Optimal Control of Stochastic Partial Differential Equations

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Abstract

In this paper, we prove the necessary and sufficient maximum principles (NSMP in short) for the optimal control of system described by a quasilinear stochastic heat equation with the control domain being convex and all the coefficients containing control variable. For that, the optimal control problem of fully coupled forward-backward doubly stochastic system is studied. We apply our NSMP to solve a kind of forward-backward doubly stochastic linear quadratic optimal control problem and an example of optimal control of SPDEs as well.

Key words: The maximum principle, fully coupled forward-backward doubly stochastic control system, convex perturbation, stochastic partial differential equations, Malliavin calculus.

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1 Introduction

In order to provide a probabilistic interpretation for the solutions of a class of quasilinear stochastic partial differential equations (SPDEs in short), Pardoux and Peng [14] introduced

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the following backward doubly stochastic differential equation (BDSDE in short):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\mathrm{d}\overleftarrow{B_s} - \int_t^T Z_s \mathrm{d}\overrightarrow{W_s}, \quad 0 \le t \le T. \tag{1.1}$$

Note that the integral with respect to $\{B_t\}$ is a "backward Itô integral" and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (for more details see [14]). Pardoux and Peng [14] have obtained the relationship between BDSDEs and a certain quasilinear stochastic partial differential equations (SPDEs in short). More precisely

$$\begin{cases} u(t,x) = h(x) + \int_t^T \left[\mathcal{L}u(s,x) + f(s,x,u(s,x),(\nabla u\sigma)(s,x))\right]\mathrm{d}s \\ \qquad + \int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x))\,\mathrm{d}\overleftarrow{B_s}, \quad 0 \le t \le T, \end{cases}$$

where $u : [0,T] \times \mathbf{R}^d \to \mathbf{R}^k$ where d, $k \in N$, and $\nabla u(s,x)$ denotes the first order derivative of $u(s,x)$ with respect to x, and

$$\mathcal{L}u = \begin{pmatrix} Lu_1 \\ \vdots \\ Lu_k \end{pmatrix},$$

with

$$L\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(x)\frac{\partial^2\phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial \phi(x)}{\partial x_i}$$

(for more details see in [14]).

In 2003, Peng and Shi [17] introduced a type of time-symmetric forward-backward stochastic differential equations, i.e., so-called fully coupled forward-backward doubly stochastic differential equations (FBDSDEs in short):

$$\begin{cases} y_t = x + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}s + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}\overrightarrow{W_s} - \int_0^t z_s \mathrm{d}\overleftarrow{B_s}, \\ Y_t = h(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}s + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}\overleftarrow{B_s} + \int_t^T Z_s \mathrm{d}\overrightarrow{W_s}. \end{cases} \tag{1.2}$$

In FBDSDEs (1.2), the forward equation is "forward" with respect to a standard stochastic integral $\mathrm{d}W_t$, as well as "backward" with respect to a backward stochastic integral $\hat{\mathrm{d}}B_t$; the coupled "backward equation" is "forward" under the backward stochastic integral $\hat{\mathrm{d}}B_t$ and "backward" under the forward one. In other words, both the forward equation and the backward one are types of BDSDE (1.1) with different directions of stochastic integrals. So (1.2) provides a very general framework of fully coupled forward-backward stochastic systems. Peng and Shi [17] proved the existence and uniqueness of solutions to FBDSDEs (1.2) with arbitrarily fixed time duration under some monotone assumptions. FBDSDEs (1.2) can provide a probabilistic interpretation for the solutions of a class of quasilinear SPDEs.

In this paper, we consider the following quasilinear SPDEs with control variable:

$$\left\{\begin{array}{l} u(t,x)=\tilde{h}(x)+\int_t^T\left[\mathcal{L}^v u(s,x)+f(s,x,u(s,x),(\nabla u\sigma)(s,x),v(s))\right]\mathrm{d}s \\ \qquad +\int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x),v(s))\,\mathrm{d}\overleftarrow{B}_s, \quad 0\leq t\leq T, \end{array}\right. \tag{1.3}$$

where $u:[0,T]\times\mathbf{R}^d\to\mathbf{R}^k$ and $\nabla u(s,x)$ denotes the first order derivative of $u(s,x)$ with respect to x, and

$$\mathcal{L}^v u=\begin{pmatrix} L^v u_1 \\ \vdots \\ L^v u_k \end{pmatrix},$$

with

$$L^v\phi(x)=\frac{1}{2}\sum_{i,j=1}^d(\sigma\sigma^*)_{ij}(x,v)\frac{\partial^2\phi(x)}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x,v)\frac{\partial\phi(x)}{\partial x_i}.$$

It is worth to pointing out that all the coefficients contain the control variable (For more details see in Section 5).

Let us describe the problem solved in this paper. Set $\mathcal{U}_{ad}$ be an admissible control set. The definitions of notations used here can be found in Section 2. The optimal control problem of SPDEs (1.3) is to find an optimal control $v^*_{(\cdot)}\in\mathcal{U}_{ad}$, such that

$$J(v^*(\cdot))\doteq\inf_{v(\cdot)\in\mathcal{U}_{ad}}J(v(\cdot)),$$

where $J(v(\cdot))$ is its cost function as follows:

$$J(v(\cdot))=\mathbf{E}\left[\int_0^T l(s,x,u(s,x),(\nabla u\sigma)(s,x),v(s))\,\mathrm{d}s+\gamma(u(0,x))\right]. \tag{1.4}$$

As we have known, stochastic control problem of the SPDEs arising from partial observation control has been studied by Mortensen [9], using a dynamic programming approach, and subsequently by Bensoussan [2], [3], using a maximum principle method. See [4], [15] and the references therein for more information. Our approach differs from the one of Bensoussan. More precisely, we relate the FBDSDEs to one kind of SPDEs with control variables where the control systems of SPDEs can be transformed to the relevant control systems of FBDSDEs. To our knowledge, this is the first time to treat the optimal control problems of SPDEs from a new perspective of FBDSDEs. It is worth mentioning that the quasilinear SPDEs in [12] Øksendal considered can just be related to our partially coupled FBDSDEs. Recently, Zhang and Shi [25], obtained the similar results, however, in their paper, the coefficients σ and g do not contain the control variable, respectively.

This paper is organized as followings. In Section 2, we state the problem and some assumptions. In Section 3 and Section 4, we give the necessary and sufficient maximum principle for fully couple forward-backward doubly stochastic control systems, respectively, in global form. In Section 5, as an application, we study the optimal control of SPDEs. For simplicity of notations, we consider the one-dimensional case. It is necessary to point out that all the results can be extended to multi-dimensional case. In Section 6 the results are illustrated by solving two problems of optimal control of LQ problem and a special SPDEs using the Malliavin calculus, respectively

2 Statement of the problem

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ be two mutually independent standard Brown motion processes, with value respectively in $\mathbf{R}^d$ and $\mathbf{R}^l$, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t^W \doteq \sigma\{W_r;\ 0 \leq r \leq t\} \bigvee \mathcal{N}, \quad \mathcal{F}_{t,T}^B \doteq \sigma\{B_r - B_t;\ t \leq r \leq T\} \bigvee \mathcal{N},$$

and

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \bigvee \mathcal{F}_{t,T}^B,\ \forall t \in [0,T].$$

Note that $\left\{\mathcal{F}_t^W; t \in [0,T]\right\}$ is an increasing filtration and $\left\{\mathcal{F}_{t,T}^B; t \in [0,T]\right\}$ is an decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing.

We denote $M^2(0,T;\mathbf{R}^n)$ the space of (class of $dP \otimes dt$ a.e equal) all $\{\mathcal{F}_t\}$-measurable n-dimensional processes v with norm of $\| \ v \ \|_M \doteq \left[\mathbf{E}\int_0^T |v(s)|^2 \mathrm{d}s\right]^{\frac{1}{2}} < \infty$. Obviously $M^2(0,T\ ;\mathbf{R}^n)$ is a Hilbert space. For any given $u \in M^2(0,T\ ;\mathbf{R}^n)$ and $v \in M^2(0,T\ ;\mathbf{R}^n)$, one can define the (standard) forward Itô's integral $\int_0^\cdot u_s \mathrm{d}\overrightarrow{W_s}$ and backward Itô's integral $\int_\cdot^T v_s \mathrm{d}\overleftarrow{B_s}$. They are both in $M^2(0,T;\mathbf{R}^n)$, (see [14] for detail).

Let $L^2(\Omega, \mathcal{F}_T, P; \mathbf{R}^n)$ denote the space of all $\{\mathcal{F}_T\}$-measurable $\mathbf{R}^n$-valued random variable ξ satisfying $\mathbf{E}|\xi|^2 < \infty$.

