{1}{2}|\sigma(y)\partial_x f(x)|^2 + B\xi_n(y) - \theta c_0\zeta(y)\right) \\ &\le \frac{1}{2}|\bar\sigma\partial_x f(x)|^2. \end{aligned}$$

Similarly, one can prove the case for $H_1 f$. □

By standard viscosity solution theory (e.g. [5]), the comparison principle holds for sub-solutions and super-solutions of

$$\partial_t u_0 = \frac{1}{2}|\bar\sigma\partial_x u_0|^2, \quad t > 0,$$

and the solution is uniquely given by the Lax formula

$$u_0(t,x) = \sup_{x'\in\mathbb{R}} \left\{ u_0(0,x') - \frac{|x-x'|^2}{2\bar\sigma^2 t} \right\}. \tag{5.5}$$

Putting together the above result and Lemmas 4.1 and 4.2, we get

Lemma 5.2.

$$\lim_{\epsilon\to 0+} \sup_{|t|+|x|+|y|<c} |u_\epsilon(t,x,y) - u_0(t,x)| = 0, \quad \forall c > 0$$

where u_0 is the solution of (3.8) and is given by (5.5).

5.0.4. *Case $\delta = \epsilon^2$.* For each $f = f(x) \in D_+$, and $\alpha = (\xi, \theta) \in \Lambda$, we choose our perturbed test function as

$$f_\epsilon(x,y) := f(x) + \epsilon g(y)$$

where $g(y) = (1-\theta)\xi(y) + \theta\zeta(y)$; $\zeta(y)$ is defined as before in (5.1). Then,

$$\begin{aligned} H_\epsilon f_\epsilon(x,y) &= \epsilon\left[(r - \frac{1}{2}\sigma^2(y))\partial_x f + \frac{1}{2}\sigma^2(y)\partial^2_{xx} f\right] + \frac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ &\quad + e^{-g(y)} B^{\partial_x f(x)} e^g(y) \\ &\le \epsilon\left[(r - \frac{1}{2}\sigma^2(y))\partial_x f + \frac{1}{2}\sigma^2(y)\partial^2_{xx} f\right] + \frac{1}{2}\sigma^2(y)|\partial_x f|^2 \\ &\quad + (1-\theta)e^{-\xi} B^{\partial_x f} e^{\xi}(y) + \theta e^{-\zeta} B^{\partial_x f} e^{\zeta}(y), \end{aligned}$$

where $B^{\partial_x f(x)}$ is the perturbed generator defined in (3.11). Recall that $\|\partial_x f\|_\infty + \|\partial^2_{xx} f\|_\infty < \infty$ by the choice of domain D_+. We can thus find a constant $c_0 > 0$ such that

$$H_\epsilon f_\epsilon(x,y) \le \frac{1}{2}|\sigma(y)\partial_x f(x)|^2 + (1-\theta)e^{-\xi} B^{\partial_x f} e^{\xi}(y) + \theta e^{-\zeta} B^{\partial_x f} e^{\zeta}(y) + \epsilon c_0.$$

Note that

$$e^{-\zeta} B^{\partial_x f(x)} e^{\zeta}(y) = B\zeta(y) + \rho\sigma(y)\nu y^\beta \partial_x f(x)\partial_y \zeta(y) + \frac{1}{2}\nu^2 y^{2\beta}|\partial_y \zeta(y)|^2,$$

where

$$B\zeta(y) = -\zeta \cdot |y-m|^\zeta + \frac{1}{2}\nu^2 y^{2\beta}\zeta(\zeta-1)|y-m|^{\zeta-2}. \tag{5.6}$$

The term $-\zeta(y)$ in $B\zeta(y)$ dominates growth in y from all other terms in $H_\epsilon f_\epsilon$ as $|y| \to \infty$. Since $\zeta(y) \to \infty$ as $|y| \to \infty$, $H_\epsilon f_\epsilon(x,y) \to -\infty$ as $|y| \to \infty$. We also have $f_\epsilon(x,y) = f(x) + \epsilon g(y) \ge f(x) - \epsilon\|\xi\|_\infty$. Therefore, for each $c > 0$, we can find $K \times \tilde{K} \in \mathcal{Q}$, such that

$$\{(x,y) : H_\epsilon f_\epsilon(x,y) \ge -c\} \cap \{(x,y) : f_\epsilon(x,y) \le c\} \subset K \times \tilde{K}, \tag{5.7}$$

verifying Condition 4.1.1.

Super-solution case follows similarly, where we define the perturbed test function as $f_\epsilon(x,y) = f(x) + \epsilon(1+\theta)\xi(y) - \epsilon\theta\zeta(y)$, for each $f \in D_-$ and $(\xi,\theta) \in \Lambda$.

Take

$$\begin{aligned} H_0(x,p;\xi,\theta) &:= \sup_{y \in E_0}\Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)e^{-\xi}B^p e^{\xi}(y) + \theta e^{-\zeta}B^p e^{\zeta}(y)\Big), \\ H_1(x,p;\xi,\theta) &:= \inf_{y \in E_0}\Big(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)e^{-\xi}B^p e^{\xi} - \theta e^{-\zeta}B^p e^{\zeta}(y)\Big) \end{aligned}$$

and

$$H_0 f(x) := \inf_{0<\theta<1} \inf_{\xi \in C_c^\infty(E_0)} H_0(x, \partial_x f; \xi, \theta),$$
$$H_1 f(x) := \sup_{0<\theta<1} \sup_{\xi \in C_c^\infty(E_0)} H_1(x, \partial_x f; \xi, \theta).$$

Conditions 4.1 and 4.2 are satisfied by these choices of H_0 and H_1. Note that, although $\frac{1}{2}|\sigma(y)p|^2$ is not bounded in y, its growth is at most $|y|^{2\sigma}$ and is dominated by the growth of $\zeta(y)$ for $|y|$ large enough.

