# On Cycles in Random Graphs

Madhav P. Desai
Department of Electrical Engineering
Indian Institute of Technology
Mumbai, India

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**Abstract**

We consider the geometric random (GR) graph on the $d-$dimensional torus with the $L_\sigma$ distance measure ($1 \leq \sigma \leq \infty$). Our main result is an exact characterization of the probability that a particular labeled cycle exists in this random graph. For $\sigma = 2$ and $\sigma = \infty$, we use this characterization to derive a series which evaluates to the cycle probability. We thus obtain an exact formula for the expected number of Hamilton cycles in the random graph (when $\sigma = \infty$ and $\sigma = 2$). We also consider the adjacency matrix of the random graph and derive a recurrence relation for the expected values of the elementary symmetric functions evaluated on the eigenvalues (and thus the determinant) of the adjacency matrix, and a recurrence relation for the expected value of the permanent of the adjacency matrix. The cycle probability features prominently in these recurrence relations.

We calculate these quantities for geometric random graphs (in the $\sigma = 2$ and $\sigma = \infty$ case) with up to $20$ vertices, and compare them with the corresponding quantities for the Erdös-Rényi (ER) random graph with the same edge probabilities. The calculations indicate that the threshold for rapid growth in the number of Hamilton cycles (as well as that for rapid growth in the permanent of the adjacency matrix) in the GR graph is lower than in the ER graph. However, as the number of vertices $n$ increases, the difference between the GR and ER thresholds reduces, and in both cases, the threshold $\sim \log(n)/n$. Also, we observe that the expected determinant can take very large values. This throws some light on the question of the maximal determinant of symmetric $0/1$ matrices.

## 1 Overview

Consider the $d-$dimensional unit torus $\mathbf{T_d} = [0,1]^d$. For $0 < r \leq 1/2$, $1 \leq \sigma < \infty$, the geometric random (GR) graph $Q_n^{(\sigma,d)}(r)$ is defined as follows. The vertex set corresponds to $n$ points $X_n = \{x_1, x_2, \ldots x_n\}$ distributed uniformly and independently in $T_d$. The set of edges $E(Q_n^{(\sigma,d)}(r))$ is defined as

$$E(Q_n^{(\sigma,d)}(r)) = \{\{x_i, x_j\} : \| x_i - x_j \|_q \leq r\}$$

where $\| \, . \, \|_q$ is the $L_q$ norm. Then, $Q_n^{(\sigma,d)}(r)$ is a random graph. In this random graph model, the presence of an edge is not necessarily independent of the presence of other edges.

Another random graph model which has been very well studied is the Erdös-Rényi (ER) random graph, which is defined as follows. Given a number $p$, $0 < p \leq 1$, let $H(n,p)$ denote the graph which has the vertex set $\{1, 2, \ldots n\}$ and an edge set consisting of edges selected with probability $p$ (a particular edge $\{i, j\}$ is present with probability $p$ and the presence of each edge is independent of the presence of other edges). The ER random graph has been extensively studied. Specifically, the asymptotic behaviour (or evolution) of this random graph has received considerable attention [1, 2]. The most celebrated result of this type [1] can be summarized as follows: if $p = p(n) = (\log n + c_n)/n$, then the random graph $G_n$ is almost surely connected (as $n \to \infty$) if $c_n \to \infty$, and is almost surely disconnected if $c_n \to -\infty$). Similar *thresholds* exist for all monotone graph properties[1] [3].

The geometric random graph appears to exhibit similar asymptotic properties. In [10], a sharp threshold for connectivity has been exhibited for the geometric random graph on the unit square ($d = 2$ and $\sigma = 2$): if $r = r(n)$ and if $\pi r(n)^2 = (\log n + c_n)/n$ then the random geometric graph is almost surely connected if $c_n \to \infty$, and is almost surely disconnected if $c_n \to -\infty$. The existence of sharp thresholds for monotone properties in geometric random graphs has been demonstrated in [9]. The monograph [5] summarizes threshold characterizations of several connectivity related properties of the geometric random graph. Upper and lower bounds on the diameter of a geometric random graph in the unit ball have been derived in [4]. The mixing times of random walks in geometric random graphs have been characterized in [6]. The limiting distribution of the eigenvalues of the adjacency matrix of a random graph has been studied in [7], [8]. An asymptotic bound for the second largest eigenvalue of the adjacency matrix of a geometric random graph has been derived in [11]. Thus, there is a large body of work on the asymptotic properties of a geometric random graph.

In the finite case, one is interested in the exact formula for the appearance of a certain property in a geometric random graph. An example of such a characterization is an exact formula for the probability of connectivity of a geometric random graph on a 1-dimensional *unit cube* [12], and an exact formula for the probability of existence of a particular labeled subgraph in the geometric random graph constructed in the $d-$dimensional unit cube using the $L_\infty$ measure [13]. We will consider the finite case, and prove an exact characterization of the probability that a labeled cycle appears in the random graph $Q_n^{(\sigma,d)}(r)$ (valid for $1 \leq \sigma \leq \infty$, and for all $d \geq 1$). Using this characterization, we show that it is possible to get exact formulas and recurrences for the computation of quantities which are related to cycle probabilities. In particular, we obtain

1. an exact formula for the appearance of a particular labeled cycle in $Q_n^{(\sigma,d)}(r)$ for $\sigma = 2$ and for $\sigma = \infty$ (the calculation of the corresponding cycle probability for $H(n,p)$ is trivial, because the edges in $H(n,p)$ are independent of each

[1] A property P is said to be monotone if, given that it holds on a graph $G$, it also holds on $G + e$, where $e$ is an edge connecting two vertices in G.

other). This formula immediately yields an expression for the expected number of Hamilton cycles in the random graph.

2. a recurrence relation for the expected values of the elementary symmetric functions evaluated at the eigenvalues of the adjacency matrix (as a special case, the expected value of the determinant of the adjacency matrix) of $H(n,p)$ and $Q_n^{(\sigma,d)}(r)$.

3. a recurrence relation for the expected values of the permanent of the adjacency matrix of $H(n,p)$ and $Q_n^{(\sigma,d)}(r)$.

