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On equivalence of the Komargodski-Seiberg action to the Volkov-Akulov action

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Abstract

Equivalence between the Komargodski-Seiberg and the Volkov-Akulov supersymmetric nonlinear Lagrangians of the Nambu-Goldstone fermions is proved in all orders in their interaction constant. Exact expression for the KS fermionic field through the VA fermion is found.

1 Introduction

Recently Komargodski and Seiberg [1] have developed the $D = 4$ $\mathcal{N} = 1$ supercurrent approach to construct the low-energy effective Lagrangian of the Nambu-Goldstone fermions [2]. The KS approach is based on the use of connections between the linear and nonlinear realizations of supersymmetry and on the formalism of constrained superfields [3], [4], [5]. This stimulates to

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deeper understanding of the NG fermions role in the minimal standard supersymmetric model, their couplings with other particles, astroparticle physics and the problem of supersymmetry breaking (see e.g. [6], [7]).

In the recent paper [8] we reported a difference between the KS Lagrangian [1] and the well known Volkov-Akulov Lagrangian [2] which put forward the question about their equivalence. The difference is a consequence of the cancellation of 4-derivative terms in the VA Lagrangian earlier observed in [9]. To study this problem a simple alghorithmic procedure was developed in [8], and the equivalence between the KS and VA Lagrangians was proved up to the first order in the interaction constant of the NG fermions.

In the present paper we prove the equivalence in all orders in the interaction constant and derive an exact expression for the KS fermion through the VA fermion using this alghorithm.

2 On a common structure of the KS and the VA Lagrangians

The algorithmic procedure [8] is based on the idea of the NG field redefinition earlier applied in [4], and in [10], [9] while studying nonlinear realizations of supersymmetric electrodynamics (see there additional refs.) However, practical realizations of the procedure run against rather tedious calculations. It seems that the procedure [8] minimizes such type of calculations, because of the special structure of bispinors in the polynomial expansion of the Majorana bispinor, representing the KS fermion, through the original VA fermion. The use of this expansion allows to find an exact expression for the KS fermion in terms of VA fermion.

The Volkov-Akulov Lagrangian [2] (details may be found in [11])

$$\mathcal{L}_{VA} = \frac{1}{a} - \frac{i}{4}\bar{\psi}^{,m}\gamma_m\psi - \frac{a}{32}[(\bar{\psi}^{,m}\gamma_m\psi)^2 - (\bar{\psi}^{,n}\gamma_m\psi)(\bar{\psi}^{,m}\gamma_n\psi)] + \frac{a^2}{3!}\sum_p (-)^p T^m_m T^n_n T^l_l, \tag{1}$$

where the $\sum_p$ corresponds to the sum in all permutations of the subindices in the products of the tensors T^m_n

$$T^m_n = -\frac{i}{4}\bar{\psi}^{,m}\gamma_n\psi, \quad \bar{\psi}^{,m} := \partial^m\bar{\psi} \ . \tag{2}$$

The KS Lagrangian [1], presented in the bispinor Majorana representation up to the total derivative term, has the form

$$\mathcal{L}_{KS} = \frac{1}{a} - \frac{i}{4}\bar{g}^{,m}\gamma_m g - \frac{a}{16}[(\bar{g}^{,m}g)^2 + (\bar{g}^{,m}\gamma_5 g)^2]$$
$$-(\frac{a}{16})^3[(\bar{g}g)^2 + (\bar{g}\gamma_5 g)^2][(\bar{g}^{,m}g_{,m})^2 + (\bar{g}^{,m}\gamma_5 g_{,m})^2], \tag{3}$$

where $g := \sqrt{2}G$ and $a := -1/f^2$, and the algebraic agreements, relations connecting bilinear covariants in the Weyl and the Majorana representations are similar to those used in [11]. One can see that each term in the VA and the KS Lagrangians contains the fermions and their derivatives only in the form of the powers $(\partial\bar{\psi}\psi)^n$ and $(\partial\bar{g}g)^n$, respectively. These combinations have various tensorial structures, but their dimensions are equal to L^{-4n}, and are inverse to the dimensions of a^n. This observation hints to seek for the expression of g_a through ψ_a just in the degrees $\partial\bar{\psi}\psi$. The conjecture results in the general polynomial [8] in the VA interaction constant a

$$g = \psi + a\chi + a^2\chi_2 + a^3\chi_3. \tag{4}$$

The maximal degree of the polynomial (4) is less than four. Actually, since the products $a^n\chi_n$ must have the same dimension as ψ, the Grassmannian Majorana bispinors χ, χ_2, χ_3 may be constructed as ψ multiplied by the above mentioned powers $(\partial\bar{\psi}\psi)^n$, i.e. they have the form $\chi_n \sim \psi(\partial\bar{\psi}\psi)^n$. These monomials are nilpotent and their maximal degree $n = 3$, because $\psi(\partial\bar{\psi}\psi)^3$ contains the maximal power of the Grassmannian bispinor ψ equal to four in the discussed case $D = 4$ $\mathcal{N} = 1$. The conjecture on the structure of the χ-bispinors (4) opens a straightforward way to the construction of the redefined KS fermion g through the VA fermion ψ, as it has been manifested in [8]. Below we present the proof of the equivalence.