To verify Lemma 4.2, we develop useful sharp estimates for H_0 and H_1 next. Denote

$$T(t)g(y) := E[g(Y_t)|Y(0) = y], \quad g \in C_b(E_0)$$

and let $\mathbb{B}$ be the weak infinitesimal generator for semigroup $\{T(t) : t \geq 0\}$ in $C_b(E_0)$. Let $D^{++}(\mathbb{B})$ denote the domain of $\mathbb{B}$ with functions strictly bounded from below by a positive constant. Similarly define notations for $\mathbb{B}^p$, the weak infinitesimal generator corresponding to the process Y^p introduced in section 3.2.2. For each $g \in D^{++}(\mathbb{B}^p) \subset C_b(E_0)$, since $\zeta > 2\sigma$, there exists compact $K \subset\subset E_0$ with

$$\sup_{y \in E_0} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y) \Big)$$
$$= \sup_{y \in K} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y) \Big).$$

For each $\epsilon > 0$, by truncating and mollifying g, we can find a $\xi := \xi_\epsilon \in C_c^\infty(E_0)$ such that

$$H_0(x, p; \xi, \theta) \leq \epsilon + \sup_{y \in K} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y) \Big).$$

Denote $p = \partial_x f(x)$. Then

(5.8)
$$H_0 f(x) \leq \inf_{0<\theta<1} \inf_{g \in D^{++}(\mathbb{B}^p)} \sup_{y \in E_0} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y) \Big).$$

Similarly, we have

(5.9)
$$H_1 f(x) \geq \sup_{0<\theta<1} \sup_{g \in D^{++}(\mathbb{B}^p)} \inf_{y \in E_0} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)\frac{\mathbb{B}^p g}{g}(y) - \theta e^{-\zeta} B^p e^{\zeta}(y) \Big).$$

We define $I_B(\cdot;p) : \mathcal{P}(E_0) \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$I_B(\mu;p) := -\inf_{g \in D^{++}(\mathbb{B}^p)} \int_{E_0} \frac{\mathbb{B}^p g}{g} d\mu \wedge \int_{E_0} e^{-\zeta(y)} B^p e^{\zeta(y)} d\mu(y).$$

However, we can find a sequence $\{g_n\} \subset D^{++}(\mathbb{B}^p)$ (take for example $g_n := e^{\zeta_n}$ where $\zeta_n \in C_c^2(E_0)$ are some smooth truncations of ζ), such that

$$\int_{E_0} e^{-\zeta(y)} B^p e^{\zeta(y)} d\mu(y) \geq \limsup_{n \to \infty} \int_{E_0} \frac{\mathbb{B}^p g_n}{g_n} d\mu.$$

Therefore we have

$$I_B(\mu;p) := -\inf_{g \in D^{++}(\mathbb{B}^p)} \int_{E_0} \frac{\mathbb{B}^p g}{g} d\mu. \tag{5.10}$$

Recall that Y^p denotes the process corresponding to generator B^p (or equivalently, $\mathbb{B}^p$). It can be directly verified that Y^p has a unique stationary distribution π^p and that Y^p is reversible with respect to it (see Appendix C of this article). Let

$$\mathcal{E}^p(f,g) := \int f \mathbb{B} g d\pi^p$$

be the Dirichlet form for Y^p. By the material in Section 7 of Stroock [28] (particularly Theorem 7.44; note that the diffusion generated by B^p has transition density with respect to Lebesgue measure – e.g. Theorem 4.3.5 of Knight [24]), we get

$$I_B(\mu;p) = \mathcal{E}^p\Big(\sqrt{\frac{d\mu}{d\pi^p}}, \sqrt{\frac{d\mu}{d\pi^p}}\Big) = \frac{\nu^2}{2} \int_0^\infty y^{2\beta} \Big|\partial_y \sqrt{\frac{d\mu}{d\pi^p}}(y)\Big|^2 \pi^p(dy) \tag{5.11}$$

(see Appendix C.3 for the last equality above). If μ in $I_B(\mu;p)$ is not absolutely continuous with respect to π^p, then the right hand quantity in (5.11) is viewed as $+\infty$. Again through Theorem 7.44 of [28], we also get that $\bar{H}_0$, defined in (3.15), can be expressed as

$$\begin{aligned} \bar{H}_0(p) &= \sup_{\mu \in \mathcal{P}(\mathbb{R}_+)} \Big(\frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2 d\mu - I_B(\mu;p)\Big) \\ &= \sup_{h \in L^2(\pi^p), \|h\|_{L^2(\pi^p)}=1} \Big(\frac{|p|^2}{2} \int_{\mathbb{R}_+} \sigma^2(y) h^2(y) \pi^p(dy) \\ &\qquad - \frac{\nu^2}{2} \int_0^\infty y^{2\beta} |\partial h(y)|^2 \pi^p(dy)\Big). \end{aligned} \tag{5.12}$$

As in Lemma 11.35 of [10],

$$\inf_{0<\theta<1} \inf_{g \in D^{++}(\mathbb{B}^p)} \sup_{y \in E_0} \Big(\frac{1}{2}|\sigma(y)p|^2 + (1-\theta)\frac{\mathbb{B}^p g}{g}(y) + \theta e^{-\zeta} B^p e^{\zeta}(y) \Big) = \bar{H}_0(p).$$

Using (5.8), this immediately gives

$$H_0 f(x) \le \bar{H}_0(\partial f(x)), \quad f \in D_+.$$

We will prove a similar inequality estimate for H_1 hence giving the following

Lemma 5.3.