These formulas can be evaluated explicitly and provide concrete information about random graphs with a finite number of vertices. For example, we observe that cycles appear earlier in GR graphs than in the ER graph. Specifically, the edge-probability threshold at which the expected number of Hamilton cycles crosses $1$ is lower in the GR graph than in the ER graph. However, the difference between the two thresholds reduces as $n$ increases. A similar observation can be made about the expected value of the permanent. The expected value of the determinant can be very different in the GR and ER models, indicating that for particular values of edge probabilities, the distribution of graphs in the GR and ER models can be very different. Another interesting observation is that as the edge probability is varied between $0$ and $1$, the expected values of the determinants of the adjacency matrix can be quite large. In effect, these expected values provide us some useful information about the largest possible determinant of a symmetric $0/1$ matrix.

## 2 Preliminaries

We introduce some notation and summarize some well known results to be used in the subsequent sections.

We use $G_n$ to denote a random graph on $n$ vertices (in one of the models described above). Then $A_{G_n} = [a_{ij}(G_n)]$ is the adjacency matrix of $G_n$, which is a symmetric random matrix with $0/1$ entries (the entries of this matrix are correlated if $G_n$ is the GR random graph).

Let $\mathbf{R}$ and $\mathbf{C}$ represent the sets of real and complex numbers respectively, and let $\mathbf{R}^d$, $\mathbf{C}^d$ denote the $d-$dimensional spaces of real and complex d-tuples. The set of integers is represented by $\mathbf{Z}$, and $\mathbf{Z}^d$ is the subset of $\mathbf{R}^d$ consisting of $d-$tuples of integers. Elements of these spaces will be denoted by bold letters such as $\mathbf{x}, \mathbf{y}, \omega$. Each $\mathbf{x}$ in any of these spaces is a $d-$tuple $(x_1, x_2, \ldots x_d)$. We will use $\mathbf{1} \in \mathbf{Z}^d$ to denote the $d-$tuple with each of its entries being $1$. If $\mathbf{x} = (x_1, x_2, \ldots x_d)$ and $\mathbf{y} = (y_1, y_2 \ldots y_d)$ are two elements of these spaces, then the *inner product* $\mathbf{x}.\mathbf{y}$ is $\sum x_j y_j$. The $L_\sigma$ norm for these spaces defined in the usual way, and for $\mathbf{x}$, $\| \mathbf{x} \|_\sigma$ denotes the $L_\sigma$ norm of $\mathbf{x}$. If $S \subset \mathbf{R}^d$, then $\Xi_S$ is the indicator function of $S$, so that

$$\Xi_S(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases}$$

For an absolutely integrable function $f\ :\ \mathbf{R}^d \rightarrow \mathbf{R}$, the Fourier transform $\hat{f}\ :\ \mathbf{R}^d \rightarrow \mathbf{C}$ is defined as

$$\hat{f}(\omega) \;=\; \int_{\mathbf{x}\in\mathbf{R}^d} e^{-i\omega.\mathbf{x}}\, f(\mathbf{x})d\mu(\mathbf{x})$$

where $d\mu(\mathbf{x})$ is the volume element in $\mathbf{R}^d$ at $\mathbf{x}$. Further, if $f(\mathbf{x}) = f(-\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^d$, then $\hat{f}(\omega) = \hat{f}(-\omega)$ for all $\omega \in \mathbf{R}^d$, and $\hat{f}$ always takes on real values. If $f$ is an absolutely integrable function with bounded support, and we define

$$f_p(\mathbf{x}) \;=\; \sum_{\mathbf{u}\in\mathbf{Z}^d} f(\mathbf{x}-\mathbf{u}) \tag{1}$$

then $f_p$ is a well defined periodic function, that is,

$$f_p(\mathbf{x}+\mathbf{u}) \;=\; f_p(\mathbf{x}) \text{ for all } \mathbf{u}\in\mathbf{Z}^d \tag{2}$$

which can be expressed by a Fourier series of the form

$$f_p(\mathbf{x}) \;=\; \sum_{\mathbf{u}\in\mathbf{Z}^d} \hat{f}(2\pi\mathbf{u})\, e^{2\pi\ i\ \mathbf{1}.\mathbf{u}} \tag{3}$$

If $f,\ g : \mathbf{R}^d \rightarrow \mathbf{R}$ are two absolutely-integrable functions, the convolution $f * g$ is also absolutely-integrable and is defined as

$$(f*g)(\mathbf{x}) \;=\; \int_{\mathbf{u}\in\mathbf{R}^d} f(\mathbf{u})g(\mathbf{x}-\mathbf{u})\, d\mu(\mathbf{u}) \tag{4}$$

and the fourier transform of $f*g$ is $\hat{f}\hat{g}$.

For $r \geq 0$, The set

$$B_{d,\sigma,r}(\mathbf{u}) \;=\; \{\mathbf{x}\in\mathbf{R}^d\ :\ \parallel \mathbf{x}-\mathbf{u} \parallel_\sigma \leq r\} \tag{5}$$

is termed the $\sigma-$*ball* of radius $r$ in $\mathbf{R}^d$, centered at $\mathbf{u}$. The volume of $B_{d,\sigma,r}(\mathbf{u})$ is denoted by $V_{d,\sigma,r}$. Clearly,

$$V_{d,\infty,r} \;=\; (2r)^d \tag{6}$$

For $\sigma = 2$ [14]

$$V_{d,2,r} \;=\; \frac{\pi^{d/2}\ r^d}{\Gamma(1\ +\ d/2)} \tag{7}$$

where $\Gamma$ is the gamma function. The surface area of $B_{d,\sigma,r}(\mathbf{u})$ is denoted by $A_{d,\sigma,r}$, and it is easy to show that $A_{d,\infty,r} = 2d(2r)^{d-1}$ and that $A_{d,2,r} = dV_{d,2,r}/r$. In $Q_n^{(\sigma,d)}(r)$, let $\beta_{d,\sigma,r}$ be the probability that two vertices $i,j$ are connected. Clearly, if $0 \leq r \leq 1/2$, $\beta_{d,\sigma,r} = V_{d,\sigma,r}$.