Definition 1. *A stochastic process $X = \{X_t; t \geq 0\}$ is called $\mathcal{F}_t$-progressively measurable, if for any $t \geq 0$, X on $\Omega \times [0,t]$ is measurable with respect to $\left(\mathcal{F}_t^W \times \mathcal{B}([0,t])\right) \vee \left(\mathcal{F}_{t,T}^B \times \mathcal{B}([t,T])\right)$.*

Under this framework, we consider the following forward-backward doubly stochastic control system

$$\begin{cases} \mathrm{d}y(t) = f(t, y(t), Y(t), z(t), Z(t), v(t))\,\mathrm{d}t \\ \qquad +g(t, y(t), Y(t), z(t), Z(t), v(t))\,\mathrm{d}\overrightarrow{W_t} - z(t)\,\mathrm{d}\overleftarrow{B_t}, \\ \mathrm{d}Y(t) = -F(t, y(t), Y(t), z(t), Z(t), v(t))\,\mathrm{d}t \\ \qquad -G(t, y(t), Y(t), z(t), Z(t), v(t))\,\mathrm{d}\overleftarrow{B_t} + Z(t)\,\mathrm{d}\overrightarrow{W_t}, \\ y(0) = x_0, \quad Y(T) = h(y(T)), \end{cases} \tag{2.1}$$

where $(y(\cdot), Y(\cdot), z(\cdot), Z(\cdot), v(\cdot)) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, $x_0 \in \mathbf{R}$, is a given constant, and

$T > 0$,

$$\begin{array}{lll} F: & [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to \mathbf{R}, \\ f: & [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to \mathbf{R}, \\ G: & [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to \mathbf{R}, \\ g: & [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to \mathbf{R}, \\ h: & \mathbf{R}\to\mathbf{R}. & \end{array}$$

Let $\mathcal{U}$ be a nonempty convex subset of $\mathbf{R}$. We define the admissible control set

$$\mathcal{U}_{ad} \doteq \left\{ v(\cdot) \in M^2(0,T;\mathbf{R});\ v(t)\in\mathcal{U},\ 0\le t\le T,\ \text{a.e., a.s.} \right\}.$$

Our optimal control problem is to minimize the cost function:

$$J(v(\cdot)) \doteq \mathbf{E}\left[\int_0^T l\left((t,y(t),Y(t),z(t),Z(t),v(t))\right)\mathrm{d}t + \Phi(y(T)) + \gamma(Y(0))\right] \tag{2.2}$$

over $\mathcal{U}_{ad}$, where

$$\begin{array}{lcl} l & : & [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}, \\ \Phi & : & \mathbf{R}\to\mathbf{R}, \\ \gamma & : & \mathbf{R}\to\mathbf{R}. \end{array}$$

An admissible control $u(\cdot)$ is called an optimal control if it attains the minimum over $\mathcal{U}_{ad}$. That is to say, we want to find a $u(\cdot)$, such that

$$J(u(\cdot)) \doteq \inf_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)).$$

(2.1) is called the state equation, the solution (y_t, Y_t, z_t, Z_t) corresponding to $u(\cdot)$ is called the optimal trajectory.

Next we will give some notations:

$$\zeta = \begin{pmatrix} y \\ Y \\ z \\ Z \end{pmatrix}, \quad A(t,\zeta) = \begin{pmatrix} -F \\ f \\ -G \\ g \end{pmatrix}(t,\zeta).$$

We use the usual inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$ in $\mathbf{R}$, $\mathbf{R}^l$, and $\mathbf{R}^d$. All the equalities and inequalities mentioned in this paper are in the sense of $dt\times dP$ almost surely on $[0,T]\times\Omega$. We assume that

(H1) $\begin{cases} \text{For each } \zeta\in\mathbf{R}^{1+1+1\times l+1\times d},\ A(\cdot,\zeta) \text{ is an } \mathcal{F}_t\text{-measurable process defined on } [0,T] \\ \text{with } A(\cdot,0)\in M^2\left(0,T;\mathbf{R}^{1+1+1\times l+1\times d}\right). \end{cases}$

(H2) $A(t,\zeta)$ and $h(y)$ satisfy Lipschitz conditions: there exists a constant $k>0$, such that

$$\begin{cases} \left|A(t,\zeta)-A\left(t,\bar{\zeta}\right)\right| \le k\left|\zeta-\bar{\zeta}\right|, & \forall\zeta,\ \bar{\zeta}\in\mathbf{R}^{1+1+1\times l+1\times d},\ \forall t\in[0,T], \\ \left|h(y)-h(\bar{y})\right| \le k\left|y-\bar{y}\right|, & \forall y,\ \bar{y}\in\mathbf{R}. \end{cases}$$

The following monotonic conditions introduced in [17], are the main assumptions in this paper.

$$\text{(H3)} \quad \begin{cases} \left\langle A(t,\zeta) - A\left(t,\bar{\zeta}\right), \zeta - \bar{\zeta}\right\rangle \leq -\mu \left|\zeta - \bar{\zeta}\right|^2, \\ \quad \forall \zeta = (y, Y, z, Z)^T, \ \bar{\zeta} = \left(\bar{y}, \bar{Y}, \bar{z}, \bar{Z}\right)^T \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}, \ \forall t \in [0,T]. \\ \langle h(y) - h(\bar{y}), y - \bar{y}\rangle \geq 0, \ \forall y, \ \bar{y} \in \mathbf{R}. \end{cases}$$

or

$$\text{(H'3)} \quad \begin{cases} \left\langle A(t,\zeta) - A\left(t,\bar{\zeta}\right), \zeta - \bar{\zeta}\right\rangle \geq \mu \left|\zeta - \bar{\zeta}\right|^2, \\ \quad \forall \zeta = (y, Y, z, Z)^T, \ \bar{\zeta} = \left(\bar{y}, \bar{Y}, \bar{z}, \bar{Z}\right)^T \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}, \ \forall t \in [0,T]. \\ \langle h(y) - h(\bar{y}), y - \bar{y}\rangle \leq 0, \ \forall y, \ \bar{y} \in \mathbf{R}, \end{cases}$$

where μ is a positive constant.

Proposition 2. *For any given admissible control $v(\cdot)$, we assume (H1), (H2) and (H3) (or (H1), (H2) and (H3)') hold. Then FBDSDEs (2.1) has the unique solution $(y_t, Y_t, z_t, Z_t) \in M^2\left(0,T;\mathbf{R}^{1+1+1\times l+1\times d}\right)$.*

The proof can be seen in [17]. The proof under the assumptions (H1), (H2) and (H$'$3) is similar.

We assume:

$$\text{(H4)} \quad \begin{cases} \text{i) } F, \ f, \ G, \ g, \ h, \ l, \ \Phi, \ \gamma \text{ are continuously differentiable} \\ \quad \text{with respect to } (y, Y, z, Z, v), \ y, \text{ and } Y; \\ \text{ii) The derivatives of } F, \ f, \ G, \ g, \ h \text{ are bounded;} \\ \text{iii) The derivatives of } l \text{ are bounded by } C\left(1 + |y| + |Y| + |z| + |Z| + |v|\right); \\ \text{iv) The derivatives of } \Phi \text{ and } \gamma \text{ with respect to } y, \ Y \text{ are bounded by} \\ \quad C\left(1+|y|\right) \text{ and } C\left(1+|Y|\right), \text{ respectively.} \end{cases}$$

Lastly, we need the following extension of Itô's formula (for more details see [14]).

Proposition 3. *Let $\alpha \in S^2\left([0,T];\mathbf{R}^k\right)$, $\beta \in M^2\left([0,T];\mathbf{R}^k\right)$, $\gamma \in M^2\left([0,T];\mathbf{R}^{k\times l}\right)$, $\delta \in S^2\left([0,T];\mathbf{R}^{k\times d}\right)$ satisfy: $\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s d\overleftarrow{B_s} + \int_0^t \delta_s d\overrightarrow{W_s}, \quad 0 \leq t \leq T$.*

Then

$$\begin{aligned} |\alpha_t|^2 = |\alpha_0|^2 + 2\int_0^t \langle \alpha_s, \beta_s\rangle \, ds + 2\int_0^t \left\langle \alpha_s, \gamma_s d\overleftarrow{B_s}\right\rangle + 2\int_0^t \left\langle \alpha_s, \delta_s d\overrightarrow{W_s}\right\rangle \\ - \int_0^t |\gamma_s|^2 \, ds + \int_0^t |\delta_s|^2 \, ds, \end{aligned}$$

$$\mathbf{E}|\alpha_t|^2 = \mathbf{E}|\alpha_0|^2 + 2\mathbf{E}\int_0^t \langle \alpha_s, \beta_s\rangle \, ds - \mathbf{E}\int_0^t |\gamma_s|^2 \, ds + \mathbf{E}\int_0^t |\delta_s|^2 \, ds.$$