$$H_1 f(x) \ge \bar{H}_0(\partial f(x)), f \in D_-, \quad H_0 f(x) \le \bar{H}_0(\partial f(x)), f \in D_+.$$

It remains to prove the estimate for H_1. By the proof of Lemma B.10 of [10],

$$\begin{aligned} \sup_{0<\theta<1} \sup_{g \in D^{++}(\mathbb{B}^p)} \inf_{y \in \mathbb{R}_+} &\Big(\frac{1}{2}|\sigma(y)p|^2 + (1+\theta)\frac{\mathbb{B}^p g}{g}(y) - \theta e^{-\zeta} B^p e^{\zeta}(y) \Big) \\ &\ge \inf_{\nu \in \mathcal{P}(\mathbb{R}_+), \langle \zeta, \nu \rangle < +\infty} \liminf_{t \to \infty} t^{-1} \log E^{\nu}[e^{\frac{1}{2}|p|^2 \int_0^t \sigma^2(Y_s^p) ds}]. \end{aligned} \tag{5.13}$$

If we show that

$$\liminf_{t \to +\infty} t^{-1} \log E[e^{\frac{1}{2}|p|^2 \int_0^t \sigma^2(Y_s^p) ds} | Y_0^p = y] \ge \bar{H}_0(p), \tag{5.14}$$

then (5.9), (5.13) and (5.14) together give us the estimate for H_1 in Lemma 5.3. The proof of (5.14) follows essentially the same argument used in Example B.14 in Appendix of [10]. Two ingredients need to be emphasized. First, for each μ with $I_B(\mu; p) < \infty$, by a mollification and truncation argument, we can find a sequence $\mu_n(dy) = \frac{e^{h_n(y)}}{\int e^{h_n} d\pi^p} d\pi^p(y)$ with $h_n + c_n \in C_c^\infty(E_0)$ for some constant c_n, such that $\lim_{n\to\infty} I_B(\mu_n; p) = I_B(\mu; p)$. Second, for every $y \in E_0$ and every $h \in C_c^\infty(E_0)$, the following ergodic theorem holds

$$\lim_{t \to \infty} \frac{1}{t} E[\int_0^t \sigma^2(\tilde{Y}_s^h) ds | \tilde{Y}_0^h = y] = \int_{-\infty}^{\infty} \sigma^2(z) d\tilde{\pi}^h(z), \tag{5.15}$$

where $d\tilde{Y}_s^h = \Big((m - \tilde{Y}_s^h) + \nu(\tilde{Y}_s^h)^\beta \partial h(\tilde{Y}_s^h) \Big) ds + \nu(\tilde{Y}_s^h)^\beta dW_s^2$ and where $\tilde{\pi}^h$ is the unique stationary distribution of $\tilde{Y}^h$.

Lemma 5.4. (5.15) *holds.*

PROOF. By Ito's formula,

$$E[\zeta(\tilde{Y}_t^h)] = E[\zeta(\tilde{Y}_0^h)] + E[\int_0^t \tilde{B}^h \zeta(\tilde{Y}_s^h)]ds,$$

where $\tilde{B}^h\zeta(y) = \Big(m - y + \nu y^\beta \partial_y h(y)\Big)\zeta'(y) + \frac{1}{2}\nu^2 y^{2\beta}\zeta''(y)$. As in (5.6), $-\zeta(y)$ is the dominating growth term in $\tilde{B}^h\zeta(y)$. Therefore, defining a family of mean occupation measure

$$\tilde{\pi}^h(t, y, A) := E[t^{-1}\int_0^t \mathbf{1}_{\{\tilde{Y}_s^h \in A\}} ds | \tilde{Y}_0^h = y],$$

we have that

$$\sup_{t>0} \int_z \zeta(z)\tilde{\pi}^h(t, y, dz) = \sup_{t>0} t^{-1} E[\int_0^t \zeta(\tilde{Y}_s^h) ds | \tilde{Y}_0^h = y] \le C(y; h(\cdot)) < \infty.$$

Hence $\{\tilde{\pi}^h(t, y, \cdot) : t > 0\}$ is tight and along convergent subsequences and corresponding limiting point $\tilde{\pi}^h$, we have

$$E[t^{-1}\int_0^t \varphi(\tilde{Y}_s^h) ds | \tilde{Y}_0^h = y] \to \int_z \varphi d\tilde{\pi}^h, \quad \varphi \in C_b(E_0). \tag{5.16}$$

Such $\tilde{\pi}^h$ is necessarily a stationary distribution satisfying $\int \tilde{B}^h \psi d\tilde{\pi}^h = 0$ for all $\psi \in C_c^2(E_0)$. Uniqueness of such probability measure can be proved by an argument similar to the one in Appendix C. We thus conclude that there is only one such $\tilde{\pi}^h$ and that convergence (5.16) occurs along the whole sequence, not just subsequences. Furthermore, the growth of σ^2 is dominated by ζ, and so by uniform integrability argument, (5.15) holds. □

From (3.16), we see that $\bar{H}_0(p)$ is convex in $p \in \mathbb{R}$. Let us denote its Legendre transform as $\bar{L}_0$, then we have the following.

LEMMA 5.5. *The unique viscosity solution to* (3.17) *is:*

$$u_0(t, x) := \sup_{x' \in \mathbb{R}} \Big\{u_0(0, x') - t\bar{L}_0\Big(\frac{x - x'}{t}\Big)\Big\}. \tag{5.17}$$

Moreover, u_ϵ *converges uniformly over compact sets in* $[0, T] \times \mathbb{R} \times E_0$ *to* u_0.