3 Exact equivalence between the KS and the VA Lagrangians

Let us start the outlined proof of the equivalence between the KS and the VA Lagrangians. The substitution of the expansion (4) in the KS Lagrangian (3) and putting it equal to the VA Lagrangian (1) yields the equations which define the bispinors χ, χ_2 and χ_3. Thus, the proof of the equivalence of the

Lagrangians is reduced to the solutions of these equations. The comparison of the terms of the same degree with respect to the constant a in the redefined KS and the original VA Lagrangians, provides the algorithmic way [8] to generate these equations. For instance, we have observed that the spinors χ_2 and χ_3 do not contribute to the terms linear in a in the redefined L_{KS}, and we obtained the equation defining the spinor χ. Then we are ready to consider the quadratic terms generating the equation for χ_2 and so on. Below we consider the procedure in more detail.

The substitution of (4) into (3) and omission of the total derivative terms yield the redefined KS kinetic term $\mathcal{K}_0$

$$\mathcal{K}_0 := -\frac{i}{4}\bar{g}^{,m}\gamma_m g = -\frac{i}{4}\bar{\psi}^{,m}\gamma_m\psi - \frac{i}{2}a(\bar{\psi}^{,m}\gamma_m\chi) - \frac{i}{2}a^2[(\bar{\psi}^{,m}\gamma_m\chi_2) + \frac{1}{2}(\bar{\chi}^{,m}\gamma_m\chi)] - \frac{i}{2}a^3[(\bar{\psi}^{,m}\gamma_m\chi_3) + (\bar{\chi}^{,m}\gamma_m\chi_2)]. \tag{5}$$

The next term $\mathcal{K}_1$ from (3), linear in a and quartic in fermionic fields, takes the form

$$\mathcal{K}_1 := -\frac{1}{16}a[(\bar{g}^{,m}g)^2 + (\bar{g}^{,m}\gamma_5 g)^2] = -\frac{1}{16}a[(\bar{\psi}^{,m}\psi)^2 + (\bar{\psi}^{,m}\gamma_5\psi)^2] - \frac{1}{16}a^3[(\bar{\psi}^{,m}\psi)(\bar{\chi}_{,m}\chi) + (\bar{\psi}^{,m}\gamma_5\psi)(\bar{\chi}^{,m}\gamma_5\chi)] \tag{6}$$

without the terms quadratic in a, because of the relation

$$(\bar{g}^{,m}g) = (\bar{\psi}^{,m}\psi) + a^2(\bar{\chi}^{,m}\chi) \tag{7}$$

and exclusion of the total derivate terms which arise from the relations $\bar{\psi}^{,m}\chi_N + \bar{\chi}_N^{,m}\psi = \partial^m(\bar{\chi}_N\psi)$. Thus, the redefined term $\mathcal{K}_1$ generates only linear and cubic in a contributions to the VA Lagrangian. The cubic in a term $\mathcal{K}_3$ gives a non-zero contribution only for zero term from the expansion (4) $g = \psi$

$$\mathcal{K}_3 = -\frac{1}{16}a^3[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2][(\bar{\psi}^{,m}\psi_{,m})^2 + (\bar{\psi}^{,m}\gamma_5\psi_{,m})^2]. \tag{8}$$

This is due to the fact that the contributions produced by χ_2 and χ_3 contain ψ^5 and higher identically vanishing factors constructed from ψ and $\bar{\psi}$.

Thus, we obtain the redefined KS Lagrangian (3) presented by the sum of (5), (6) and (8)

$$\mathcal{L}_{KS} = \frac{1}{a} + \mathcal{K}_0 + \mathcal{K}_1 + \mathcal{K}_3. \tag{9}$$

Matching (9) with the VA Lagrangian (1) generates the desired equations for the χ, χ_2, χ_3 from (4).