More generally, if $\phi \in C^2\left(\mathbf{R}^k\right)$,

$$\phi\left(\alpha_t\right)=\phi\left(\alpha_0\right)+\int_0^t\left\langle\phi^{'}\left(\alpha_s\right),\beta_s\right\rangle ds+\int_0^t\left\langle\phi^{'}\left(\alpha_s\right),\gamma_s d\overleftarrow{B}_s\right\rangle+\int_0^t\left\langle\phi^{'}\left(\alpha_s\right),\delta_s d\overrightarrow{W}_s\right\rangle$$
$$-\frac{1}{2}\int_0^t Tr\left[\phi^{''}\left(\alpha_s\right)\gamma_s\gamma^*\right]ds+\frac{1}{2}\int_0^t Tr\left[\phi^{''}\left(\alpha_s\right)\delta_s\delta_s^*\right]ds.$$

Here $S^2\left(0,T;\mathbf{R}^k\right)$ denotes the space of (classes of $dP\otimes dt$ a.e. equal) all $\mathcal{F}_t$-progressively measurable k-dimensional processes v with

$$\mathbf{E}\left(\sup_{0\leq t\leq T}|v(t)|^2\right)<\infty.$$

3 A Necessary Maximum Principle for Optimal Forward-backward Doubly Stochastic Control system

We consider the forward-backward doubly stochastic control system (2.1) and the cost function (2.2). Let $u\left(\cdot\right)$ be an optimal control and $\left(y\left(\cdot\right),Y\left(\cdot\right),z\left(\cdot\right),Z\left(\cdot\right)\right)$ be the corresponding trajectory. Let $v\left(\cdot\right)$ be any given admissible control such that $u\left(\cdot\right)+v\left(\cdot\right)\in\mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is convex, then for any $0\leq\rho\leq1$, $u_\rho\left(\cdot\right)=u\left(\cdot\right)+\rho v\left(\cdot\right)$ is also in $\mathcal{U}_{ad}$.

We introduce the following variational equation of FBDSDEs:

$$\begin{cases}
\mathrm{d}y^1(t)=[f_y(t,y(t),Y(t),z(t),Z(t),u(t))\,y^1(t)\\
\qquad +f_Y(t,y(t),Y(t),z(t),Z(t),u(t))\,Y^1(t)\\
\qquad +f_z(t,y(t),Y(t),z(t),Z(t),u(t))\,z^1(t)\\
\qquad +f_Z(t,y(t),Y(t),z(t),Z(t),u(t))\,Z^1(t)\\
\qquad +f_v(t,y(t),Y(t),z(t),Z(t),u(t))\,v(t)]\mathrm{d}t\\
\qquad +[g_y(t,y(t),Y(t),z(t),Z(t),u(t))\,y^1(t)\\
\qquad +g_Y(t,y(t),Y(t),z(t),Z(t),u(t))\,Y^1(t)\\
\qquad +g_z(t,y(t),Y(t),z(t),Z(t),u(t))\,z^1(t)\\
\qquad +g_Z(t,y(t),Y(t),z(t),Z(t),u(t))\,Z^1(t)\\
\qquad +g_v(t,y(t),Y(t),z(t),Z(t),u(t))\,v(t)]\mathrm{d}\overrightarrow{W}_t-z^1(t)\,\mathrm{d}\overleftarrow{B}_t,\\
\mathrm{d}Y^1(t)=-[F_y(t,y(t),Y(t),z(t),Z(t),u(t))\,y^1(t)\\
\qquad +F_Y(t,y(t),Y(t),z(t),Z(t),u(t))\,Y^1(t)\\
\qquad +F_z(t,y(t),Y(t),z(t),Z(t),u(t))\,z^1(t)\\
\qquad +F_Z(t,y(t),Y(t),z(t),Z(t),u(t))\,Z^1(t)\\
\qquad +F_v(t,y(t),Y(t),z(t),Z(t),u(t))\,v(t)]\mathrm{d}t\\
\qquad -[G_y(t,y(t),Y(t),z(t),Z(t),u(t))\,y^1(t)\\
\qquad +G_Y(t,y(t),Y(t),z(t),Z(t),u(t))\,Y^1(t)\\
\qquad +G_z(t,y(t),Y(t),z(t),Z(t),u(t))\,z^1(t)\\
\qquad +G_Z(t,y(t),Y(t),z(t),Z(t),u(t))\,Z^1(t)\\
\qquad +G_v(t,y(t),Y(t),z(t),Z(t),u(t))\,v(t)]\mathrm{d}\overleftarrow{B}_t+Z^1(t)\,\mathrm{d}\overrightarrow{W}_t,\\
y^1(0)=0,\quad Y^1(t)=h_y(y(T))\,y^1(T).
\end{cases}\tag{3.1}$$

From (H3), (H4) and Proposition 2, it is easy to check that (3.1) satisfies (H1), (H2) and (H3). Then there exists a unique quadruple of $\left(y^1(t), Y^1(t), z^1(t), Z^1(t)\right)$ satisfying FBDSDEs (3.1).

We denote by $\left(y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t)\right)$ the trajectory of FBDSDEs (2.1) corresponding to $u_\rho(\cdot)$ as followings.

$$\begin{cases} \mathrm{d}y_\rho(t) = f\left(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u_\rho(t)\right)\mathrm{d}t \\ \qquad +g\left(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u_\rho(t)\right)\mathrm{d}\overrightarrow{W}_t - z_\rho(t)\,\mathrm{d}\overleftarrow{B}_t, \\ \mathrm{d}Y_\rho(t) = -F\left(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u_\rho(t)\right)\mathrm{d}t \\ \qquad -G\left(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u_\rho(t)\right)\mathrm{d}\overleftarrow{B}_t + Z_\rho(t)\,\mathrm{d}\overrightarrow{W}_t, \\ y_\rho(0) = x_0, \quad Y_\rho(T) = h\left(y_\rho(T)\right), \end{cases}$$

Then we will study the solutions depending on parameter to forward-backward doubly stochastic control system.

Lemma 4. *Assume that (H1)-(H4) hold. Then we have*

$$\begin{aligned} \lim_{\rho\to 0}\frac{y_\rho(t)-y(t)}{\rho} &= y^1(t), \\ \lim_{\rho\to 0}\frac{Y_\rho(t)-Y(t)}{\rho} &= Y^1(t), \\ \lim_{\rho\to 0}\frac{z_\rho(t)-z(t)}{\rho} &= z^1(t), \\ \lim_{\rho\to 0}\frac{Z_\rho(t)-Z(t)}{\rho} &= Z^1(t), \end{aligned}$$

where the limits are in $M^2(0,T)$.

Proof. Firstly, we show the continuous dependence of solutions with respect to the parameter ρ. Let

$$\begin{aligned} \hat{y}(t) &= y_\rho(t) - y(t), \\ \hat{Y}(t) &= Y_\rho(t) - Y(t), \\ \hat{z}(t) &= z_\rho(t) - z(t), \\ \hat{Z}(t) &= Z_\rho(t) - Z(t). \end{aligned}$$

We have

$$\begin{cases} \mathrm{d}\hat{y}(t) = [f(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t) + \rho v(t)) \\ \qquad -f(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad +f(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad -f(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}t \\ \qquad +[g(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t) + \rho v(t)) \\ \qquad -g(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad +g(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad -g(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}\overrightarrow{W_t} - \hat{z}(t)\,\mathrm{d}\overleftarrow{B_t}, \\ \mathrm{d}\hat{Y}(t) = -[F(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t) + \rho v(t)) \\ \qquad -F(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad +F(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t) \\ \qquad -F(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}t \\ \qquad -[G(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t) + \rho v(t)) \\ \qquad -G(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad +G(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)) \\ \qquad -G(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}\overleftarrow{B_t} + \hat{Z}(t)\,\mathrm{d}\overrightarrow{W_t}, \\ \hat{y}(0) = 0, \quad \hat{Y}(T) = h(y_\rho(T)) - h(y(T)). \end{cases}$$