PROOF. We know that u_0 defined by (5.17) solves (3.17) by the Lax formula. That u_0 is the unique solution follows from standard viscosity comparison principle with convex Hamiltonians. The convergence result follows from multi-scale viscosity convergence results developed in section 4– Lemmas 4.1 and 4.2. □

6. Large deviation, asymptotic for option prices and implied volatilities. We finish the proof of Theorem 2.1, Corollary 2.1 and Theorem 2.2.

6.1. *A large deviation theorem.*

PROOF OF THEOREM 2.1. From the previous section we have $u_\epsilon(t,x,y) \to u_0(t,x)$ as $\epsilon \to 0$ for each fixed $(t,x,y) \in [0,T] \times \mathbb{R} \times E_0$. All we need is exponential tightness of $\{X_{\epsilon,\delta,t}\}$ to apply Bryc's lemma and to conclude our proof. This is obtained as follows.

Let $f(x) = \log(1+x^2)$ and $\zeta(y)$ be defined as in (5.1). Take

$$f_\epsilon(x,y) = \begin{cases} f(x) + \epsilon^3\zeta(y) & \text{for the case } \delta = \epsilon^4 \\ f(x) + \epsilon\zeta(y) & \text{for the case } \delta = \epsilon^2. \end{cases}$$

Note that $f(x)$ is an increasing function of $|x|$ and $\zeta(\cdot) \geq 0$, therefore, for any $c > 0$ there exists a compact set $K_c \subset \mathbb{R}$ such that $f_\epsilon(x,y) > c$ when $x \notin K_c$. We next compute $H_\epsilon f_\epsilon(x,y)$ (see (3.5)). Observe that since $\|\partial_x f\|_\infty + \|\partial^2_{xx} f\|_\infty < \infty$, by our choice of $\zeta(\cdot)$, $H_\epsilon f_\epsilon(x,y) \to -\infty$ as $|y| \to \infty$. Therefore $\sup_{x\in R, y\in R} H_\epsilon f_\epsilon(x,y) = C < \infty$. For simplicity, we denote $X_{\epsilon,\delta,t}$ by $X_{\epsilon,t}$. The P and E below denote probability and expectation conditioned on (X,Y) starting at (x,y).

$$\begin{aligned} &P(X_{\epsilon,t} \notin K_c) e^{(c - f_\epsilon(x,y) - tC)/\epsilon} \\ &\leq E\left[\exp\left\{\frac{f_\epsilon(X_{\epsilon,t}, Y_{\epsilon,t})}{\epsilon} - \frac{f_\epsilon(x,y)}{\epsilon} - \int_0^t e^{-\frac{f_\epsilon(X_{\epsilon,s}, Y_{\epsilon,s})}{\epsilon}} A_\epsilon e^{\frac{f_\epsilon(X_{\epsilon,s}, Y_{\epsilon,s})}{\epsilon}} ds\right\}\right] \\ &\leq 1. \end{aligned}$$

In the above inequalities, the term within expectation in the second line is a non-negative local martingale (and hence a supermartingale), see [8] [Lemma 4.3.2]. We apply the optional sampling theorem to get the last inequality above. Therefore

$$\epsilon \log P(X_{\epsilon,t} \notin K_c) \leq tC + f_\epsilon(x,y) - c \leq const - c,$$

giving us exponential tightness of $X_{\epsilon,t}$.

Let $u_0^{h,r}$ denote the limit of $u_{\epsilon,\delta}$ when $u_{\epsilon,\delta}(0,x,y) = h(x)$ and $\delta = \epsilon^r$, $r = 2,4$. Applying Bryc's lemma we get, $\{X_{\epsilon,\epsilon^r,t}\}$ for $r = 2,4$ satisfies a LDP with speed $1/\epsilon$ and rate function

$$I_r(x; x_0, t) := \sup_{h \in C_b(R)} \{h(x) - u_0^{h,r}(t, x_0)\}. \tag{6.1}$$

In Appendix D we check that $I_2(x;x_0,t) = t\bar{L}_0\left(\frac{x_0-x}{t}\right)$ where $\bar{L}$ is the Legendre transform of $\bar{H}_0$ defined in (3.15); and $I_4 = \frac{|x_0-x|^2}{2\bar{\sigma}^2 t}$.

□

6.2. *Option prices.*

PROOF OF COROLLARY 2.1. We follow the proof of Corollary 1.3 in [11] and show that $\lim_{\epsilon\to 0^+} \epsilon \log E\left[(S_{\epsilon,t}-K)^+\right]$ is bounded above and below by $-I_r(\log K; x_0, t)$.

Recall that we are considering out-of-the-money call options and hence $x_0 < \log K$ (see (2.4)). Since our rate functions $I_r(x;x_0,t)$, for both $r=2,4$, are non-negative, convex functions with $I_r(x_0;x_0,t)=0$, they are consequently monotonically increasing functions of x when $x \geq x_0$. Using this fact and the continuity of the rate functions, the proof of the lower bound follows verbatim from the proof in [11]. We refer the reader to [11] for details.