The Bessel's function of the first kind [15] with parameter $\nu$ is denoted by $J_\nu$. The following result is well known:

$$\hat{\Xi}_{B_{d,2,r}(0)}(\omega) \;=\; (2\pi r)^{d/2}\, \frac{J_{d/2}(r \parallel \omega \parallel_2)}{\sqrt{\parallel \omega \parallel_2}} \tag{8}$$

## 3 The probability that a particular labeled cycle appears in $G_n$

A labeled cycle in $G_n$ of length $q \leq n$ is a sequence of vertices $\mathbf{y_1}, \mathbf{y_2}, \ldots \mathbf{y_q}$ such that $\{\mathbf{y_i}, \mathbf{y_{i+1}}\} \in E(G_n)$ for $i = 1, 2, \ldots q-1$, and $\{\mathbf{y_q}, \mathbf{y_1}\} \in E(G_n)$. Let $\Theta(G_n, q)$ denote the probability that this labeled cycle is present in $G_n$. In both the GR and ER graph, this probability does not depend on the particular labeled cycle whose existence is in question. Thus, when $G_n$ is either an ER or a GR graph,

$$\Theta(G_n, q) \; = \Theta(G_m, q), \;\; n, m \geq q. \tag{9}$$

When $G_n = H(n,p)$, $\Theta(G_n, q)$ can be calculated very easily. Let $n > 0$ and $1 < q \leq n$. If $G_n = H(n,p)$, then the existence of a $q-$cycle in $G_n$ implies the presence of $q$ edges if $q > 2$, and $q-1$ edges if $q = 2$. In the ER random graph $H(n,p)$, the presence of an edge is independent of the presence of the others. Thus,

$$\Theta(H(n,p), q) = \begin{cases} p & \text{if q = 2} \\ p^q & \text{if q > 2} \end{cases} \tag{10}$$

In the case of the geometric random graph $Q_n^{(\sigma,d)}(r)$, things are more complicated because the edges are not necessarily independent. Our main result is an exact characterization of $\Theta(Q_n^{(\sigma,d)}(r)$ for any $\sigma$, $d$.

**Theorem 1** *Let $0 < r \leq 1/2$, and $q > 1$. Then*

$$\Theta(Q_n^{(\sigma,d)}(r), q) \; = \; \begin{cases} \beta_{d,\sigma,r} & \text{if } q = 2 \\ \sum_{\mathbf{m} \in \mathbf{Z}^d} \; \hat{\Xi}^q_{B_{d,\sigma,r}(0)}(2\pi \mathbf{m}) & \text{if } q > 2 \end{cases} \tag{11}$$

**Proof:** Let $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_q$ be the $q > 1$ random points which form the labeled cycle of length $q$ (these points are uniformly distributed in $T_d$). Then, $\Theta(Q_n^{(\sigma,d)}(r), q)$ is equal to the probability that for $i = 1, 2, \ldots q-1$,

$$\| \, \mathbf{x}_i - \mathbf{x}_{i+1} \, \|_\sigma \leq \; r \tag{12}$$

and $\| \, \mathbf{x}_q - \mathbf{x}_1 \, \|_\sigma \leq \; r$. Clearly, if $q = 2$, then the required probability is just $\beta_{d,\sigma,r}$.

Assume that $q > 2$. We decompose $\Theta(Q_n^{(\sigma,d)}(r), q)$ as follows:

$$\begin{aligned} \Theta(Q_n^{(\sigma,d)}(r), q) \;\; &= \;\; \Pr(\| \, \mathbf{x}_i - \mathbf{x}_{i+1} \, \|_\sigma \leq r, \; i = 1, 2, \ldots q-1, \text{ and } \; \| \, \mathbf{x}_1 - \mathbf{x}_q \, \|_\sigma \leq r) \\ &= \;\; \Pr(\| \, \mathbf{x}_1 - \mathbf{x}_q \, \|_\sigma \leq r \; / \; \; \| \, \mathbf{x}_i - \mathbf{x}_{i+1} \, \|_\sigma \leq r, \; i = 1, 2, \ldots q-1) \\ & \quad\; \times \Pr(\| \, \mathbf{x}_i - \mathbf{x}_{i+1} \, \|_\sigma \leq r, \; i = 1, 2, \ldots q-1). \end{aligned} \tag{13}$$

Clearly, since we are looking at i.i.d. points on the unit torus $T_1$, the events $\| \, \mathbf{x}_1 - \mathbf{x}_2 \, \|_\sigma \leq r$, $\| \, \mathbf{x}_2 - \mathbf{x}_3 \, \|_\sigma \leq r$, $\ldots \| \, \mathbf{x}_{q-1} - \mathbf{x}_q \, \|_\sigma \leq r$ are independent of each other, and the probability of occurence of each is $\beta_{d,\sigma,r}$. Hence,

$$\Pr(\| \, x_i - x_{i+1} \, \|_\sigma \leq r, \; i = 1, 2, \ldots q-1) \; = \; \beta_{d,\sigma,r}^{q-1}. \tag{14}$$

Thus, we can write

$$\Theta(Q_n^{(\sigma,d)}(r), q) \;=\; A_{d,\sigma,q}(r) \times \beta_{d,\sigma,r}^{q-1} \tag{15}$$

where

$$A_{d,\sigma,q}(r) = \Pr(\| x_1 - x_q \|_\sigma \leq r \;/\; \| x_i - x_{i+1} \|_\sigma \leq r,\; i = 1, 2, \ldots q-1).$$

We can interpret $A_{d,\sigma,q}(r)$ in the following manner. Consider a random walk in $\mathbf{R}^d$ starting from the origin $\mathbf{w_1} = \mathbf{0}$. A point $\mathbf{u_1}$ is chosen uniformly in the ball $B_{d,\sigma,r}(\mathbf{0})$. The walk then moves to $\mathbf{w_2} = \mathbf{w_1} + \mathbf{u_1}$. Continuing in this manner, if the current point is $\mathbf{w_k}$, the walk moves to $\mathbf{w_{k+1}} = \mathbf{w_k} + \mathbf{u_k}$ where $\mathbf{u_k}$ is chosen uniformly in the ball $B_{d,\sigma,r}(\mathbf{0})$. Since all points $\mathbf{m} \in \mathbf{Z}^d$ map to the origin $\mathbf{0}$ in the unit torus,

$$A_{d,\sigma,q}(r) \;=\; \Pr\left(\, \mathbf{w_q} + \mathbf{m} \in B_{d,\sigma,r}(\mathbf{0}) \text{ for some } \mathrm{m} \in \mathbf{Z}^{\mathrm{d}}\right) \tag{16}$$

Each $\mathbf{u_i}$ is generated uniformly from $B_{d,\sigma,r}(\mathbf{0})$, and thus, the probability density function of each $\mathbf{u_i}$ is

$$p_u(\mathbf{x}) \;=\; \frac{\Xi_{B_{d,\sigma,r}(0)}(\mathbf{x})}{\beta_{d,\sigma,r}} \tag{17}$$