We will prove $\left(\hat{y}(t), \hat{Y}(t), \hat{z}(t), \hat{Z}(t)\right)$ converge to 0 in $M^2(0,T)$ as $\rho \to 0$. Applying Itô's formula to $\left\langle \hat{y}(t), \hat{Y}(t) \right\rangle$ on $[0,T]$, and by (H4) it follows that

$$\begin{aligned}
& \mathbf{E}\left\langle \hat{y}(T), h(y_\rho(T)) - h(y(T)) \right\rangle \\
= & \mathbf{E}\int_0^T \left\langle A(t,\xi_\rho) - A(t,\xi), \xi_\rho - \xi \right\rangle \mathrm{d}t \\
& -\mathbf{E}\int_0^T \hat{y}(t)\left[F(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)+\rho v(t))\right. \\
& \left. -F(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t))\right]\mathrm{d}t \\
& +\mathbf{E}\int_0^T \hat{Y}(t)\left[f(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)+\rho v(t))\right. \\
& \left. -f(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t))\right]\mathrm{d}t \\
& -\mathbf{E}\int_0^T \hat{z}(t)\left[G(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)+\rho v(t))\right. \\
& \left. -G(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t))\right]\mathrm{d}t \\
& +\mathbf{E}\int_0^T \hat{Z}(t)\left[g(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t)+\rho v(t))\right. \\
& \left. -g(t, y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t))\right]\mathrm{d}t \\
\leq & -\mu\mathbf{E}\int_0^T \left[|\hat{y}(t)|^2 + \left|\hat{Y}(t)\right|^2 + |\hat{z}(t)|^2 + \left|\hat{Z}(t)\right|^2\right]\mathrm{d}t \\
& +\frac{\mu}{4}\mathbf{E}\int_0^T \left[|\hat{y}(t)|^2 + \left|\hat{Y}(t)\right|^2 + |\hat{z}(t)|^2 + \left|\hat{Z}(t)\right|^2\right]\mathrm{d}t \\
& +\frac{1}{\mu}\rho^2 C\mathbf{E}\int_0^T |v(t)|^2\,\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
\xi_\rho(t) &= (y_\rho(t), Y_\rho(t), z_\rho(t), Z_\rho(t), u(t))^T, \\
\xi(t) &= (y(t), Y(t), z(t), Z(t), u(t))^T, \\
A(t,\xi) &= \begin{pmatrix} -F(t,\xi) \\ f(t,\xi) \\ -G(t,\xi) \\ g(t,\xi) \end{pmatrix}, \quad A(t,\xi_\rho) = \begin{pmatrix} -F(t,\xi_\rho) \\ f(t,\xi_\rho) \\ -G(t,\xi_\rho) \\ g(t,\xi_\rho) \end{pmatrix}.
\end{aligned}$$

Thus we get

$$\mathbf{E}\int_0^T \left[|\hat{y}(t)|^2 + \left|\hat{Y}(t)\right|^2 + |\hat{z}(t)|^2 + \left|\hat{Z}(t)\right|^2\right]\mathrm{d}t \leq \rho^2 C\mathbf{E}\int_0^T |v(t)|^2\,\mathrm{d}t.$$

Then we have $\left(\hat{y}(t), \hat{Y}(t), \hat{z}(t), \hat{Z}(t)\right)$ converge to 0 in $M^2(0,T)$ as ρ tends to 0. Next we set

$$\begin{aligned}
\triangle y(t) &= \frac{y_\rho(t)-y(t)}{\rho},\\
\triangle Y(t) &= \frac{Y_\rho(t)-Y(t)}{\rho},\\
\triangle z(t) &= \frac{z_\rho(t)-z(t)}{\rho},\\
\triangle Z(t) &= \frac{Z_\rho(t)-Z(t)}{\rho},
\end{aligned}$$

then

$$\begin{cases}
\mathrm{d}\triangle y(t) = \frac{f(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-f(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho}\mathrm{d}t\\
\qquad +\frac{g(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-g(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho}\mathrm{d}\overrightarrow{W}_t\\
\qquad -\triangle z(t)\,\mathrm{d}\overleftarrow{B}_t,\\
-\mathrm{d}\triangle Y(t) = \frac{F(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-F(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho}\mathrm{d}t\\
\qquad +\frac{G(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-G(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho}\mathrm{d}\overleftarrow{B}_t\\
\qquad -\triangle Z(t)\,\mathrm{d}\overrightarrow{W}_t,\\
\triangle y(0)=0, \quad \triangle Y(T) = \frac{h(y_\rho(T)-h(y(T)))}{\rho}.
\end{cases}$$

The above equations can be expressed as follows

$$\begin{cases}
\mathrm{d}\triangle y(t) = \bar{f}(t,\triangle y(t),\triangle Y(t),\triangle z(t),\triangle Z(t),v(t))\,\mathrm{d}t\\
\qquad +\bar{g}(t,\triangle y(t),\triangle Y(t),\triangle z(t),\triangle Z(t),v(t))\,\mathrm{d}\overrightarrow{W}_t\\
\qquad -\triangle z(t)\,\mathrm{d}\overleftarrow{B}_t,\\
-\mathrm{d}\triangle Y(t) = \bar{F}(t,\triangle y(t),\triangle Y(t),\triangle z(t),\triangle Z(t),v(t))\,\mathrm{d}t\\
\qquad +\bar{G}(t,\triangle y(t),\triangle Y(t),\triangle z(t),\triangle Z(t),v(t))\,\mathrm{d}\overleftarrow{B}_t\\
\qquad -\triangle Z(t)\,\mathrm{d}\overrightarrow{W}_t,\\
\triangle y(0)=0, \quad \triangle Y(T) = \frac{h(y_\rho(T))-h(y(T))}{\rho},
\end{cases}$$

where $\bar{\theta}=\bar{f},\ \bar{F},\ \bar{g},\ \bar{G}$, respectively,

$$\bar{\theta}(t,\triangle y,\triangle Y,\triangle z,\triangle Z,v) = A^\theta(t)\triangle y + B^\theta(t)\triangle Y + C^\theta(t)\triangle z + D^\theta(t)\triangle Z + E^\theta(t)v,$$

and

$$A^{\theta}(t) = \begin{cases} \frac{\theta(t,y_\rho(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-\theta(t,y(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))}{y_\rho(t)-y(t)}, & y_\rho(t)-y(t)\neq 0, \\ 0, \quad \text{otherwise}; \end{cases}$$

$$B^{\theta}(t) = \begin{cases} \frac{\theta(t,y(t),Y_\rho(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-\theta(t,y(t),Y(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))}{Y_\rho(t)-Y(t)}, & Y_\rho(t)-Y(t)\neq 0, \\ 0, \quad \text{otherwise}; \end{cases}$$

$$C^{\theta}(t) = \begin{cases} \frac{\theta(t,y(t),Y(t),z_\rho(t),Z_\rho(t),u(t)+\rho v(t))-\theta(t,y(t),Y(t),z(t),Z_\rho(t),u(t)+\rho v(t))}{z_\rho(t)-z(t)}, & z_\rho(t)-z(t)\neq 0, \\ 0, \quad \text{otherwise}; \end{cases}$$

$$D^{\theta}(t) = \begin{cases} \frac{\theta(t,y(t),Y(t),z(t),Z_\rho(t),u(t)+\rho v(t))-\theta(t,y(t),Y(t),z(t),Z(t),u(t)+\rho v(t))}{Z_\rho(t)-Z(t)}, & Z_\rho(t)-Z(t)\neq 0, \\ 0, \quad \text{otherwise}; \end{cases}$$

$$E^{\theta}(t) = \begin{cases} \frac{\theta(t,y(t),Y(t),z(t),Z(t),u(t)+\rho v(t))-\theta(t,y(t),Y(t),z(t),Z(t),u(t))}{\rho v(t)}, & \rho v(t)\neq 0, \\ 0, \quad \text{otherwise}. \end{cases}$$

From the continuous dependence of solutions with respect to the parameter ρ, we obtain

$$\begin{aligned} \lim_{\rho\to 0} A^{\theta}(t) &= \theta_y(t,y(t),Y(t),z(t),Z(t),u(t)), \\ \lim_{\rho\to 0} B^{\theta}(t) &= \theta_Y(t,y(t),Y(t),z(t),Z(t),u(t)), \\ \lim_{\rho\to 0} C^{\theta}(t) &= \theta_z(t,y(t),Y(t),z(t),Z(t),u(t)), \\ \lim_{\rho\to 0} D^{\theta}(t) &= \theta_Z(t,y(t),Y(t),z(t),Z(t),u(t)), \\ \lim_{\rho\to 0} E^{\theta}(t) &= \theta_v(t,y(t),Y(t),z(t),Z(t),u(t)). \end{aligned}$$

According to the continuous dependence of solutions with respect to the parameter and the uniqueness of solutions of FBDSDE (3.1), the solutions $(\triangle y(t), \triangle Y(t), \triangle z(t), \triangle Z(t))$ converge to $(y^1(t), Y^1(t), z^1(t), Z^1(t))$ in $M^2\left(0,T;R^{1+1+1\times l+1\times d}\right)$ as $\rho\to 0$. The proof is completed.□

Now we give the variational inequality.

