

On Higher Derivatives as Constraints in Field Theory: a Geometric Perspective

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Abstract

We formalize geometrically the idea that the (de Donder) Hamiltonian formulation of a higher derivative Lagrangian field theory can be constructed understanding the latter as a first derivative theory subjected to constraints.

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1 Introduction

Let $\pi : E \rightarrow M$ be a fiber bundle, $\pi_l : J^l \rightarrow M$ its l -th jet bundle, $l = 0, 1, 2, \dots$, and $\mathcal{L} \in \Lambda^n(J^k)$ a basic n -form on J^k , $n = \dim M$, $k > 1$. \mathcal{L} may be interpreted as a Lagrangian density defining the k th derivative action functional $\mathcal{S} : s \mapsto \int_M (j_k s)^* \mathcal{L}$ on sections s of π . The associated calculus of variations, and, in particular, the Euler-Lagrange equations, have a nice geometric (and homological) formulation in terms of the so called \mathcal{C} -spectral sequence [1]. The Hamiltonian counterpart of the theory is very well established in the case $k = 1$. In particular, there are universally accepted field theoretic, geometric definitions of the Legendre transform and the Hamilton equations (see, for instance, [2] for a recent review). On the other hand, only recently a geometric

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formulation of the (Hamilton-like, higher derivative) de Donder field theory [3] has been proposed by the author which is natural, i.e. it is independent on any structure other than the action functional itself [4]. Such a formulation is based on a generalization to higher derivative Lagrangian field theory of the mixed Lagrangian-Hamiltonian formalism by Skinner and Rusk [5, 6, 7]. In such a theory the Legendre transform is not defined a priori but it is rather a consequence of the field equations.

Aldaya and de Azcarraga have suggested that higher derivative Hamiltonian field theory can be introduced understanding higher derivative Lagrangian field theory as a first order theory with (vakonomic) constraints [8]. However, they work in local coordinates and not all their conclusions have an intrinsic, geometric meaning. The aim of this short communication is to show that the idea by Aldaya and de Azcarraga can be given a precise, and natural, geometric formulation. In particular, momenta in higher derivative field theory can be mathematically understood as Lagrange multipliers in an equivalent first derivative theory subjected to (vakonomic) constraints.

2 The Constraint Bundle

We assume that the reader is familiar with Lagrangian and Hamiltonian formalisms on fiber bundles [2]. We refer to [4] and [9] for notations, conventions, and the basic differential geometric constructions we will use in the following.

Let \mathcal{L} be as in the introduction. It is well known that J^k is naturally embedded in $J^1\pi_{k-1}$, the first jet bundle of π_{k-1} . Denote by \dots, x^i, \dots coordinates on M , by \dots, u_I^α, \dots jet coordinates on J^k , ($I = i_1 \dots i_r$ being a multi-index denoting multiple differentiation of the field variables \dots, u^α, \dots , $i_1, \dots, i_r = 1, \dots, n$, $|I| := r \leq k$) and by $\dots, u_{J,i}^\alpha, \dots$ jet coordinates on $J^1\pi_{k-1}$, $|J| \leq k-1$. The embedding $e : J^k \hookrightarrow J^1\pi_{k-1}$ reads locally

$$e^*(u_{J,i}^\alpha) = u_{J,i}^\alpha, \quad |J| \leq k-1.$$

In particular $J^k \longrightarrow J^{k-1}$ is an affine subbundle of $J^1\pi_{k-1} \longrightarrow J^{k-1}$. \mathcal{L} can be understood as a first derivative Lagrangian density, say \mathcal{L}' , on the constraint subbundle J^k of $J^1\pi_{k-1}$. Sections σ of π_{k-1} satisfying the constraint, i.e. whose first jet prolongation $j_1\sigma$ takes values in $J^k \subset J^1\pi_{k-1}$, are precisely those of the form $\sigma = j_{k-1}s$ for some section s of π . In other words, considering J^k as a constraint subbundle of $J^1\pi_{k-1}$ is the same as introducing new variables corresponding to derivatives of the fields and then impose the obvious differential relations among them. Therefore, the variational problem defined by \mathcal{L}' is equivalent to the original one, and, in principle, we can apply the Lagrange multiplier method to find solutions. To do this, we should, first of all, 1) choose an extension of \mathcal{L}' to the whole $J^1\pi_{k-1}$ and 2) present $J^k \subset J^1\pi_{k-1}$ as the zero locus of a (sufficiently regular) morphism of the bundle $J^1\pi_{k-1} \longrightarrow J^{k-1}$, with values in a vector bundle $V \longrightarrow J^{k-1}$ [10]. Since neither 1) nor 2) can be done in a natural