The upper bound follows from [11] once we justify the following limit: for any $p>1$,

$$\lim_{\epsilon\to 0^+} \epsilon \log E[S^p_{\epsilon,\delta,t}] = 0 \quad \text{for both } \delta=\epsilon^4 \text{ and } \delta=\epsilon^2. \tag{6.2}$$

Recall the operator $A_{\epsilon,\delta}$ defined at the beginning of section 3. By a slight abuse of notation, we can use $A_{\epsilon,\delta}$ to denote the operator acting on the unbounded function e^{px} given below:

$$A_{\epsilon,\delta}e^{px} = \epsilon\Big((r-\frac{1}{2}\sigma^2(y))pe^{px} + \frac{1}{2}\sigma^2(y)p^2e^{px}\Big).$$

Let

$$M_t := \exp\left\{pX_{\epsilon,\delta,t} - pX_{\epsilon,\delta,0} - \int_0^t e^{-pX_{\epsilon,\delta,s}} A_{\epsilon,\delta} e^{pX_{\epsilon,\delta,s}} ds\right\}.$$

Then M_t is a non-negative local martingale (supermartingale), this follows from the proof of [8] [Lemma 4.3.2]. By the optional sampling theorem

$$EM_t \leq 1.$$

Recall that $X_{\epsilon,\delta,t} = \log S_{\epsilon,\delta,t}$, then

$$
\begin{aligned}
&E[S^{\frac{p}{2}}_{\epsilon,\delta,t}] = E[e^{\frac{p}{2}X_{\epsilon,\delta,t}}] \\
&\qquad \le (EM_t)^{1/2}\left(E\left[\exp\left\{+pX_{\epsilon,\delta,0} + \int_0^t e^{-pX_{\epsilon,\delta,s}}A_{\epsilon,\delta}e^{pX_{\epsilon,\delta,s}}ds\right\}\right]\right)^{1/2}
\end{aligned} \tag{6.3}
$$

(by Hölder's inequality)

$$
\le 1\cdot e^{\frac{1}{2}px_0}\left(E\left[\exp\left\{\int_0^t e^{-pX_{\epsilon,\delta,s}}A_{\epsilon,\delta}e^{pX_{\epsilon,\delta,s}}ds\right\}\right]\right)^{1/2}.
$$

We simplify and bound the right hand side of the above inequality:

$$
\begin{aligned}
&E\left[\exp\left\{\int_0^t e^{-pX_{\epsilon,\delta,s}}A_{\epsilon,\delta}e^{pX_{\epsilon,\delta,s}}ds\right\}\right] \\
&\qquad = E\left[\exp\left\{\int_0^t \epsilon\Big((r-\frac{1}{2}\sigma^2(Y_{\epsilon,\delta,s}))p + \frac{1}{2}\sigma^2(Y_{\epsilon,\delta,s})p^2\Big)ds\right\}\right] \\
&\qquad = e^{\epsilon rpt}E\left[\exp\{\delta(p^2-p)\int_0^{\epsilon t/\delta}\sigma^2(Y_{\epsilon,\delta,\frac{\delta}{\epsilon}u})du\}\right]
\end{aligned}
$$

(by change of variable $u = \frac{\epsilon}{\delta}s$; recall that $\delta = \epsilon^2$ or ϵ^4)

$$
= e^{\epsilon rpt}E\left[\exp\{\delta(p^2-p)\int_0^{\epsilon t/\delta}\sigma^2(Y_u)du\}\right]
$$

where Y_u is the process with generator B given in (1.2). By convexity of exponential functions we get

$$
\begin{aligned}
&E\left[\exp\left\{\int_0^t e^{-pX_{\epsilon,\delta,s}}A_{\epsilon,\delta}e^{pX_{\epsilon,\delta,s}}ds\right\}\right] \\
&\qquad \le e^{\epsilon rpt}E\left[\frac{\delta}{t\epsilon}\int_0^{\epsilon t/\delta}\exp\{t\epsilon(p^2-p)\sigma^2(Y_u)\}du\right].
\end{aligned} \tag{6.4}
$$

Since $\delta = \epsilon^2$ or ϵ^4, $\epsilon/\delta \to \infty$ as $\epsilon \to 0$. Therefore, by the ergodicity of Y and $\exp\{t(p^2-p)\sigma^2(y)\} \in L^1(d\pi)$ (this follows from an argument similar to proof of Lemma 5.4; note that $\sigma < 1-\beta$ by Assumption 1.1.3), the right hand side of the above inequality (6.4) is uniformly bounded for all $\epsilon > 0$. Putting this together with (6.3) we get (6.2). ☐

6.3. *Implied volatilities.*

Proof of Theorem 2.2. Recall that $X_{\epsilon,t} = \log S_{\epsilon,t}$ and $x_0 = \log S_0$. Note that we have dropped the subscript δ in the notation and the dependence on $\delta = \epsilon^4$ or ϵ^2 should be understood by context. Our first step is to show that

$$\lim_{\epsilon \to 0^+} \sigma_\epsilon(t, \log K, x_0)\sqrt{\epsilon t} = 0. \tag{6.5}$$

Once we have shown this, the rest of the proof is identical to that of Corollary 1.4 in [11].

By definition of implied volatility

$$\begin{aligned} E[(S_{\epsilon,t} - K)^+] =& e^{r\epsilon t} S_0 \Phi\left(\frac{x_0 - \log K + r\epsilon t + \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon \sqrt{\epsilon t}}\right) \\ & - K\Phi\left(\frac{x_0 - \log K + r\epsilon t - \frac{1}{2}\sigma_\epsilon^2 \epsilon t}{\sigma_\epsilon \sqrt{\epsilon t}}\right) \end{aligned} \tag{6.6}$$

where Φ is the Gaussian cumulative distribution function. Let $l \geq 0$ be the limit of $\sigma_\epsilon\sqrt{\epsilon t}$ along a converging subsequence. If $\lim_{\epsilon \to 0^+}$ of the left-hand-side of (6.6) is 0, then l satisfies

$$S_0 \Phi\left(\frac{x_0 - \log K}{l} + \frac{l}{2}\right) - K\Phi\left(\frac{x_0 - \log K}{l} - \frac{l}{2}\right) = 0.$$

The only solution of the above equation is $l = 0$ and thus we get (6.5).