Then, the probability density function of $\mathbf{w_k}$ is the $k-1$ fold convolution

$$p_k(\mathbf{x}) \;=\; (p_u * p_u * \ldots * p_u)(\mathbf{x}) \tag{18}$$

and the Fourier transform of $p_k$ is

$$\hat{p}_k(\omega) \;=\; \left(\frac{\hat{\Xi}_{B_{d,\sigma,r}(\mathbf{0})}(\omega)}{\beta_{d,\sigma,r}}\right)^{k-1} \tag{19}$$

Define the periodic function $s_r(\mathbf{x})$ as follows

$$s_r(\mathbf{x}) \;=\; \sum_{\mathbf{m} \in \mathbf{Z}^d} \Xi_{B_{d,\sigma}(r)}(\mathbf{x} - \mathbf{m}) \tag{20}$$

Then, since $r \leq 1/2$,

$$A_{d,\sigma,q}(r) \;=\; \int_{\mathbf{x} \in \mathbf{R}^d} s_r(\mathbf{x}) p_q(\mathbf{x}) d\mu(\mathbf{x}) \tag{21}$$

The periodic function $s_r(\mathbf{x})$ has a Fourier series representation

$$s_r(\mathbf{x}) \;=\; \sum_{\mathbf{m} \in \mathbf{Z}^d} c_{\mathbf{m}} e^{2\pi\; i\; \mathbf{m}.\mathbf{x}} \tag{22}$$

with

$$c_{\mathbf{m}} \;=\; \hat{\Xi}_{B_{d,\sigma,r}(\mathbf{0})}(2\pi \mathbf{m}) \tag{23}$$

Thus,

$$A_{d,\sigma,q}(r) \;=\; \int_{\mathbf{x} \in \mathbf{R}^d} \sum_{\mathbf{m} \in \mathbf{Z}^d} c_{\mathbf{m}} p_q(\mathbf{x}) e^{2\pi i \mathbf{m}.\mathbf{x}} d\mu(\mathbf{x}) \tag{24}$$

Observe that $p_q(\mathbf{x}) = 0$ when $\| \mathbf{x} \|_\sigma > qr$. Thus the integral in Eq. (24) can be considered to be over a compact set, and the order of summation and integration can then be exchanged [16], and we can write

$$A_{d,\sigma,q}(r) = \sum_{\mathbf{m}\in\mathbf{Z}^d} \int_{\mathbf{x}\in\mathbf{R}^d} c_{\mathbf{m}} p_q(\mathbf{x}) e^{2\pi i \mathbf{m}.\mathbf{x}} d\mu(\mathbf{x}) \tag{25}$$

For any absolutely integrable $f : \mathbf{R}^d \to \mathbf{R}$, we have

$$\int_{\mathbf{x}\in\mathbf{R}^d} f(\mathbf{x}) d\mu(\mathbf{x}) = \hat{f}(\mathbf{0})$$

Also, by the frequency shift property, the Fourier transform of $f(x)e^{i\mathbf{a}.\mathbf{x}}$ is $\hat{f}(\omega - \mathbf{a})$. Using these facts, we obtain

$$A_{d,\sigma,q}(r) = \sum_{\mathbf{m}\in\mathbf{Z}^d} \hat{\Xi}_{B_{d,\sigma,r}(\mathbf{0})}(2\pi\mathbf{m})\, \hat{p}_q(-2\pi\mathbf{m}) \tag{26}$$

From Eq. (19) and Eq. (26), the theorem follows. □

Using Theorem 1, we can obtain series representations for $\Theta$ in terms of the Fourier transform $\hat{\Xi}_{B_{d,\sigma,r}(0)}(\omega)$. This Fourier transform is relatively easy to compute for $\sigma = \infty$ and for $\sigma = 2$.

**Corollary 1** *Let $n > 0$, $0 < r \leq 1/2$, and $1 < q \leq n$.*

$$\Theta(Q_n^{(\infty,d)}(r), q) = \begin{cases} (2r)^d & \text{if q} = 2 \\ (2r)^{dq} \left( 1 + 2\sum_{k=1}^{\infty} (\mathrm{sinc}(2\pi kr))^q \right)^d & \text{if q} > 2 \end{cases} \tag{27}$$

**Proof:** Since $\beta_{d,\infty,r} = (2r)^d$, the first part of Eq. (27) (for $q = 2$) follows from Theorem 1.

Assume that $q > 2$. Since we are using the $L_\infty$ norm, each of the $d$ projections of the points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_q$ must induce a cycle in $T_1$. Since the projections are independent of each other, it follows that

$$\Theta(Q_n^{(\infty,d)}(r), q) = \left(\Theta(Q_n^{(\infty,1)}(r), q)\right)^d. \tag{28}$$

It is easy to see that

$$\hat{\Xi}_{B_{1,\infty,r}(0)}(\omega) = 2r\,\mathrm{sinc}(\omega r) \tag{29}$$

Using Eq. (29) and Eq. (28) together with Theorem 1 we obtain the required expression (we have used $\mathrm{sinc}(x) = \mathrm{sinc}(-x)$ to rewrite the series). □

**Corollary 2** *Let $n > 0$, $d > 1$, and $1 < q \leq n$. Then*

$$\Theta(Q_n^{(2,d)}(r), q) = \begin{cases} V_{d,2,r} & \text{if q} = 2 \\ V_{d,2,r}^q + (2\pi r)^{dq/2} \sum_{k=1}^{\infty} \psi_d(k) \left( \frac{J_{d/2}(2\pi\, r\, \sqrt{k})}{\left(2\pi\, \sqrt{k}\right)^{1/2}} \right)^q & \text{if q} > 2 \end{cases} \tag{30}$$

*where $\psi_d(k)$ is the number of solutions $\mathbf{x} \in \mathbf{Z}^d$ to the equation $\| \mathbf{x} \|_2 = k$.*

**Proof:** The proof follows immediately from Eq. (8) and Theorem 1. □

**Remark:** In order to compute the series in Eq. (30), we need to evaluate the function $\psi_d(k)$. The following recurrence can be used:

$$\psi_1(k) \;=\; \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k \neq 0 \text{ and } k = m^2 \text{ for some } m \in \mathbf{Z} \\ 0 & \text{otherwise} \end{cases}$$

and if $d > 1$,

$$\psi_d(k) \;=\; \sum_{0 \leq m \leq \sqrt{k}} \psi_{d-1}(k - m^2)$$

## 4 The expected number of Hamilton cycles in $Q_n^{(2,d)}(r)$

The Hamilton cycle problem in geometric random graphs has been studied in [17], in which the authors show that the threshold for the existence of a Hamilton cycle in a geometric random graph (in the unit cube) is the same as that for 2-connectivity. The number of Hamilton cycles in a random graph[2] [18] also shows a sharp thresholding property.