way, we prefer to change a bit our strategy.

Instead of $J^1\pi_{k-1}$, consider $J^1\pi_k$, the first jet bundle of π_k . There is a natural projection $p : J^1\pi_k \rightarrow J^k$. Moreover, we can draw a diagram

$$\begin{array}{ccc} X_k & \hookrightarrow & J^1\pi_k \\ \downarrow & & \downarrow \\ J^k & \hookrightarrow & J^1\pi_{k-1} \end{array},$$

where $X_k := p^{-1}(J^k)$. We understand X_k as a constraint subbundle in $J^1\pi_k$. Notice that \mathcal{L} is naturally extended to $J^1\pi_k$ (and, in particular, X_k) as $p^*(\mathcal{L})$. Moreover, $X_k \subset J^1\pi_k$ can be presented as the zero locus of a morphism $\psi : J^1\pi_k \rightarrow V$ of the bundle $J^1\pi_k \rightarrow J^k$, with values in a vector bundle $V \rightarrow J^k$, as follows. Let $\theta_0 \in J^k$ and $\theta \in J^1\pi_k$ be a point over it, i.e., the projection $J^1\pi_k \rightarrow J^k$ sends θ to θ_0 . θ can be understood as an n -dimensional subspace $L(\theta)$ in $T_\theta J^k$ transversal to the fiber F of π_k through θ_0 (see, for instance, [11]), or, which is the same, as a linear map $\Pi(\theta) : T_\theta J^k \rightarrow T_{\theta_0} J^k$, with the following two properties: 1) $\Pi(\theta)$ is a projector, i.e., $\Pi(\theta) \circ \Pi(\theta) = \Pi(\theta)$, 2) $\ker \Pi(\theta) = T_\theta F$. Then $L(\theta) = \text{im } \Pi(\theta)$. If θ has jet coordinates \dots, u_I^α, \dots , $|I| \leq k$, then

$$\Pi(\theta) = \left(\frac{\partial}{\partial x^i} + \sum_{|I| \leq k} u_{I,i}^\alpha \frac{\partial}{\partial u_I^\alpha} \right) \otimes dx^i.$$

Now, there is a canonical geometric structure on J^k , the so called Cartan distribution [11]. The Cartan plane $\mathcal{C}(\theta_0) \subset T_{\theta_0} J^k$ at θ_0 can be described as the kernel of a canonical linear map $U(\theta_0) : T_{\theta_0} J^k \rightarrow T_{\bar{\theta}_0} J^{k-1}$, $\bar{\theta}_0 \in J^{k-1}$ being the image of θ_0 under the projection $J^k \rightarrow J^{k-1}$ [12]. In local coordinates $U(\theta_0)$ is given by

$$U(\theta_0) = \sum_{|I| \leq k-1} \frac{\partial}{\partial u_I^\alpha} \otimes (du_I^\alpha - u_{I,i}^\alpha dx^i).$$

We can also compose $\Pi(\theta)$ and $U(\theta_0)$, to check whether $L(\theta) \subset \mathcal{C}(\theta_0)$. In local coordinates

$$\begin{aligned} U(\theta_0) \circ \Pi(\theta) &= \sum_{|I| \leq k-1} (du_I^\alpha - u_{I,i}^\alpha dx^i) \left(\frac{\partial}{\partial x^j} + \sum_{|J| \leq k} u_{J,j}^\beta \frac{\partial}{\partial u_J^\beta} \right) \frac{\partial}{\partial u_I^\alpha} \otimes dx^j \\ &= \sum_{|I| \leq k-1} (u_{I,i}^\alpha - u_{I,i}^\alpha) \frac{\partial}{\partial u_I^\alpha} \otimes dx^i. \end{aligned} \quad (1)$$

We conclude that $L(\theta) \subset \mathcal{C}(\theta_0)$ iff $\dots, u_{I,i}^\alpha = u_{I,i}^\alpha, \dots$, $|I| \leq k-1$, i.e., $\theta \in X_k$.