The equation for the spinor χ, obtained in [8],

$$i(\bar{\psi}^{,m}\gamma_m\chi) = -\frac{1}{8}[(\bar{\psi}^{,m}\psi)^2 + (\bar{\psi}^{,m}\gamma_5\psi)^2] + \frac{1}{16}[(\bar{\psi}^{,m}\gamma_m\psi)^2 - (\bar{\psi}^{,n}\gamma_m\psi)(\bar{\psi}^{,m}\gamma_n\psi)], \tag{10}$$

was simplified by cancellation of the common divisor $\bar{\psi}^{,m}$ to

$$\gamma_m\chi = \frac{i}{8}[\psi(\bar{\psi}_{,m}\psi) + \gamma_5\psi(\bar{\psi}_{,m}\gamma_5\psi)] - \frac{i}{16}[\gamma_m\psi(\bar{\psi}^{,n}\gamma_n\psi) - \gamma_n\psi(\bar{\psi}^{,n}\gamma_m\psi)]. \tag{11}$$

Multiplication of Eq. (13) by γ^m resulted in the general solution

$$\chi = -\frac{i}{32}[(\gamma_m\psi)(\bar{\psi}^{,m}\psi) + (\gamma_m\gamma_5\psi)(\bar{\psi}^{,m}\gamma_5\psi)] - \frac{i}{64}[3\psi(\bar{\psi}^{,m}\gamma_m\psi) + (\Sigma_{mn}\psi)(\bar{\psi}^{,n}\gamma^m\psi)]. \tag{12}$$

We repeated here the derivation of the solution for χ, because the same procedure will be applied to obtain general solutions of the equations for the spinors χ_2 and χ_3.

The equation for χ_2 is derived by matching the terms quadratic in a from (9) and (1)

$$(\bar{\psi}^{,m}\gamma_m\chi_2) = -\frac{1}{2}(\bar{\chi}^{,l}\gamma_l\chi) + \frac{1}{3!2}\sum_p(-)^p(\bar{\psi}^{,m}\gamma_m\psi)T_n^nT_l^l, \tag{13}$$

where the $\sum_p$ corresponds to the sum in all the permutations of the subindices in the products of the tensors T_n^n, T_l^l, $(\bar{\psi}^{,m}\gamma_m\psi)$. This equation defines χ_2 through the VA fermion ψ and the solution (12) for χ.

The terms cubic in a from (9) have to be mutually cancelled, because such terms are absent in the VA Lagrangian (1). It gives the third equation which defines the remaining spinor χ_2

$$(\bar{\psi}^{,m}\gamma_m\chi_3) = -(\bar{\chi}^{,l}\gamma_l\chi_2) + \frac{i}{4}[(\bar{\psi}^{,m}\psi)(\bar{\chi}_{,m}\chi) + (\bar{\psi}^{,m}\gamma_5\psi)(\bar{\chi}_{,m}\gamma_5\chi)] + \frac{2i}{16^3}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2][(\bar{\psi}^{,m}\psi_{,m})^2 + (\bar{\psi}^{,m}\gamma_5\psi_{,m})^2]. \tag{14}$$

This equation defines χ_3 through the solution (12) for χ and unknown yet spinor χ_2. To solve Eq. (13) for χ_2 we observe that all its terms, besides the term $(\bar{\chi}^{,l}\gamma_l\chi)$, contain the common divisor $\bar{\psi}^{,m}$. However, the divisor $\bar{\psi}^{,m}$ is hidden in $(\bar{\chi}^{,l}\gamma_l\chi)$ as it follows from the solution (12) for χ. Then we factorize this divisor in (12) and present χ in the form

$$\chi_a = A_{lab}\bar{\psi}^{,lb}, \tag{15}$$

where the spinor matrix A_{lab}, carrying a 4- vector index l, has the form

$$A_{lab} := \frac{i}{32}[(\gamma_l\psi)_a\psi_b + (\gamma_l\gamma_5\psi)_a(\gamma_5\psi)_b] + \frac{i}{64}[3\psi_a(\gamma_l\psi)_b + (\Sigma_{nm}\psi)_a(\gamma^n\psi)_b]. \tag{16}$$

The substitution of (15) into Eq. (13) transforms it to the factorized form

$$(\bar{\psi}^{,m}\gamma_m\chi_2) = \frac{1}{2}\bar{\psi}^{,mb}(\bar{\chi}^{,l}\gamma_l A_m)_b + \frac{1}{3!2}\sum_p(-)^p(\bar{\psi}^{,m}\gamma_m\psi)T^n_nT^l_l \tag{17}$$

analogous with (10). After cancellation of the divisor $\bar{\psi}^{,m}$ of this equation we obtain solution for the bispinor χ_{2a}

$$\chi_{2a} = -\frac{1}{8}\gamma^{mb}_{a.}(\bar{\chi}^{,l}\gamma_l A_m)_b - \frac{1}{3!8}\sum_p(-)^p(\gamma^m\gamma_m\psi)_aT^n_nT^l_l. \tag{18}$$