We therefore need to prove

$$\lim_{\epsilon \to 0^+} E\left[(S_{\epsilon,t} - K)^+\right] = 0. \tag{6.7}$$

By (1.5a) we have

$$S_{\epsilon,t} - K = S_0 - K + \epsilon \int_0^t r S_{\epsilon,t} dt + \sqrt{\epsilon} \int_0^t S_{\epsilon,t} \sigma(Y_{\epsilon,t}) dW_t^{(1)}.$$

It can be verified that $E[(S_{\epsilon,t} - K) - (S_0 - K)]^2 \to 0$, as $\epsilon \to 0$, for both cases $\delta = \epsilon^4$ and $\delta = \epsilon^2$. Therefore

$$\lim_{\epsilon \to 0^+} E[(S_{\epsilon,t} - K)^+] = E[(S_0 - K)^+] = 0$$

as $S_0 < K$ (this is an out-of-the-money call option).

□

In the following appendix, we collect some material regarding 1-D diffusions Y and technical but elementary estimates.

APPENDIX A: POSITIVITY OF THE Y PROCESS

In this section we prove positivity of the Y process when $\frac{1}{2} < \beta < 1$ in (1.1b). Assume $m > 0$ and $Y_0 > 0$. Recall the scale function $s(y)$ defined in the introduction and let $S(y) = \int_1^y s(y)dy$. By Lemma 6.1(ii) in Karlin and Taylor [23], to prove that Y_t remains positive a.s. for all $t \geq 0$, it is sufficient to show that

$$\lim_{\epsilon \to 0^+} S(\epsilon) = -\infty.$$

For $0 < \epsilon \ll 1$,

$$-S(\epsilon) = \int_\epsilon^1 s(y)dy = \int_\epsilon^1 \exp\left\{-\int_1^y \frac{2(m-z)}{\nu^2|z|^{2\beta}}dz\right\}dy$$
$$= C\int_\epsilon^1 \exp\left\{\frac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}dy$$

(where C is a positive constant and $2\beta - 1, 1-\beta > 0$)

$$= \int_{2\epsilon}^1 (\textit{positive integrand})dy + C\int_\epsilon^{2\epsilon} \exp\left\{\frac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}dy$$
$$\geq C\epsilon \exp\left\{\frac{2m}{\nu^2(2\beta-1)(2\epsilon)^{2\beta-1}}\right\} \to +\infty$$

as $\epsilon \to 0^+$, provided $m > 0$. Therefore $\lim_{\epsilon\to 0^+} S(\epsilon) = -\infty$.

APPENDIX B: GROWTH ESTIMATES FOR SOLUTIONS TO POISSON EQUATIONS

Assume χ satisfies the Poisson equation

$$B\chi(y) = \frac{1}{2}|p|^2(\bar{\sigma}^2 - \sigma^2(y)),$$

where $\overline{\sigma}^2$, defined in (3.7), is the average of $\sigma^2(y)$ with respect to the invariant distribution $\pi(dy)$, given in (1.4), of the Y process. In this section we find growth estimates for χ.

The right hand side of the above Poisson equation is centered with respect to the invariant distribution $\pi(dy) = \frac{m(y)}{Z}dy$ (given in (1.4)) and so

$$\int_0^\infty m(z)(\bar{\sigma}^2 - \sigma^2(z))dz = 0; \tag{B.1}$$

where

$$m(y) = \frac{1}{\nu^2 y^{2\beta}} \exp\left\{\int_1^y \frac{2(m-z)}{\nu^2 z^{2\beta}}dz\right\}.$$

By (1.3),

$$\begin{aligned}\chi(y) &:= \int dS(y)\int_0^y |p|^2(\bar{\sigma}^2-\sigma^2(z))dM(z)\\ &= \int \frac{1}{y^{2\beta}m(y)}\left[\int^y \frac{|p|^2 m(z)(\bar{\sigma}^2-\sigma^2(z))}{\nu^2}dz\right]dy\end{aligned}$$

is a solution up to a constant, and so

$$\begin{aligned}\chi'(y) &= \frac{|p|^2}{\nu^2 y^{2\beta} m(y)}\left[\int_0^y m(z)(\bar{\sigma}^2-\sigma^2(z))dz\right]\\ &= -\frac{|p|^2}{\nu^2 y^{2\beta} m(y)}\left[\int_y^\infty m(z)(\bar{\sigma}^2-\sigma^2(z))dz\right].\end{aligned}$$

The last equality is by the centering condition (B.1). Given the bounds on $\sigma(y)$ in Assumption 1.1.3, we can compute the following bounds where the constants, denoted by c, are positive and vary from line to line.

$$\begin{aligned}|\chi'(y)| &\le \frac{c|p|^2}{\nu^2 y^{2\beta} m(y)}\int_y^\infty z^{2\sigma} m(z)dz\\ &= \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}}}\int_y^\infty z^{2\sigma-2\beta}e^{-\alpha z^{1-2\beta}}e^{-\frac{z^{2-2\beta}}{\nu^2(1-\beta)}}dz\end{aligned}$$

where $\alpha = \frac{2m}{\nu^2(2\beta-1)} > 0$. Bounding $e^{-\alpha z^{1-2\beta}}$ above by 1 we get

$$\begin{aligned}|\chi'(y)| &\le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}}}\int_y^\infty z^{2\sigma-2\beta}e^{-\frac{z^{2-2\beta}}{\nu^2(1-\beta)}}dz\\ &= \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}}}\int_{y^{2-2\beta}}^\infty u^{\frac{2\sigma-1}{2-2\beta}}\exp\left\{-\frac{u}{\nu^2(1-\beta)}\right\}du\end{aligned}$$