Lemma 5. *Assume that (H1)-(H4) hold. Then we have*

$$\begin{aligned} &\mathbf{E}\Phi_y(y(T))\,y^1(T)+\mathbf{E}\gamma_Y(Y(0))\,Y^1(0) \\ &+\mathbf{E}\int_0^T [l_y(t,y(t),Y(t),z(t),Z(t),u(t))\,y^1(t) \\ &+l_Y(t,y(t),Y(t),z(t),Z(t),u(t))\,Y^1(t) \\ &+l_z(t,y(t),Y(t),z(t),Z(t),u(t))\,z^1(t) \\ &+l_Z(t,y(t),Y(t),z(t),Z(t),u(t))\,Z^1(t) \\ &+l_v(t,y(t),Y(t),z(t),Z(t),u(t))\,v(t)]\,dt \\ &\geq 0. \end{aligned}$$

Proof From Lemma 4 and (H4), we can get

$$\begin{aligned}
\lim_{\rho\to 0}\frac{\mathbf{E}\left[\Phi\left(y_\rho\left(T\right)\right)-\Phi\left(y\left(T\right)\right)\right]}{\rho} &= \mathbf{E}\Phi_y\left(y\left(T\right)\right)y^1\left(T\right),\\
\lim_{\rho\to 0}\frac{\mathbf{E}\left[\gamma\left(Y_\rho\left(0\right)\right)-\gamma\left(Y\left(0\right)\right)\right]}{\rho} &= \mathbf{E}\gamma_Y\left(Y\left(0\right)\right)Y^1\left(0\right),
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\rho\to 0}\rho^{-1}\mathbf{E}\int_0^T\left[l\left(t,y_\rho\left(t\right),Y_\rho\left(t\right),z_\rho\left(t\right),Z_\rho\left(t\right),u\left(t\right)+\rho v\left(t\right)\right)\right.\\
&\left.-l\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)\right]\mathrm{d}t\\
=\;&\mathbf{E}\int_0^T\left[l_y\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)y^1\left(t\right)\right.\\
&+l_Y\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)Y^1\left(t\right)\\
&+l_z\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)z^1\left(t\right)\\
&+l_Z\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)Z^1\left(t\right)\\
&\left.+l_v\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right)\right)v\left(t\right)\right]\mathrm{d}t.
\end{aligned}$$

On the other hand, since $u\left(\cdot\right)$ is an optimal control, it follows that

$$\rho^{-1}\left[J\left(u\left(\cdot\right)+\rho v\left(\cdot\right)\right)-J\left(u\left(\cdot\right)\right)\right]\geq 0.$$

Therefore the desired result is obtained. □

Now we introduce the adjoint equation by virtue of dual technique and Hamilton function for our problem. From the variational inequality obtained in Lemma 5, the maximum principle can be proved by using Itô's formula. The adjoint equations are

$$\begin{cases} \mathrm{d}p(t) = [F_Y(t, y(t), Y(t), z(t), Z(t), u(t))\, p(t) \\ \qquad -f_Y(t, y(t), Y(t), z(t), Z(t), u(t))\, q(t) \\ \qquad +G_Y(t, y(t), Y(t), z(t), Z(t), u(t))\, k(t) \\ \qquad -g_Y(t, y(t), Y(t), z(t), Z(t), u(t))\, h(t) \\ \qquad -l_Y(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}t \\ \qquad +[F_Z(t, y(t), Y(t), z(t), Z(t), u(t))\, p(t) \\ \qquad -f_Z(t, y(t), Y(t), z(t), Z(t), u(t))\, q(t) \\ \qquad +G_Z(t, y(t), Y(t), z(t), Z(t), u(t))\, k(t) \\ \qquad -g_Z(t, y(t), Y(t), z(t), Z(t), u(t))\, h(t) \\ \qquad -l_Z(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}\overrightarrow{W_t} - k_t \mathrm{d}\overleftarrow{B_t}, \\ \mathrm{d}q(t) = [F_y(t, y(t), Y(t), z(t), Z(t), u(t))\, p(t) \\ \qquad -f_y(t, y(t), Y(t), z(t), Z(t), u(t))\, q(t) \\ \qquad +G_y(t, y(t), Y(t), z(t), Z(t), u(t))\, k(t) \\ \qquad -g_y(t, y(t), Y(t), z(t), Z(t), u(t))\, h(t) \\ \qquad -l_y(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}t \\ \qquad +[F_z(t, y(t), Y(t), z(t), Z(t), u(t))\, p(t) \\ \qquad -f_z(t, y(t), Y(t), z(t), Z(t), u(t))\, q(t) \\ \qquad +G_z(t, y(t), Y(t), z(t), Z(t), u(t))\, k(t) \\ \qquad +G_z(t, y(t), Y(t), z(t), Z(t), u(t))\, k(t) \\ \qquad -g_z(t, y(t), Y(t), z(t), Z(t), u(t))\, h(t) \\ \qquad -l_z(t, y(t), Y(t), z(t), Z(t), u(t))]\mathrm{d}\overleftarrow{B_t} + h_t \mathrm{d}\overrightarrow{W_t}, \\ p(0) = -\gamma_Y(Y(0)), \quad q(T) = -h_y(y(T))\, P(T) + \Phi_y(y(T)). \end{cases} \tag{3.2}$$

It is easy to check that FBDSDEs (3.2) satisfies (H1), (H2) and (H$'$3), so it has a unique solution $(p(t), q(t), k(t), h(t)) \in M^2\left(0, T; \mathbf{R}^{1+1+l+d}\right)$.

We define the Hamiltonian function H as follows:

$$\begin{aligned} & H(t, y(t), Y(t), z(t), Z(t), v(t), p(t), q(t), k(t), h(t)) \\ \doteq\; & \langle q(t), f(t, y(t), Y(t), z(t), Z(t), v(t))\rangle \\ & - \langle p(t), F(t, y(t), Y(t), z(t), Z(t), v(t))\rangle \\ & - \langle k(t), G(t, y(t), Y(t), z(t), Z(t), v(t))\rangle \\ & + \langle h(t), g(t, y(t), Y(t), z(t), Z(t), v(t))\rangle \\ & + l(t, y(t), Y(t), z(t), Z(t), v(t)). \end{aligned} \tag{3.3}$$

FBDSDEs (3.2) can be rewritten as

$$\begin{cases} \mathrm{d}p(t) = -H_Y \mathrm{d}t - H_Z \mathrm{d}\overrightarrow{W_t} - k(t)\, \mathrm{d}\overleftarrow{B_t}, \\ \mathrm{d}q(t) = -H_y \mathrm{d}t - H_z \mathrm{d}\overleftarrow{B_t} + h(t)\, \mathrm{d}\overrightarrow{W_t}, \\ q(T) = -h_y(y(T))\, p(T) + \Phi_y(y(T)), \\ p(0) = -\gamma_Y(Y(0)), \qquad 0 \le t \le T, \end{cases} \tag{3.4}$$

where $H_\beta = H_\beta(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t))$, $\beta = y, Y, z, Z$, respectively.

At last, we can claim the first and major result in this paper.

Theorem 6. *(***Necessary maximum principle***) Let $u(\cdot)$ be an optimal control and let $(y(\cdot), Y(\cdot), z(\cdot), Z(\cdot))$ be the corresponding trajectory. Then we have*

$$\begin{aligned} \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v - u(t)\rangle &\geq 0, \\ a.e.,\ a.s.,\ t &\in [0, T],\ \forall v \in \mathcal{U}, \end{aligned} \tag{3.5}$$

where $(p(t), q(t), k(t), h(t))$ is the solution of the adjoint equation (3.2).

Proof. Applying Itô's formula to $\langle y^1(t), q(t)\rangle + \langle Y^1(t), p(t)\rangle$ on $[0, T]$, we have

$$\begin{aligned} & \mathbf{E}\left[\left\langle y^1(T), q(T)\right\rangle + \left\langle Y^1(T), p(T)\right\rangle - \left\langle y^1(0), q(0)\right\rangle - \left\langle Y^1(0), p(0)\right\rangle\right] \\ & +\mathbf{E}\int_0^T [l_y(t, y(t), Y(t), z(t), Z(t), u(t))\, y^1(t) \\ & +l_Y(t, y(t), Y(t), z(t), Z(t), u(t))\, Y^1(t) \\ & +l_z(t, y(t), Y(t), z(t), Z(t), u(t))\, z^1(t) \\ & +l_Z(t, y(t), Y(t), z(t), Z(t), u(t))\, Z^1(t) \\ & +l_v(t, y(t), Y(t), z(t), Z(t), u(t))\, v(t)]\mathrm{d}t \\ = & \ \mathbf{E}\int_0^T [\langle q(t), f_v(t, y(t), Y(t), z(t), Z(t), u(t))\, v(t)\rangle \\ & -\langle p(t), F_v(t, y(t), Y(t), z(t), Z(t), u(t))\, v(t)\rangle \\ & -\langle k(t), G_v(t, y(t), Y(t), z(t), Z(t), u(t))\, v(t)\rangle \\ & +\langle h(t), g_v(t, y(t), Y(t), z(t), Z(t), u(t))\, v(t)\rangle \\ & +\langle v(t), l_v(t, y(t), Y(t), z(t), Z(t), u(t))\rangle]\mathrm{d}t. \end{aligned}$$

From the variational inequality in Lemma 5 and noting (3.3), for any $v(\cdot) \in \mathcal{U}_{ad}$ such that $u(\cdot) + v(\cdot) \in \mathcal{U}_{ad}$, we have

$$\mathbf{E}\int_0^T \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v(t)\rangle\, \mathrm{d}t \geq 0.$$

For $\forall v \in \mathcal{U}$, we set

$$v(t) = \begin{cases} 0, & t \in [0, t], \\ v, & t \in [t, , t + \varepsilon], \\ 0, & t \in [t + \varepsilon, T]. \end{cases}$$