In view of its coordinate expression, $U(\theta_0) \circ \Pi(\theta)$ can be understood as an element in $V_{\bar{\theta}_0} J^{k-1} \otimes T_x^* M$, where $V_{\bar{\theta}_0} J^{k-1} = \ker d_{\bar{\theta}_0} \pi_{k-1} \subset T_{\bar{\theta}_0} J^{k-1}$ is the π_{k-1} -vertical tangent

space to J^{k-1} at the point $\bar{\theta}_0$ and $x = \pi_k(\theta) \in M$. Therefore, the map $\theta \mapsto U(\theta_0) \circ \Pi(\theta)$ can be understood as an affine morphism $\psi : J^1\pi_k \rightarrow V$ of the bundle $J^1\pi_k \rightarrow J^k$, with values in the (pull-back) vector bundle

$$V := VJ^{k-1} \otimes_M T^*M \times_{J^{k-1}} J^k \rightarrow J^k$$

whose fiber over θ_0 is $V_{\bar{\theta}_0} J^{k-1} \otimes T_x^*M$. Formula (1) then shows that $\theta \in X_k$ iff $\psi(\theta) = 0$. Formula (1) also shows that ψ has fiber-wise maximal rank at the points of X_k , and in this sense, will be referred to as a *regular morphism* [10]. We have thus proved the following

Theorem 1 $X_k \subset J^1\pi_k$ is the zero locus of a canonical regular morphism of the affine bundle $J^1\pi_k \rightarrow J^k$ with values in a canonical vector bundle $V \rightarrow J^k$.

Notice that $VJ^{k-1} \otimes_M T^*M \rightarrow J^{k-1}$ is the model vector bundle for the affine bundle $J^1\pi_{k-1} \rightarrow J^{k-1}$.

Corollary 2 A smooth function $F \in C^\infty(J^1\pi_k)$ vanishes on the constraint subbundle X_k iff there exists a morphism $\lambda : J^1\pi_k \rightarrow V^*$, with values in the dual bundle, such that $\langle \lambda, \psi \rangle = 0$.

The above corollary shows that variables in the fiber of $V^* \rightarrow J^k$ basically play the role of Lagrange multipliers (see below for details).

3 Higher Derivatives as Constraints

Consider the first derivative action functional $\mathbf{S}' : \sigma \mapsto \int_M \sigma^* \mathcal{L}$ on sections of π_k constrained by X_k , i.e., we restrict \mathbf{S}' to those sections σ such that $\text{im } j_1\sigma \subset X_k$ (notice that, without the constraints, \mathbf{S}' would actually be a zeroth derivative action functional and, therefore, a very trivial one). The variational problem defined in this way is equivalent to the original one. In fact, similarly as above, sections σ of π_k such that $\text{im } j_1\sigma \subset X_k$ are precisely those of the form $\sigma = j_k s$ for some section s of π . In view of Theorem 1, we can use the method of Lagrange multipliers to find extremals. In the present case, the method consists in searching for extremals of a new, unconstrained, first derivative, action functional $\mathbf{S}_1 : \Sigma \mapsto \int_M (j_1\Sigma)^* \mathcal{L}_1$ on an augmented space of sections Σ . More precisely, Σ is a section of the bundle $V^\dagger := V^* \otimes_M \Lambda^n T^*M \rightarrow M$, which, in the following, we denote by q . Notice that, by construction, points of V and points of V^\dagger over the same point θ_0 of J^k can be paired to give a top form over M at $\pi_k(\theta_0)$. We denote by $\langle \cdot, \cdot \rangle$ such pairing. Since $\text{Hom}(\Lambda^1(M), \Lambda^n(M)) \simeq \Lambda^{n-1}(M)$ we have

$$V^\dagger \simeq V^* J^{k-1} \otimes_M \Lambda^{n-1} T^*M \times_{J^{k-1}} J^k$$

and it identifies naturally with $J^\dagger\pi_{k-1} \times_{J^{k-1}} J^k$, $J^\dagger\pi_{k-1}$ being the reduced multi-momentum bundle of π_{k-1} [2] (see also [4]).