(by change of variable $u = z^{2-2\beta}$)

$$\le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2 e^{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}}}\left[y^{2\sigma-1}\exp\left\{-\frac{y^{2-2\beta}}{\nu^2(1-\beta)}\right\}\right].$$

In the last inequality we used $\int_a^\infty \left[u^{\frac{2\sigma-1}{2-2\beta}}e^{-\frac{u}{\nu^2(1-\beta)}}\right]du \le \nu^2(1-\beta)a^{\frac{2\sigma-1}{2-2\beta}}e^{-\frac{a}{\nu^2(1-\beta)}}$ (since $\frac{2\sigma-1}{2-2\beta} < 0$). Therefore

$$|\chi'(y)| \le \frac{c|p|^2 e^{\alpha y^{1-2\beta}}}{\nu^2}y^{2\sigma-1} \sim c|p|^2 y^{2\sigma-1} \quad \text{as } y\to\infty,$$

since $e^{\alpha y^{1-2\beta}} \sim O(1)$ as $y \to \infty$.

APPENDIX C: Y^P PROCESS

Fix $p \in \mathbb{R}$. Denote $\mu_p(y) := (m-y)+\rho p\sigma(y)\nu y^\beta$ and let Y^p be the process with generator

$$B^p g = \mu_p(y)\partial_y g + \frac{1}{2}\nu^2 y^{2\beta}\partial^2_{yy} g, \qquad g \in C^2_c(E_0).$$

In this section we calculate the unique stationary distribution and Dirichlet form of the process Y^p and we show that it is a reversible process. To this end, we first compute the scale function and speed measure.

The scale function and speed measure for the Y^p process are given by:

$$s_p(y) = \exp\left\{-\int_1^y \frac{2\mu_p(z)}{\nu^2 z^{2\beta}}\right\} \quad \text{and} \quad m_p(y) = \frac{2}{\nu^2 y^{2\beta} s_p(y)}.$$

Evaluating the integral in $s_p(y)$ we get (the C below denotes a positive constant that varies from line to line):

$$s_p(y) = \begin{cases} C\exp\left\{-\frac{2m\log y}{\nu^2} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)} - \frac{2\rho p}{\nu}J\right\} & \text{if } \beta = \frac{1}{2} \\ C\exp\left\{\frac{2m}{\nu^2(2\beta-1)y^{2\beta-1}} + \frac{y^{2-2\beta}}{\nu^2(1-\beta)} - \frac{2\rho p}{\nu}J\right\} & \text{if } \beta \in 0 \cup (\frac{1}{2}, 1). \end{cases}$$

where

$$J(y) = \int^y \frac{\sigma(z)}{z^\beta} dz.$$

Due to bounds on σ given in Assumption 1.1.3, there exist $C_1, C_2 > 0$ such that

$$C_1 y^{1-\beta} \leq J(y) \leq C_2 y^{1-\beta+\sigma},$$

where $\begin{cases} 0 < 1-\beta \leq 1-\beta+\sigma \leq 1 & \text{if } \frac{1}{2} \leq \beta < 1 \\ 1 = 1-\beta \leq 1-\beta+\sigma < 2 & \text{if } \beta = 0 \end{cases}$. Therefore

$$\begin{cases} \frac{1}{s_p(y)} \to 0 \text{ when } y \to 0 \text{ or } y \to \infty & \text{if } \frac{1}{2} \leq \beta < 1 \\ \frac{1}{s_p(y)} \to 0 \text{ when } |y| \to \infty & \text{if } \beta = 0. \end{cases} \tag{C.1}$$

Define for $y \in E_0$,

$$\begin{aligned} S_p(y) :&= \int_1^y s_p(z) dz \\ &= C\int_1^y \exp\left\{\frac{2m}{\nu^2(2\beta-1)z^{2\beta-1}} + \frac{z^{2-2\beta}}{\nu^2(1-\beta)} - \frac{2\rho p}{\nu}J(z)\right\} dz \end{aligned}$$

where $C > 0$. Observe that $S_p(y) \to -\infty$ as y approaches the left end point of E_0 and $S_p(y) \to +\infty$ as $y \to \infty$.

C.1. Stationary distribution. Let π^p be an invariant distribution of the process Y^p. Suppose it has density function $\Psi(y)$, i.e. $d\pi^p(y) = \Psi(y)dy$, then Ψ is uniquely determined as the solution of

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\nu^2 y^{2\beta}\Psi(y)\right) - \frac{\partial}{\partial y}\left(\mu_p(y)\Psi(y)\right) = 0,$$

satisfying $\Psi(y) \geq 0$ for all y and $\int_{E_0}\Psi(y)dy = 1$. Solving the above differential equation, we get $\Psi(y) = m_p(y)\left[C_1 S_p(y) + C_2\right]$. Since Ψ is non-negative and $S_p(y) \to -\infty$ as y approaches the left boundary of E_0, we take $C_1 = 0$. The other constant C_2 is uniquely determined by the condition $\int_{E_0}\Psi(y)dy = 1$. Therefore π^p is the unique invariant distribution of Y^p and is given by

$$d\pi^p(y) = \Psi(y)dy = \frac{m_p(y)}{Z_1}dy = \frac{2}{Z_1\nu^2 y^{2\beta}s_p(y)}dy \quad \text{for } y \in E_0, \tag{C.2}$$

where $Z_1 = \int_{E_0} m_p(y)dy$.