Then we have

$$\mathbf{E}\int_t^{t+\varepsilon} \langle H_v(t, y(t), Y(t), z(t), Z(t), u(t), p(t), q(t), k(t), h(t)), v\rangle\, \mathrm{d}t \geq 0.$$

Notice the fact that

$$\mathbf{E}\int_t^{t+\varepsilon}\left\langle H_v\left(t,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right),p\left(t\right),q\left(t\right),k\left(t\right),h\left(t\right)\right),u\left(t\right)\right\rangle \mathrm{d}t=0.$$

Differentiating with respect to ε at $\varepsilon=0$ gives

$$\begin{array}{rcl}\mathbf{E}\left\langle H_v\left(t,,y\left(t\right),Y\left(t\right),z\left(t\right),Z\left(t\right),u\left(t\right),p\left(t\right),q\left(t\right),k\left(t\right),h\left(t\right)\right),v-u\left(t\right)\right\rangle & \geq & 0,\\ \text{a.e., a.s., } t & \in & \left[0,T\right].\end{array}$$

The proof is completed. $\square$

4 A Sufficient Maximum Principle for Optimal Forward-backward Doubly Stochastic Control system

In this section, we investigate a sufficient maximum principle for the optimal control problem stated in Section 2. For simplicity of notations, we use the subscript label.

Theorem 7. *(**Sufficient maximum principle**). Let $\left(\tilde{u}_t;\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t\right)_{0\leq t\leq T}$ be an quintuple and suppose there exists a solution $\left(\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)_{0\leq t\leq T}$ of the corresponding adjoint forward-backward doubly stochastic equation (3.2) such that for arbitrary admissible control $v\left(\cdot\right)\in\mathcal{U}_{ad}$, we have*

$$\mathbf{E}\int_0^T\left\langle\tilde{k}_t,\left(Y_t-\tilde{Y}_t\right)\right\rangle^2dt<\infty, \tag{4.1}$$

$$\mathbf{E}\int_0^T\left\langle\tilde{p}_t,\left(Z_t-\tilde{Z}_t\right)\right\rangle^2dt<\infty, \tag{4.2}$$

$$\mathbf{E}\int_0^T\left\langle\tilde{h}_t,\left(y_t-\tilde{y}_t\right)\right\rangle^2dt<\infty, \tag{4.3}$$

$$\mathbf{E}\int_0^T\left\langle\tilde{q}_t,\left(z_t-\tilde{z}_t\right)\right\rangle^2dt<\infty, \tag{4.4}$$

$$\mathbf{E}\int_0^T\left\langle\left(Y_t-\tilde{Y}_t\right),H_Z\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)\right\rangle^2dt<\infty, \tag{4.5}$$

$$\mathbf{E}\int_0^T\left\langle\tilde{p}_t,\left(G\left(t,y_t,Y_t,z_t,Z_t\right)-G\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t\right)\right)\right\rangle^2dt<\infty, \tag{4.6}$$

$$\mathbf{E}\int_0^T\left\langle\left(y_t-\tilde{y}_t\right),H_z\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)\right\rangle^2dt<\infty, \tag{4.7}$$

$$\mathbf{E}\int_0^T\left\langle\tilde{q}_t,\left(g\left(t,y_t,Y_t,z_t,Z_t\right)-g\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t\right)\right)\right\rangle^2dt<\infty. \tag{4.8}$$

Further, suppose that for all $t \in [0,T]$, $H\left(s, y, Y, z, Z, v, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)$ *is convex in* (y, Y, z, Z, v), *and* $\gamma(Y)$ *is convex in* Y *and* Φ *is convex in* y, *moreover the following conditions holds*

$$\mathbf{E}\left[H\left(t, \tilde{y}_s, \tilde{Y}_s, \tilde{z}_s, \tilde{Z}_s, \tilde{u}_s, \tilde{p}_s, \tilde{q}_s, \tilde{k}_s, \tilde{h}_s\right)\right] = \inf_{v \in U} \mathbf{E}\left[H\left(t, \tilde{y}_s, \tilde{Y}_s, \tilde{z}_s, \tilde{Z}_s, v, \tilde{p}_s, \tilde{q}_s, \tilde{k}_s, \tilde{h}_s\right)\right]. \quad (4.9)$$

Then $\tilde{u}_t$ *is an optimal control.*

Proof. Let $(y_t, Y_t, z_t, Z_t, v_t) = \left(y_t^{(v)}, Y_t^{(v)}, z_t^{(v)}, Z_t^{(v)}, v_t\right)$ be an arbitrary quintuple. According to the definition of the cost function (2.2), we have

$$\begin{aligned} J(v(\cdot)) - J(\tilde{u}(\cdot)) &= \mathbf{E}\int_0^T \left[l(t, y_t, Y_t, z_t, Z_t, v_t) - l\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right] \mathrm{d}t \\ &\quad + \mathbf{E}\left[\Phi(y_T) - \Phi(\tilde{y}_T)\right] + \mathbf{E}\left[\gamma(Y_0) - \gamma\left(\tilde{Y}_0\right)\right] \\ &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3, \end{aligned}$$

where

$$\begin{aligned} \mathbf{I}_1 &= \mathbf{E}\int_0^T \left[l(t, y_t, Y_t, z_t, Z_t, v_t) - l\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right] \mathrm{d}t \\ \mathbf{I}_2 &= \mathbf{E}\left[\Phi(y_T) - \Phi(\tilde{y}_T)\right] \\ \mathbf{I}_3 &= \mathbf{E}\left[\gamma(Y_0) - \gamma\left(\tilde{Y}_0\right)\right]. \end{aligned}$$

Now applying Itô's formula to $\left\langle \tilde{p}_t, Y_t - \tilde{Y}_t \right\rangle + \langle \tilde{q}_t, y_t - \tilde{y}_t \rangle$ on $[0, T]$, we get

$$
\begin{aligned}
& \left\langle \tilde{p}_T, Y_T - \tilde{Y}_T \right\rangle + \langle \tilde{q}_T, y_T - \tilde{y}_T \rangle - \left\langle \tilde{p}_0, Y_0 - \tilde{Y}_0 \right\rangle - \langle \tilde{q}_0, y_0 - \tilde{y}_0 \rangle \\
= \ & \langle \Phi_y(\tilde{y}_T), y_T - \tilde{y}_T \rangle + \left\langle \gamma_Y\left(\tilde{Y}_0\right), Y_0 - \tilde{Y}_0 \right\rangle \\
= \ & \int_0^T \left\langle \left(Z_t - \tilde{Z}_t\right), \left(-H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& - \int_0^T \left\langle \tilde{k}_t, \left(G(t, y_t, Y_t, z_t, Z_t, v_t) - G\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle (z_t - \tilde{z}_t), \left(-H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle \tilde{h}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left(Y_t - \tilde{Y}_t\right) \mathrm{d}\tilde{p}_t + \int_0^T \tilde{p}_t \mathrm{d}\left(Y_t - \tilde{Y}_t\right) + \int_0^T (y_t - \tilde{y}_t) \mathrm{d}\tilde{q}_t + \int_0^T \tilde{q}_t \mathrm{d}(y_t - \tilde{y}_t) \\
= \ & \int_0^T \left\langle \left(Z_t - \tilde{Z}_t\right), \left(-H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& - \int_0^T \left\langle \tilde{k}_t, \left(G(t, y_t, Y_t, z_t, Z_t, v_t) - G\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle (z_t - \tilde{z}_t), \left(-H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle \tilde{h}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle \left(Y_t - \tilde{Y}_t\right), \left(-H_Y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle \left(Y_t - \tilde{Y}_t\right), \left(-H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \mathrm{d}\overrightarrow{W_t} \right\rangle \\
& - \int_0^T \left\langle \tilde{k}_t, \left(Y_t - \tilde{Y}_t\right) \mathrm{d}\overleftarrow{B_t} \right\rangle \\
& - \int_0^T \left\langle \tilde{p}_t, \left(F(t, y_t, Y_t, z_t, Z_t, v_t) - F\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\
& - \int_0^T \left\langle \tilde{p}_t, \left(G(t, y_t, Y_t, z_t, Z_t, v_t) - G\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \mathrm{d}\overleftarrow{B_t} \right\rangle \\
& + \int_0^T \left\langle \tilde{p}_t, \left(Z_t - \tilde{Z}_t\right) \mathrm{d}\overrightarrow{W_t} \right\rangle \\
& + \int_0^T \left\langle (y_t - \tilde{y}_t), \left(-H_y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \right\rangle \mathrm{d}t \\
& + \int_0^T \left\langle (y_t - \tilde{y}_t), \left(-H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right) \mathrm{d}\overleftarrow{B_t} \right\rangle \\
& + \int_0^T \left\langle (y_t - \tilde{y}_t), \tilde{h}_t \mathrm{d}W_t \right\rangle \\
& + \int_0^T \tilde{q}_t \left(f(t, y_t, Y_t, z_t, Z_t, v_t) - f\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \mathrm{d}t
\end{aligned}
$$

where we claim that

$$\begin{cases} Y_T - \tilde{Y}_T = h(y_T) - h(\tilde{y}_T) = h_y(\tilde{y}(T))(y(T) - \tilde{y}(T)), \\ y_0 - \tilde{y}_0 = x_0 - x_0 = 0, \\ \tilde{p}_0 = -\gamma_Y(Y_0), \\ \tilde{q}_T = \Phi_y(\tilde{y}_T) - h_y(\tilde{y}(T))\tilde{p}(T). \end{cases}$$