Using $\Theta(G_n, n)$, we can directly get the expected number of Hamilton cycles in $G_n$. Denote the expected number of Hamilton cycles in the random graph $G_n$ by $\tau(G_n)$. For $n > 2$, the number of labeled Hamilton cycles in a complete graph on $n$ vertices is $(n-1)!/2$. It follows that, for $n > 2$,

$$\tau(G_n) \;=\; \Theta(G_n, n)\;(n-1)!/2 \tag{31}$$

because the probability of each such labeled cycle being present is $\Theta(G_n, n)$.

Consider the threshold for $G_n$ defined as the smallest edge-probability such that $\tau(G_n) \geq 1$. We can use Corollaries 1 and 2 to compute this threshold when $G_n = Q_n^{(2,d)}(r)$ and $G_n = Q_n^{\infty,d}(r)$, and contrast this threshold with that for the ER graph $H(n,p)$. In Figure 1, we show the thresholds obtained for $H(n,p)$ and $Q_n^{(2,\sigma)}(r)$. The computed threshold for the geometric random graph is lower than that for the ER graph. However, the difference between the two thresholds reduces as $n$ increases. Asymptotically, the threshold for the appearance of a Hamilton cycle seems to be similar in the GR graph and the ER random graph (this threshold is of the order $\log(n)/n$ [17]). An explanation for this is that as $n$ increases, the end points of a path of length $n$ become less correlated (recall the random walk argument used in the proof of Theorem 1), and thus, the probability of an edge between the end points of the path is close to the edge probability.

[2] The random graph model used in [18] starts with an empty graph on $n$ vertices, and produces a sequence of graphs by adding new edges with equal probability. A threshold is then a position in the sequence at which a property becomes true with high probabililty.

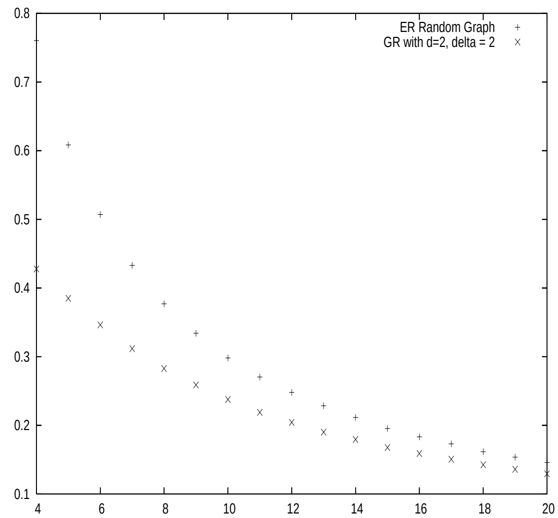


Figure 1: Threshold for $\tau(G_n) \geq 1$ plotted as a function of the $n$ for the ER graph and for the GR graph with $d = 2$, $\sigma = 2$

# 5 The expected value of the determinant and the permanent of $A_{G_n}$

Let $F_{G_n}(x)$ be the matrix $xI + A_{G_n}$. Define the two polynomials

$$\Lambda_{G_n}(x) \; = \; \det(F_{G_n}(x))), \tag{32}$$

and

$$\Gamma_{G_n}(x) \; = \; \mathrm{per}(F_{G_n}(x)). \tag{33}$$

The polynomials $\Lambda_{G_n}(x)$ and $\Gamma_{G_n}(x)$ have coefficients which are random variables. In particular, the coefficients in $\Lambda_{G_n}$ are symmetric functions of the eigenvalues of $A_{G_n}$. Define

$$\bar{\Lambda}_{G_n}(x) \; = \; E(\Lambda_{G_n}(x)) \tag{34}$$

and

$$\bar{\Gamma}_{G_n}(x) \; = \; E(\Gamma_{G_n}(x)) \tag{35}$$

where the expectation of a polynomial $p(x)$ is the polynomial $\bar{p}(x)$ whose coefficients are the expectations of the corresponding coefficients in $p(x)$.

The coefficient of $x^k$ in $\bar{\Lambda}_{G_n}(x)$ is the expected value of the elementary symmetric function of degree $n-k$ evaluated at the eigenvalues of $A_{G_n}$. In particular, the constant term in $\bar{\Lambda}_{G_n}(x)$ is the expected value of the determinant of $A_{G_n}$, so that the expected value of the determinant of $A_{G_n}$ is $\bar{\Lambda}_{G_n}(0)$. The coefficient of $x^k$ in $\bar{\Gamma}_{G_n}(x)$ is the expected number of cycle covers across all subgraphs of $G_n$ with $n - k$ vertices. Also, the expected value of the permanent of $G_n$ is $\bar{\Gamma}_{G_n}(0)$.

There is a strong connection between cycles and permutations, and between permutations and determinants (and permanents). We expect that the characterization of $\Theta(G_n, q)$ will help determine the behaviour of the determinant (and permanent). More concretely, we show that

**Theorem 2** *Let $G_n$ be a random graph on $n > 0$ vertices ($G_n$ is either the ER graph or the GR graph). Then, for $n \geq 1$, the polynomials $\bar{\Lambda}_{G_n}(x)$ and $\bar{\Gamma}_{G_n}(x)$ satisfy the recurrence relations*

$$\bar{\Lambda}_{G_n}(x) \;=\; x\bar{\Lambda}_{G_{n-1}}(x) \;+\; \sum_{q=2}^{n} (-1)^{q-1} \frac{n-1!}{n-q!} \;\Theta(G_n, q)\, \bar{\Lambda}_{G_{n-q}}(x) \tag{36}$$

*and*

$$\bar{\Gamma}_{G_n}(x) \;=\; x\bar{\Gamma}_{G_{n-1}}(x) \;+\; \sum_{q=2}^{n} \frac{n-1!}{n-q!}\, \Theta(G_n, q)\, \bar{\Gamma}_{G_{n-q}}(x) \tag{37}$$

*with initial conditions $\bar{\Lambda}_{G_0}(x) = \bar{\Gamma}_{G_0}(x) = 1$.*

**Proof:** We start with the following formulas for the determinant and the permanent. If $B = [b_{ij}]$ is an $n \times n$ matrix, then

$$\det(B) \;=\; \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n} b_{i\sigma(i)} \tag{38}$$

and

$$\mathrm{per}(B) \;=\; \sum_{\sigma \in S_n} \prod_{i=1}^{n} b_{i\sigma(i)} \tag{39}$$

where $S_n$ is the group of permutations of $\{1, 2, \ldots n\}$.