C.2. Reversibility. Let $\varphi, \psi \in C_c^2(E_0)$, then

$$\begin{aligned}\int_{E_0}\psi B^p\varphi d\pi^p &= \frac{1}{Z_1}\int_{E_0}\psi\left[\frac{1}{2}\nu^2 y^{2\beta}\varphi'' + \mu_p(y)\varphi'\right]\frac{2}{\nu^2 y^{2\beta}s_p(y)}dy \\ &= \frac{1}{Z_1}\int_{E_0}\psi\left[\varphi'' e^{\int^y \frac{2\mu_p(y)}{\nu^2 y^{2\beta}}} + \frac{2\mu_p}{\nu^2 y^{2\beta}}\varphi' e^{\int^y \frac{2\mu_p(y)}{\nu^2 y^{2\beta}}}\right]dy \\ &= \frac{1}{Z_1}\int_{E_0}\psi\frac{d}{dy}\left(\frac{\varphi'}{s_p(y)}\right)dy.\end{aligned}$$

Integrating by parts twice and using the boundary conditions (C.1), we get

$$\int_{E_0}\psi B^p\varphi d\pi^p = \frac{1}{Z_1}\int_{E_0}\varphi\frac{d}{dy}\left(\frac{\psi'}{s_p(y)}\right)dy = \int_{E_0}\varphi B^p\psi d\pi^p.$$

C.3. Dirichlet form. By similar calculations as before, when proving reversibility, we get: for $f, g \in L^2(\pi^p)$,

$$\begin{aligned}\mathcal{E}^p(f,g) &:= -\int_{E_0} fB^p g d\pi^p \\ &= -\frac{1}{Z_1}\int_{E_0} f(y)\frac{d}{dy}\left(\frac{g'(y)}{s_p(y)}\right)dy \\ &= \frac{1}{Z_1}\int_{E_0} f'(y)g'(y)\frac{1}{s_p(y)}dy \\ &= \frac{\nu^2}{2}\int_{E_0} y^{2\beta}f'(y)g'(y)d\pi^p(y),\end{aligned}$$

where we integrated by parts once and used (C.1) in the second last line.

APPENDIX D: RATE FUNCTION FORMULAS

Recall the following characterization of the rate functions given in (6.1):

$$I_r(x;x_0,t) = \sup_{h\in C_b(\mathbb{R})}\{h(x) - u_0^{h,r}(t,x_0)\},$$

where $r = 2, 4$ correspond to the two regimes $\delta = \epsilon^2$ and $\delta = \epsilon^4$ respectively. The $u_0^{h,r}$ are given in (5.17) and (5.5) respectively as

$$u_0^{h,2}(t,x_0) = \sup_{x'\in\mathbb{R}}\left\{h(x') - t\bar{L}\left(\frac{x_0 - x'}{t}\right)\right\},$$
$$u_0^{h,4}(t,x_0) = \sup_{x'\in\mathbb{R}}\left\{h(x') - \left(\frac{|x_0 - x'|^2}{2\bar{\sigma}^2 t}\right)\right\}.$$

For notational convenience, we will drop the subscript r in I_r, and, in the case $r = 4$ we will denote the term $\left(\frac{|x_0-x'|^2}{2\bar{\sigma}^2 t}\right)$ by $t\bar{L}\left(\frac{x_0-x'}{t}\right)$. The rate functions can then be rewritten as

$$I(x;x_0,t) = \sup_{h\in C_b(\mathbb{R})}\inf_{x'\in\mathbb{R}}\left\{h(x) - h(x') + t\bar{L}\left(\frac{x_0 - x'}{t}\right)\right\}$$

for both regimes $r = 2$ and $r = 4$.

LEMMA D.1.

$$I(x;x_0,t) = t\bar{L}\left(\frac{x_0 - x}{t}\right).$$

PROOF. Note that for both cases $r = 2, 4$, $\bar{L}_0$ is convex, $\bar{L}_0(0) = 0$ and $\bar{L}_0$ is a non-negative function. This is obvious for the case $r = 4$. We can deduce this in the $r = 2$ case since $\bar{H}_0(p)$ (defined in (3.15)) is convex and $\bar{H}_0(0) = 0$.

Re-write

$$\begin{aligned} I(x;x_0,t) &= t\bar{L}_0\left(\frac{x_0 - x}{t}\right) + \sup_{h\in C_b(\mathbb{R})}\inf_{x'\in\mathbb{R}}\{h(x) - h(x') \\ &\qquad + t\bar{L}_0\left(\frac{x_0 - x'}{t}\right) - t\bar{L}_0\left(\frac{x_0 - x}{t}\right)\} \\ &= t\bar{L}_0\left(\frac{x_0 - x}{t}\right) + J \end{aligned}$$

where $J = \sup_{h \in C_b(\mathbb{R})} J_h$ and
$J_h = \inf_{x' \in \mathbb{R}} \left\{ h(x) - h(x') + t\bar{L}_0\left(\frac{x_0 - x'}{t}\right) - t\bar{L}_0\left(\frac{x_0 - x}{t}\right) \right\}$. Taking $x' = x$ in the inf we get $J_h \leq 0$ and therefore

$$J \leq 0. \tag{D.1}$$

Note that x_0 and x are fixed. Define a function $h^* \in C_b(\mathbb{R})$ as follows:

$$h^*(x') = t\bar{L}_0\left(\frac{x_0 - x'}{t}\right) \wedge t\bar{L}_0\left(\frac{x_0 - x}{t}\right).$$

Then

$$J_{h^*} = 0,$$

and consequently

$$J \geq 0. \tag{D.2}$$

By (D.1) and (D.2), $J = 0$ and we get

$$I(x; x_0, t) = t\bar{L}_0\left(\frac{x_0 - x}{t}\right).$$

□

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