The Lagrangian density \mathcal{L}_1 is defined by

$$(j_1\Sigma)^*\mathcal{L}_1 = \sigma^*\mathcal{L} + \langle \Sigma, \psi \circ j_1\sigma \rangle \in \Lambda^n(M),$$

where Σ is a section of $q : V^\dagger \rightarrow M$, and σ is the section of π_k given by projecting Σ onto J^k . Describe \mathcal{L}_1 locally. To this aim, let \mathcal{L} be locally given by

$$\mathcal{L} = L[x, u]d^n x,$$

where $L[x, u] := L(\dots, x^i, \dots, u_I^\alpha, \dots)$, $|I| \leq k$, is a local function on J^k and $d^n x := dx^1 \wedge \dots \wedge dx^n$. Moreover, let $\dots, p_\alpha^{J \cdot j}, \dots$ be standard, dual coordinates on $J^\dagger\pi_{k-1}$ corresponding to jet coordinates \dots, u_J^α, \dots on J^{k-1} , $|J| \leq k-1$. It is easy to see that, locally, $\mathcal{L}_1 = L_1[x, u, p, u., p.]d^n x$ where

$$\begin{aligned} L_1[x, u, p, u., p.] &:= L_1(\dots, x^i, \dots, u_I^\alpha, \dots, p_\alpha^{J \cdot j}, \dots, u_{I \cdot i}^\alpha, \dots, p_\alpha^{J \cdot j \cdot i}, \dots) \\ &= L[x, u] + \sum_{|I| \leq k-1} p_\alpha^{I \cdot i} (u_{I \cdot i}^\alpha - u_{Ii}^\alpha), \end{aligned} \quad (2)$$

$\dots, p_\alpha^{J \cdot j \cdot i}, \dots$ being jet coordinates corresponding to coordinates $\dots, p_\alpha^{J \cdot j}, \dots$ on V^\dagger . Formula (2) shows that the $p_\alpha^{J \cdot j}$'s, i.e., variables in the fiber of $V^\dagger \rightarrow J^k$ play the role of Lagrange multipliers.

Now, consider the Euler-Lagrange-Hamilton equations [4] determined by \mathbf{S} . They are the higher derivative, field theoretic analogue of the equations of motions of a Lagrangian mechanical system proposed by Skinner and Rusk in [5, 6]. Recall that the Euler-Lagrange-Hamilton equations are imposed precisely on sections of $V^\dagger \rightarrow M$ and are of the PD-Hamilton type (see [9] for the definition and main properties of PD-Hamiltonian systems and their PD-Hamilton equations). Moreover, the PD-Hamiltonian system determining them is an exact form. Therefore, the Euler-Lagrange-Hamilton equations are the Euler-Lagrange equations determined by a suitable Lagrangian density. The latter coincides with \mathcal{L}_1 up to total divergences. Indeed, the Euler-Lagrange equations determined by \mathbf{S}_1 locally read

$$\left(\begin{array}{c} \frac{\delta}{\delta u_I^\alpha} \\ \frac{\delta}{\delta p_\alpha^{J \cdot j}} \end{array} \right) L_1 = 0.$$