By Davis inequality, under the conditions (4.1)-(4.8), we can ensure that the stochastic integral with respect to the Brownian motion have zero expectation. Moreover, by virtue of convexity of Φ and γ, we instantly get

$$\begin{aligned} \mathbf{I}_2 + \mathbf{I}_3 &= \mathbf{E}\left[\Phi(y_T) - \Phi(\tilde{y}_T)\right] + \mathbf{E}\left[\gamma(Y_0) - \gamma\left(\tilde{Y}_0\right)\right] \\ &\geq \mathbf{E}\left\langle \Phi_y(\tilde{y}_T), y_T - \tilde{y}_T \right\rangle + \mathbf{E}\left\langle \gamma_Y\left(\tilde{Y}_0\right), Y_0 - \tilde{Y}_0 \right\rangle \\ &= -\mathbf{E}\int_0^T \left\langle \left(Y_t - \tilde{Y}_t\right), H_Y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) \right\rangle \mathrm{d}t \\ &\quad -\mathbf{E}\int_0^T \left\langle \tilde{p}_t, \left(F(t, y_t, Y_t, z_t, Z_t, v_t) - F\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\ &\quad -\mathbf{E}\int_0^T \left\langle (y_t - \tilde{y}_t), H_y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) \right\rangle \mathrm{d}t \\ &\quad +\mathbf{E}\int_0^T \left\langle \tilde{q}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\ &\quad -\mathbf{E}\int_0^T \left\langle \left(Z_t - \tilde{Z}_t\right), H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) \right\rangle \mathrm{d}t \\ &\quad -\mathbf{E}\int_0^T \left\langle \tilde{k}_t, \left(G(t, y_t, Y_t, z_t, Z_t, v_t) - G\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\ &\quad -\mathbf{E}\int_0^T \left\langle (z_t - \tilde{z}_t), H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) \right\rangle \mathrm{d}t \\ &\quad +\mathbf{E}\int_0^T \left\langle \tilde{h}_t, \left(g(t, y_t, Y_t, z_t, Z_t, v_t) - g\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t\right)\right) \right\rangle \mathrm{d}t \\ &= -\Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5, \end{aligned}$$

where

$$
\begin{aligned}
\Xi_1 &= \mathbf{E}\int_0^T \left\langle H_y\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right),(y_t-\tilde{y}_t)\right\rangle \mathrm{d}t \\
&\quad +\mathbf{E}\int_0^T \left\langle H_Y\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right),\left(Y_t-\tilde{Y}_t\right)\right\rangle \mathrm{d}t \\
&\quad +\mathbf{E}\int_0^T \left\langle H_z,\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right),(z_t-\tilde{z}_t)\right\rangle \mathrm{d}t \\
&\quad +\mathbf{E}\int_0^T \left\langle H_Z\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right),\left(Z_t-\tilde{Z}_t\right)\right\rangle \mathrm{d}t \\
\Xi_2 &= -\mathbf{E}\int_0^T \left\langle \tilde{p}_t, F\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-F\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle \mathrm{d}t \\
\Xi_3 &= \mathbf{E}\int_0^T \left\langle \tilde{q}_t, g\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-g\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle \mathrm{d}t \\
\Xi_4 &= -\mathbf{E}\int_0^T \left\langle \tilde{k}_t, G\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-G\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle \mathrm{d}t \\
\Xi_5 &= \mathbf{E}\int_0^T \left\langle \tilde{h}_t, g\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-g\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle \mathrm{d}t.
\end{aligned}
$$

Noting the definition of H and $\mathbf{I}_1$, we have

$$
\begin{aligned}
\mathbf{I}_1 &= \mathbf{E}\int_0^T \left[l\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-l\left(t,\tilde{y}_t,\tilde{Y}_s,\tilde{z}_s,\tilde{Z}_s,\tilde{u}_t\right)\right]\mathrm{d}t \\
&= \mathbf{E}\int_0^T \left[H\left(t,y_t,Y_t,z_t,Z_t,v_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)-H\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)\right]\mathrm{d}t \\
&\quad -\mathbf{E}\int_0^T \left[\left\langle \tilde{q}_t, f\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-f\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle\right]\mathrm{d}t \\
&\quad +\mathbf{E}\int_0^T \left[\left\langle \tilde{p}_t, F\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-F\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle\right]\mathrm{d}t \\
&\quad +\mathbf{E}\int_0^T \left[\left\langle \tilde{k}_t, G\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-G\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle\right]\mathrm{d}t \\
&\quad -\mathbf{E}\int_0^T \left[\left\langle \tilde{h}_t, g\left(t,y_t,Y_t,z_t,Z_t,v_t\right)-g\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t\right)\right\rangle\right]\mathrm{d}t \\
&= \Xi_6-\Xi_2-\Xi_3-\Xi_4-\Xi_5,
\end{aligned}
$$

where

$$
\Xi_6=\mathbf{E}\int_0^T \left[H\left(t,y_t,Y_t,z_t,Z_t,v_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)-H\left(t,\tilde{y}_t,\tilde{Y}_t,\tilde{z}_t,\tilde{Z}_t,\tilde{u}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{h}_t\right)\right]\mathrm{d}t.
$$

On the one hand, by the virtue of convexity of $H\left(t, y, Y, z, Z, v, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)$ with respect to (y, Y, z, Z, v), we obtain

$$
\begin{aligned}
& H\left(t, y_t, Y_t, z_t, Z_t, v_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) - H\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) \\
\geq\; & H_y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(y_t - \tilde{y}_t) \\
& + H_Y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\left(Y_t - \tilde{Y}_t\right) \\
& + H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(z_t - \tilde{z}_t) \\
& + H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\left(Z_t - \tilde{Z}_t\right) \\
& + H_u\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(v_t - \tilde{u}_t)
\end{aligned}
\tag{4.10}
$$

On the other hand, we know

$$
\mathbf{E}\left[H_u\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(v_t - \tilde{u}_t)\right] \geq 0.
$$

Consequently, associating with (4.10), we claim that

$$
\begin{aligned}
\Xi_6 \;=\; & \mathbf{E}\int_0^T \left[H\left(t, y_t, Y_t, z_t, Z_t, v_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right) - H\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\right] \mathrm{d}t \\
\geq\; & \mathbf{E}\int_0^T H_y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(y_t - \tilde{y}_t)\, \mathrm{d}t \\
& + \mathbf{E}\int_0^T H_Y\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\left(Y_t - \tilde{Y}_t\right) \mathrm{d}t \\
& + \mathbf{E}\int_0^T H_z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)(z_t - \tilde{z}_t)\, \mathrm{d}t \\
& \mathbf{E}\int_0^T H_Z\left(t, \tilde{y}_t, \tilde{Y}_t, \tilde{z}_t, \tilde{Z}_t, \tilde{u}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t, \tilde{h}_t\right)\left(Z_t - \tilde{Z}_t\right) \mathrm{d}t \\
=\; & \Xi_1.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
J(v(\cdot)) - J(u(\cdot)) &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 \\
&= \Xi_6 - \Xi_2 - \Xi_3 - \Xi_4 - \Xi_5 \\
&\quad - \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 \\
&\geq \Xi_1 - \Xi_2 - \Xi_3 - \Xi_4 - \Xi_5 \\
&\quad - \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_5 \\
&= 0.
\end{aligned}
$$

Since $\forall v(\cdot) \in \mathcal{U}_{ad}$ is arbitrary, we say that $\tilde{u}(\cdot)$ is an optimal control. The proof is complete. $\square$