Each permutation $\sigma \in S_n$ can be uniquely decomposed into a set of disjoint cycles on $\{1, 2, \ldots n\}$. Each cycle $C$ in the disjoint cycle-decomposition of a permutation is of the form $(i_1 i_2 \ldots i_q)$, where $\sigma(i_r) \;=\; i_{r+1}, r = 1, 2, \ldots q-1$ and $\sigma(i_q) \;=\; i_1$. The sign of the cycle $C$ is $sign(C) = (-1)^{|C|-1}$, where $|C|$ is the number of elements in $C$. The sign of the permutation is then the product of signs of the cycles into which $\sigma$ is decomposed. We will say that the pair $(i, j) \in C$ if $i, j$ are consecutive elements in the cycle $C$ ($i_0$ is considered to be after $i_q$). Then, given $\sigma$, we have

$$\prod_{i=1}^{n} b_{i\sigma(i)} \;=\; \prod_{C \in \sigma} \prod_{(i,j) \in C} b_{ij} \tag{40}$$

For a cycle $C$, We define

$$w_B(C) \;=\; \prod_{(i,j) \in C} b_{ij} \tag{41}$$

Then,

$$\det(B) \;=\; \sum_{\sigma \in S_n} \prod_{C \in \sigma} (-1)^{|C|-1} w_B(C) \tag{42}$$

and

$$\mathrm{per}(B) \;=\; \sum_{\sigma\in S_n}\prod_{C\in\sigma} w_B(C). \tag{43}$$

Let $B = F_{G_n}(x)$. For a cycle $C = (i_1 i_2 \dots i_q)$ in some permutation, we see that if $q > 1$, then

$$E(w_B(C)) \;=\; \Theta(G_n, q) \tag{44}$$

and if $q = 1$, then

$$E(w_B(C)) \;=\; x. \tag{45}$$

For convenience, we set $\Theta(G_n, 1) = x$.

Also, if $C_1, C_2, \dots C_t$ are vertex-disjoint cycles in $G_n$, then the presence of $C_i$ is independent of the presence of $C_j$ for $j \neq i$, and

$$E(\prod_{i=1}^{t} w_B(C_i)) \;=\; \prod_{i=1}^{t} E(w_B(C_i)). \tag{46}$$

It follows that

$$\bar{\Lambda}_{G_n}(x) \;=\; \sum_{\sigma\in S_n}\prod_{C\in\sigma}(-1)^{|C|-1}\Theta(G_n, |C|). \tag{47}$$

Similarly,

$$\bar{\Gamma}_{G_n}(x) \;=\; \sum_{\sigma\in S_n}\prod_{C\in\sigma}\Theta(G_n, |C|) \tag{48}$$

The counting of permutations $\sigma \in S_n$ can be carried out by fixing the cycle $C$ which contains $1$ and counting permutations of elements not in $C$. For $1 \leq q \leq n$, Let $D_q$ be the set of cycles of length $q$ which contain $1$. We observe that

$$|D_q| \;=\; (q-1)! \begin{pmatrix} n-1 \\ q-1 \end{pmatrix},$$

because each cycle in $D_q$ is determined by the choice of $q-1$ elements (other than $1$) out of $n-1$ elements, and there are $q-1!$ distinct cycles on $q$ elements.

Let $\mathbf{N} = \{1, 2, \dots, n\}$ and let $P(A)$ be the set of permutations of the set $A \subset \mathbf{N}$. Then, we can write

$$\sum_{\sigma\in S_n}\prod_{C\in\sigma}(-1)^{|C|-1}\,\Theta(G_n, |C|) \tag{49}$$

as

$$\sum_{q=1}^{n}\left(\sum_{C\in D_q}(-1)^{|C|-1}\,\Theta(G_n, q)\left(\sum_{\sigma\in P(\mathbf{N}-C)}\prod_{D\in\sigma}(-1)^{|D|-1}\,\Theta(G_n, |D|)\right)\right) \tag{50}$$

where the innermost summation over $P(A)$ is taken to be $1$ if $A = \phi$. Since $|C| = q$ for each $C \in D_q$, we can rewrite Eq. (50) (using Eq. (9) to replace $\Theta(G_n, |D|)$ by $\Theta(G_{n-q}, |D|)$) as

$$\sum_{q=1}^{n}\begin{pmatrix} n-1 \\ q-1 \end{pmatrix}(q-1)!\,(-1)^{q-1}\,\Theta(G_n, q)\left(\sum_{\sigma\in P(\mathbf{N}-C)}\prod_{D\in\sigma}(-1)^{|D|-1}\,\Theta(G_{n-q}, |D|)\right). \tag{51}$$

The inner summation in Eq. (51) is just $\bar{\Lambda}_{G_{n-q}}(x)$, and thus, the recurrence relation for $\bar{\Lambda}_{G_n}(x)$ follows. The recurrence relation for $\bar{\Gamma}_{G_n}(x)$ can be shown to hold in a similar manner, completing the proof of Theorem 2. □

**Remark:** The result in Theorem 2 holds for any random graph $G_n$ in which the probability of appearance of a labeled cycle depends only on its length and the probability of appearance of a set of vertex-disjoint cycles is the product of probabilities of appearance of the elements in this set.

For $n > 0$, $0 < k \leq n$, let $F_{n,k}(t_1, t_2, \ldots t_n)$ denote the elementary symmetric function

$$F_{n,k}(t_1, t_2, \ldots t_n) \;=\; \sum_{\{i_1, i_2, \ldots i_k\} \in \{1,2,\ldots,n\}} t_{i_1} t_{i_2} \ldots t_{i_k} \tag{52}$$

For $k = 0$, define $F_{n,k} = 1$, and define $F_{n,k} = 0$ if $n < k$ or if $k < 0$. Now, let $\hat{F}_{n,k}$ denote the expected value of $F_{n,k}$ evaluated on the $n$ eigenvalues of $A_{G_n}$. Then, the expected value of the determinant of $A_{G_n}$ is just $\hat{F}_{n,n}$. Then, we have the following corollary of Theorem 2.