Now,

$$\left(\begin{array}{c} \frac{\delta}{\delta u_I^\alpha} \\ \frac{\delta}{\delta p_\alpha^{J \cdot j}} \end{array} \right) L_1 = \left(\begin{array}{c} \frac{\partial}{\partial u_I^\alpha} - \frac{d}{dx^i} \frac{\partial}{\partial u_{I \cdot i}^\alpha} \\ \frac{\partial}{\partial p_\alpha^{J \cdot j}} - \frac{d}{dx^i} \frac{\partial}{\partial p_\alpha^{J \cdot j \cdot i}} \end{array} \right) L_1 = \left(\begin{array}{c} \frac{\partial L}{\partial u_I^\alpha} - \delta_{Jj}^I p_\alpha^{J \cdot j} - p_\alpha^{I \cdot i \cdot j} \\ u_{J \cdot j}^\alpha - u_{Jj}^\alpha \end{array} \right)$$

which is the left hand side of Euler-Lagrange-Hamilton equations determined by \mathbf{S} . Summarizing, we have proved the following

Theorem 3 *The Euler-Lagrange equations determined by \mathbf{S}_1 coincide with the Euler-Lagrange-Hamilton equations determined by \mathbf{S} .*

Recall that the Euler-Lagrange-Hamilton equations cover the Euler-Lagrange equations in the sense that solutions of the former are surjectively mapped to solutions of the latter by projection onto E [4]. We then duly recover the Lagrange multiplier theorem in the present case (see, for instance, [10], see also [13]).

4 The Hamiltonian Sector

Let us now have a look at the Hamiltonian counterpart of the field theory defined by \mathbf{S}_1 . Let $J^\dagger q$ be the reduced multimomentum bundle of $q : V^\dagger \rightarrow M$ and $\dots, P_\alpha^{I,i}, \dots, Q_{J,j}^{\alpha,i}, \dots$ be dual coordinates on it corresponding to coordinates $\dots, u_I^\alpha, \dots, p_\alpha^{J,j}, \dots$ on V^\dagger , respectively, $|I| \leq k$, $|J| \leq k-1$. The Legendre transform $F\mathcal{L}_1 : J^1 q \rightarrow J^\dagger q$ is the fiber-derivative of \mathcal{L}_1 [2]. Clearly, $F\mathcal{L}_1$ is actually independent of \mathcal{L} . Locally,

$$F\mathcal{L}_1^*(P_\alpha^{I,i}) = \begin{cases} p_\alpha^{I,i} & \text{if } |I| \leq k-1 \\ 0 & \text{if } |I| = k \end{cases},$$

$$F\mathcal{L}_1^*(Q_{J,j}^{\alpha,i}) = 0.$$

This shows that $\text{im } F\mathcal{L}_1 \simeq V^\dagger$ and that, if we understand this isomorphism, $F\mathcal{L}_1 : J^1 q \rightarrow V^\dagger$ is nothing but the canonical projection. In particular, $F\mathcal{L}_1 : J^1 q \rightarrow \text{im } F\mathcal{L}_1$ is a surjective submersion with connected fibers and, therefore, \mathcal{L}_1 induces on $\text{im } F\mathcal{L}_1 \simeq V^\dagger$ a unique PD-Hamiltonian system ω such that $F\mathcal{L}_1^*(\omega) = d\Theta_{\mathcal{L}_1}$, $\Theta_{\mathcal{L}_1}$ being the Poincaré-Cartan n -form determined by \mathcal{L}_1 on $J^1 q$ [2]. A direct computation shows that ω is locally given by

$$\omega = \sum_{|I| \leq k-1} dp_\alpha^{I,i} \wedge du_I^\alpha \wedge d^{n-1}x_i + d\left(\sum_{|I| \leq k-1} p_\alpha^{I,i} u_{Ii}^\alpha - L[x, u]\right) \wedge d^n x,$$

where $d^{n-1}x_i := i_{\partial/\partial x^i} d^n x$, and that the corresponding PD-Hamilton equations (de Donder-Weyl equations) are nothing but Euler-Lagrange equations determined by \mathbf{S}_1 . We have thus proved the following

Theorem 4 *The de Donder-Weyl equations and the Euler-Lagrange equations determined by \mathbf{S}_1 coincide.*

Thus, despite the Legendre transform is far from being an isomorphism, the Hamiltonian counterpart of the theory is basically identical to the Lagrangian one.

We conclude remarking that the geometric formulation of the (Hamilton-like, higher derivative) de Donder field theory can be recovered from ω exactly as in [4]. This completes the program of the paper.

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