The similar procedure may be repeated to solve Eq. (14). To avoid further complications let us present the term $(\bar{\chi}^{,l}\gamma_l\chi_2)$ in Eq. (14) as

$$(\bar{\chi}^{,l}\gamma_l\chi_2) = (\bar{\chi}^{,l}_2\gamma_l\chi) + \partial^l(\bar{\chi}\gamma_l\chi_2) \tag{19}$$

and omit the terms which have the form of total derivatives. This trick allows to rewrite the relation (14) in the factorized form

$$(\bar{\psi}^{,m}\gamma_m\chi_3) = \bar{\psi}^{,mb}(\bar{\chi}^{,l}_2\gamma_l A_m)_b + \frac{i}{4}[(\bar{\psi}^{,m}\psi)(\bar{\chi}_{,m}\chi) + (\bar{\psi}^{,m}\gamma_5\psi)(\bar{\chi}_{,m}\gamma_5\chi)] + \frac{2i}{16^3}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2][(\bar{\psi}^{,m}\psi_{,m})^2 + (\bar{\psi}^{,m}\gamma_5\psi_{,m})^2]. \tag{20}$$

Again, as in the previous cases, one can cancel the same divisor $\bar{\psi}^{,m}$ and get the general solution for the bispinor χ_3

$$\chi_{3a} = -\frac{1}{4}\gamma^{mb}_{a.}(\bar{\chi}_2^{,l}\gamma_l A_m)_b - \frac{i}{16}[(\gamma_m\psi)_a(\bar{\chi}^{,m}\chi) + (\gamma_m\gamma_5\psi)_a(\bar{\chi}^{,m}\gamma_5\chi)] - \frac{i}{2(16)^3}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2][(\gamma_m\psi^{,m})_a(\bar{\psi}^{,l}\psi_{,l}) + (\gamma_m\gamma_5\psi^{,m})_a(\bar{\psi}^{,l}\gamma_5\psi_{,l})]. \tag{21}$$

The comparison of the solutions (18) and (21) for the bispinors χ_{Ja} (with $J = 2, 3$) shows that each of them contains similar combinations $(\gamma^m\bar{\chi}_J^{,l}\gamma_l A_m)_a$. Using the definition (16) of the matrix A^m_{ab} allows to present this composite bispinor $(\gamma^m\bar{\chi}_J^{,l}\gamma_l A_m)_a$ in the explicit form

$$(\gamma^m\bar{\chi}_J^{,l}\gamma_l A_m)_a = -\frac{i}{32}[(\bar{\chi}_J^{,m}\psi)(\gamma_m\psi)_a - (\bar{\chi}_J^{,k}\Sigma_{km}\psi)(\gamma^m\psi)_a + (\bar{\chi}_J^{,m}\gamma_5\psi)(\gamma_m\gamma_5\psi)_a - (\bar{\chi}_J^{,k}\Sigma_{km}\gamma_5\psi)(\gamma^m\gamma_5\psi)_a] - \frac{i}{64}[12(\bar{\chi}_J^{,m}\gamma_m\psi)\psi_a - (\bar{\chi}_J^{,m}\gamma_m\Sigma_{kl}\psi)(\Sigma^{lk}\psi)_a]. \tag{22}$$

Thus, we find all sought-for Majorana bispinors $\chi,\ \chi_2, \chi_3$ from the relation (4) which expresses the KS fermion through the VA fermion

$$g_a := \sqrt{2}G_a = \psi_a + a(A_m\bar{\psi}^{,m})_a - \frac{a^2}{8}[\,(\gamma^m\bar{\chi}^{,l}\gamma_l A_m)_a + \frac{1}{3!}\sum_p(-)^p(\gamma^m\gamma_m\psi)_a T^n_n T^l_l\,] - \frac{a^3}{4}[\,(\gamma^m\bar{\chi}_2^{,l}\gamma_l A_m)_a + \frac{i}{4}[(\gamma_m\psi)_a(\bar{\chi}^{,m}\chi) + (\gamma_m\gamma_5\psi)_a(\bar{\chi}^{,m}\gamma_5\chi)] + \frac{2i}{(16)^3}[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2][(\gamma_m\psi^{,m})_a(\bar{\psi}^{,l}\psi_{,l}) + (\gamma_m\gamma_5\psi^{,m})_a(\bar{\psi}^{,l}\gamma_5\psi_{,l})]\,]. \tag{23}$$

The solution (23) is presented in a compactified form. When substituted into (23), the solutions (12), (18) for the Majorana bispinors χ, χ_2 together with the expressions for matrix T^m_n (2), A_m (16) and for the composite bispinors (22) yield the representation for the KS fermion G only through the VA fermion ψ and its derivatives.

4 Conclusion

The equivalence between the Komargodski-Seiberg and the Volkov-Akulov $D = 4$ $\mathcal{N} = 1$ supersymmetric actions for the Nambu-Goldstone fermions is proved. The representation for the KS fermionic field in terms of the VA fermion is found. This representation has a rather complicated form, but some simplifications may be achieved using the Fierz rearrengements.

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