**Corollary 3** *For the random graph $G_n$, if $n > 0$, and $0 < k \leq n$, then*

$$\hat{F}_{n,k} \;=\; \hat{F}_{n-1,k} \;+\; \sum_{q=2}^{n} (-1)^q \frac{n-1!}{n-q!}\, \Theta(G_n, q)\, \hat{F}_{n-q,k-q} \tag{53}$$

**Proof:** Follows from Theorem 2 by noting that the coefficient of $x^k$ in $\bar{\Lambda}$ is $\hat{F}_{n,n-k}$. □

Note that in both models, if the edge probability is 1, then $\Theta(G_n, q) = 1$, and $G_n$ is always the complete graph, so that the expected value of the determinant of $G_n$ is $(-1)^{n-1} \times (n-1)$. Using Theorem 2, we obtain the following identity for $n > 0$:

$$n \;=\; 1 \;+\; \sum_{q=2}^{n} \frac{n-1!}{n-q!} \times ((n-q) - 1) \tag{54}$$

Also, the permanent of the complete graph on $n$ vertices is the number of derangements of the set $\mathbf{N} = \{1, 2, \ldots n\}$. Thus, the recurrence proved in Theorem 2 yields the following identity for the number of derangements $d_n$ of $\mathbf{N}$

$$d_n \;=\; \sum_{q=2}^{n} \frac{n-1!}{n-q!}\, d_{n-q}, \tag{55}$$

with the initial conditions $d_1 = 0$, and $d_0 = 1$.

We use these ecurrence relations to compute these expected values for $n \leq 20$ in the GR and ER models[3]. Some interesting conclusions can be drawn from these calculations.

[3]The recurrence relations were directly computed using *long double* precision arithmetic. For higher values of $n$ one would need to use higher precision arithmetic.

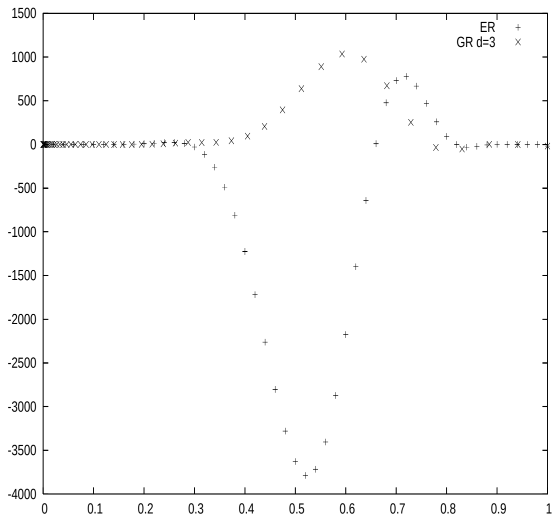


Figure 2: The expected value of the determinant plotted as a function of the edge probability for $n = 20$ in the ER and GR (with $d = 3$, $\sigma = \infty$) models.

Consider the plot in Figure 2, in which we compare the behaviour of the determinant of $G_{20}$ as a function of the edge probability. The graph has been plotted for the ER graph and for the GR graph with $d = 3$. The behaviour of the determinant in the two models is quite different, and clearly, so is the distribution of $G_n$.

In Figure 3, we show a plot of the expected value of the permanent of $A_{G_n}$ (for $n = 20$) as a function of the edge probability in the ER and GR ($d = 1$) models. We can also define a threshold for the expected permanent as the smallest edge probability for which the expected value of the permanent is $\geq 1$. A comparison of this threshold for the GR and ER graphs shows that this threshold is lower for the GR graph, but the two thresholds come closer as $n$ increases (see Figure 4). Thus, the permanent of the GR graph grows more rapidly than that of the ER graph. This is expected since a labeled cycle is more likely in the GR graph.

## 5.1 Graphs with large determinants

Looking at Figure 2, we see that for intermediate values of the edge probability, large magnitudes appear in the plots of the expected value of the determinant. For instance, we observe that, in the ER random graph with $n = 20$, the largest absolute value of the determinant is $3787.81$, and this provides a lower bound on the maximal determinant of a symmetric $20 \times 20$ $0/1$ matrix .

For a general (non-symmetric) $n \times n$ $0/1$ matrix, the determinant is bounded above by $(n+1)^{(n+1)/2}/2^n$ [19]. The number of (possibly non-symmetric) $n \times n$ $0/1$ matrices which achieve this bound is also known for $n \leq 9$ [20]. However, similar characterizations of the determinants of *symmetric* $0/1$ matrices are not so common. For example,

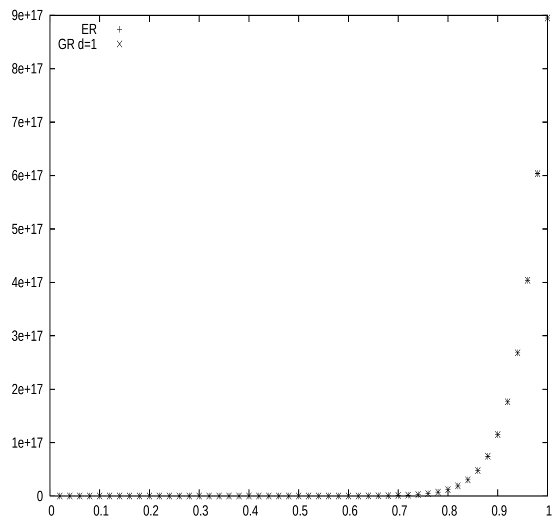


Figure 3: The expected value of the permanent plotted as a function of the edge probability for $n = 20$ in the ER and GR (with $d = 1$, $\sigma = \infty$) models.

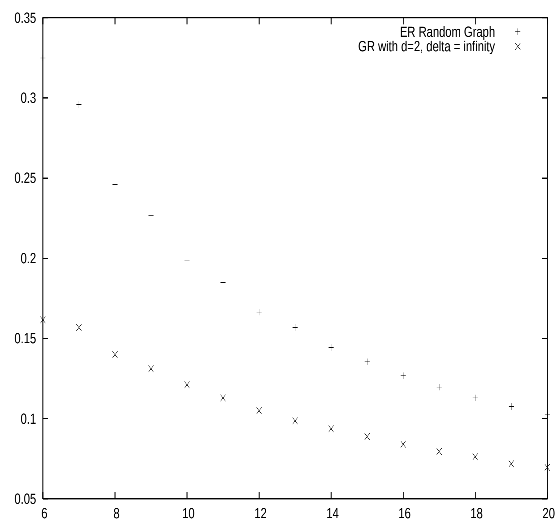


Figure 4: Threshold for the expected value of the permanent plotted as a function of the $n$ for the ER graph and for the GR graph with $d = 2$, $\sigma = \infty$

in [21], the authors show that for $n \geq 7$, the maximal determinant of the adjacency matrix of a $(n-3)$-regular graph on $n$ vertices is $(n-3)3^{[n/4]-1}$. For $n = 20$, this works out to be $1377$ which is less than the largest observed determinant value in the evolution of $H(20, p)$.

Thus, the recurrence formula for the expected value of the determinant seems to provide some useful information about the maximal determinant of a class of symmetric $0/1$ matrices (in effect, we have a lower bound on the largest value of such determinants). Also, if the expected determinant is large, then it may be possible to find a symmetric $0/1$ matrix with large determinant by using a Monte Carlo sampling approach. An estimate of the second moment of the determinant of the random graph will throw more light on this possibility.

# 6 Conclusions

We have derived an exact characterization of the probability of existence of a labeled cycle in geometric random graphs on a unit torus with an arbitrary number of dimensions, and with an arbitrary $L_\sigma$ distance metric). This cycle probability can be calculated in terms of the Fourier transform of the indicator function of a ball in $L_\sigma$. Explicit expressions for this Fourier transform can be easily computed in the $\sigma = \infty$ and $\sigma = 2$ case.

From the cycle probability, one gets the expected number of Hamilton cycles in the geometric random graph. These exact expressions complement the asymptotic threshold results for the existence of Hamilton cycles in geometric random graphs (as in [17]). We observe that as the edge probability increases, a Hamilton cycle appears earlier in the GR graph than in the ER graph.

The cycle probabilities can also be used to find the expected values of the determinant (and more generally, the expected values of the elementary symmetric functions evaluated at the eigenvalues of the adjacency matrix) and the permanent of the adjacency matrix of the random graph. We obtain recurrence relations for these quantities and illustrate them by a few calculations. In particular, the determinant exhibits very different behaviour in the two models. Also, large magnitudes of the determinant are observed in the evolution of the random graphs. This throws some light on the as yet unresolved question of the maximal determinant of symmetric 0/1 matrices.

# References


[1] P. Erdös, A. Rényi, “On Random Graphs I”, Publicaciones Mathematicae **6**, pp. 290-297.

[2] B. Bollobas, *Random Graphs*, Cambridge University Press, 2001.

[3] E. Friedgut, G. Kalai, “Every Monotone Graph Property has a Sharp Threshold”, *Proceedings of the American Mathematical Society* Vol. 124 (1996), pp. 2993-3002.

[4] R. Ellis, J. Martin, C. Yan, “Random Geometric Graph Diameter in the Unit Ball”, *Algorithmica* Vol. 47 (2007): pp. 421438.

[5] M.D. Penrose, *Random Geometric Graphs*, Oxford University Press, Oxford, (2003).

[6] C. Avin, G. Ercal, “On the cover time and mixing time of random geometric graphs”, Theoretical Computer Science, Vol. 380 (2007), pp. 2-22.

[7] P. Blackwell, M. Edmondson-Jones, Jonathan Jordan, “Spectra of adjacency matrices of random geometric graphs”, Preprint.

[8] S. Rai, “The Spectrum of a Random Geometric Graph is Concentrated”, *Journal of Theoretical Probability*, Vol. 20, No. 2, June 2007.

[9] A. Goel, S. Rai, Bhaskar Krishnamachari, “Monotone Properties of Random Geometric Graphs have Sharp Thresholds”, *Annals of Applied Probability* Vol. 15 (2005), No. 4, pp. 2535-2552.

[10] P. Gupta, P. R. Kumar, “The Capacity of Wireless Networks”, *IEEE Transactions on Information Theory* Vol. IT-46 (2000), No. 2, pp. 388-404.

[11] S. Boyd, A. Ghosh, B. Prabhakar, D. Shah, “Mixing Times for Random Walks on Geometric Random Graphs”, *SIAM Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, Vancouver, January 2005.

[12] M. Desai, D. Manjunath, “On the connectivity in finite ad-hoc networks”, *IEEE Communication Letters*, 6 (2002), pp. 437-439.

[13] M. Desai, D. Manjunath, “On Range Matrices and Wireless Networks in d-Dimensions”, *Proceedings of WIOPT 2005*, pp. 190-196. *IEEE Communication Letters*, 6 (2002), pp. 437-439.

[14] E.W. Weisstein, ”Unit Sphere,” From MathWorld–A Wolfram Web Resource (http://mathworld.wolfram.com/UnitSphere.html).

[15] E.W. Weisstein, ”Bessel Function of the First Kind.” From MathWorld–A Wolfram Web Resource (http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html).

[16] W.H. Young, “On the Integration of Fourier Series”, *Proc. London Math. Soc.* 9 (1911), pp. 449-462.

[17] J. Balogh, B. Bollobas, M. Krivelevich, M. Walters, T. Muller “Hamilton Cycles in Random Geometric Graphs,” *Preprint*, 2010.

[18] C. Cooper, A.M. Frieze, “On the Number of Hamilton Cycles in a Random Graph”, *J. Graph Theory*, Vol. 13, No.6, pp. 719-735 (1989).

[19] D.K. Faddeev, I.S. Sominskii, *Problems in Higher Algebra* (transl. by J.L. Brenner), W.H. Freeman and Co., San Francisco London (1965).

[20] M. Zivkovic, “Classification of Small (0,1) Matrices,” *Linear Algebra and its Applications*, 414 (2006), pp. 310-346.

[21] S. Fallat, P. Van Den Driessche, “Maximal Determinant of $(0,1)$ matrices with Certain Constant Row and Column Sums”, *Linear and Multilinear Algebra*, vol. 42, pp. 303-318.