5 Applications to optimal control problems of stochastic partial differential equations

In this section, we will give necessary and sufficient maximum principles for optimal control of SPDEs. Let us first give some notations from [14]. For convenience, all the variables in this section are one-dimensional. It is necessary to point out that all the results in this section can be extended in multi-dimensional case, but we use the notations in general case. From now on $C^k(\mathbf{R};\mathbf{R})$, $C^k_{l,b}(\mathbf{R};\mathbf{R})$, $C^k_p(\mathbf{R};\mathbf{R})$ will denote respectively the set of functions of class C^k from $\mathbf{R}$ into $\mathbf{R}$, the set of those functions of class C^k whose partial derivatives of order less than or equal to k are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions of class C^k which, together with all their partial derivatives of order less than or equal to k, grow at most like a polynomial function of the variable x at infinity. We consider the following quasilinear SPDEs with control variable:

$$\begin{cases} u(t,x) = \tilde{h}(x) + \int_t^T \left[\mathcal{L}^v u(s,x) + f(s,x,u(s,x),(\nabla u\sigma)(s,x),v(s))\right] \mathrm{d}s \\ \qquad + \int_t^T g(s,x,u(s,x),(\nabla u\sigma)(s,x),v(s))\, \mathrm{d}\overleftarrow{B}_s, \quad 0 \le t \le T, \end{cases} \tag{5.1}$$

where $u : [0,T] \times \mathbf{R} \to \mathbf{R}$ and $\nabla u(s,x)$ denotes the first order derivative of $u(s,x)$ with respect to x, and

$$\mathcal{L}^v u = \begin{pmatrix} L^v u_1 \\ \vdots \\ L^v u_k \end{pmatrix},$$

with $L\phi(x) = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{ij}(x,v)\frac{\partial^2\phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x,v)\frac{\partial\phi(x)}{\partial x_i}$. In the present paper, we set $d = k = 1$, and

$$\begin{aligned} b &: \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \\ \sigma &: \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \\ f &: [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \\ g &: [0,T] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}, \\ \tilde{h} &: \mathbf{R} \to \mathbf{R}. \end{aligned}$$

In order to assure the existence and uniqueness of solutions for (5.1) and (5.3) below, we give the following assumptions for sake of completeness (see [14] for more details).

(A1)

$$\begin{cases} b \in C^3_{l,b}\left(\mathbf{R}\times\mathbf{R};\mathbf{R}\right), \quad \sigma \in C^3_{l,b}\left(\mathbf{R}\times\mathbf{R};\mathbf{R}\right), \quad \tilde{h} \in C^3_p\left(\mathbf{R};\mathbf{R}\right), \\ f\left(t,\cdot,\cdot,\cdot,v\right) \in C^3_{l,b}\left(\mathbf{R}\times\mathbf{R}\times\mathbf{R};\mathbf{R}\right), \quad f\left(\cdot,x,y,z,v\right) \in M^2\left(0,T;\mathbf{R}\right), \\ g\left(t,\cdot,\cdot,\cdot,v\right) \in C^3_{l,b}\left(\mathbf{R}\times\mathbf{R}\times\mathbf{R};\mathbf{R}\right), \quad g\left(\cdot,x,y,z,v\right) \in M^2\left(0,T;\mathbf{R}\right), \\ \forall t \in [0,T], \, x\in\mathbf{R}, \, y \in \mathbf{R}, \, z\in\mathbf{R}, \, v\in\mathbf{R}. \end{cases}$$

(A2) There exist some constant $c > 0$ and $0 < \alpha < 1$ such that for all $(t, x, y_i, z_i, v) \in [0,T]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$, $(i = 1, 2)$,

$$\begin{cases} \left|f\left(t,x,y_1,z_1,v\right) - f\left(t,x,y_2,z_2,v\right)\right|^2 \le c\left(\left|y_1-y_2\right|^2 + \left|z_1-z_2\right|^2\right), \\ \left|g\left(t,x,y_1,z_1,v\right) - g\left(t,x,y_2,z_2,v\right)\right|^2 \le c\left|y_1-y_2\right|^2 + \alpha\left|z_1-z_2\right|^2. \end{cases}$$

Let $\mathcal{U}_{ad}$ be an admissible control set. The optimal control problem of SPDE (5.1) is to find an optimal control $v^*_{(\cdot)} \in \mathcal{U}_{ad}$, such that

$$J\left(v^*\left(\cdot\right)\right) \doteq \inf_{v(\cdot)\in\mathcal{U}_{ad}} J\left(v\left(\cdot\right)\right),$$

where $J\left(v\left(\cdot\right)\right)$ is the cost function as follows:

$$J\left(v\left(\cdot\right)\right) = \mathbf{E}\left[\int_0^T l\left(s,x,u\left(s,x\right),\left(\nabla u\sigma\right)\left(s,x\right),v\left(s\right)\right)\mathrm{d}s + \gamma\left(u\left(0,x\right)\right)\right]. \tag{5.2}$$

Here we assume l and γ satisfy (H4). We can transform the optimal control problem of SPDEs (5.1) into one of the following FBDSDEs with control variable:

$$\begin{cases} X^{t,x}\left(s\right) = x + \int_t^s b\left(X^{t,x}\left(r\right),v\left(r\right)\right)\mathrm{d}r + \int_t^s \sigma\left(X^{t,x}\left(r\right),v\left(r\right)\right)\mathrm{d}\overrightarrow{W}_r, \\ Y^{t,x}\left(s\right) = \tilde{h}\left(X^{t,x}\left(T\right)\right) + \int_s^T f\left(r,X^{t,x}\left(r\right),Y^{t,x}\left(r\right),Z^{t,x}\left(r\right),v\left(r\right)\right)\mathrm{d}r \\ \qquad + \int_s^T g\left(r,X^{t,x}\left(r\right),Y^{t,x}\left(r\right),Z^{t,x}\left(r\right),v\left(r\right)\right)\mathrm{d}\overleftarrow{B}_r \\ \qquad - \int_s^T Z^{t,x}\left(r\right)\mathrm{d}\overrightarrow{W}_r, \qquad 0\le t\le s\le T, \end{cases} \tag{5.3}$$

where $\left(X^{t,x}\left(\cdot\right),Y^{t,x}\left(\cdot\right),Z^{t,x}\left(\cdot\right),v\left(\cdot\right)\right) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$, $x \in \mathbf{R}$. The corresponding optimal control problem of FBDSDEs (5.3) is to find an optimal control $v^*\left(\cdot\right) \in \mathcal{U}_{ad}$, such that

$$J\left(v^*\left(\cdot\right)\right) \doteq \inf_{v(\cdot)\in\mathcal{U}_{ad}} J\left(v\left(\cdot\right)\right),$$

where $J\left(v\left(\cdot\right)\right)$ is the cost function the same as (5.2):

$$J\left(v\left(\cdot\right)\right) = \mathbf{E}\left[\int_0^T l\left(s,X\left(s\right),Y\left(s\right),Z\left(s\right),v\left(s\right)\right)\mathrm{d}s + \gamma\left(Y\left(0\right)\right)\right].$$

Now we consider the following adjoint FBDSDEs involving the four unknown processes $\left(p\left(t\right),q\left(t\right),k\left(t\right),h\left(t\right)\right)$:

$$\begin{cases} \mathrm{d}p\left(t\right) = \left(f_Y p\left(t\right) + g_Y k\left(t\right) - l_Y\right)\mathrm{d}t + \left(f_Z p\left(t\right) - g_Z k\left(t\right) - l_Z\right)\mathrm{d}\overrightarrow{W}_t - k\left(t\right)\mathrm{d}\overleftarrow{B}_t, \\ \mathrm{d}q\left(t\right) = \left(f_X p\left(t\right) - b_X q\left(t\right) + g_X k\left(t\right) - \sigma_X h\left(t\right) - l_X\right)\mathrm{d}t + h\left(t\right)\mathrm{d}\overrightarrow{W}_t, \\ p\left(0\right) = -\gamma_Y\left(Y\left(0\right)\right), \quad q\left(T\right) = -\tilde{h}_X\left(X\left(T\right)\right)p\left(T\right), \qquad 0\le t\le T. \end{cases} \tag{5.4}$$

It is easy to see that the first equation of (5.4) is a "forward" BDSDE, so it is uniquely solvable by virtue of the result in [14]. The second equation of (5.4) is a standard BSDE, so it is uniquely solvable by virtue of the result in [13]. Therefore we know that (5.4) has a unique solution $(p(\cdot), q(\cdot), k(\cdot), h(\cdot)) \in M^2(0,T;\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R})$. Define the Hamilton function as follows:

$$\begin{aligned}\bar{H}(t,X,Y,Z,v,p,q,k,h) &= H(t,X,Y,0,Z,v,p,q,k,h) \\ &= l(t,X,Y,Z,v) - k\cdot g(t,X,Y,Z,v) \\ &\quad +q\cdot b(X,v) - p\cdot f(t,X,Y,Z,v) + h\cdot\sigma(X,v).\end{aligned} \tag{5.5}$$

We now formulate a maximum principle for the optimal control system of (5.3).

Theorem 8. *Suppose (A1)-(A2) hold. Let $(X(\cdot), Y(\cdot), Z(\cdot), u(\cdot))$ be an optimal control and its corresponding trajectory of (5.3), $(p(\cdot), q(\cdot), k(\cdot), h(\cdot))$ be the solution of (5.4). Then the maximum principle holds, that is, for $t\in[0,T]$, $\forall v\in\mathcal{U}_{ad}$,*

$$\left\langle \bar{H}(t,X(t),Y(t),Z(t),v^*(t),p(t),q(t),k(t),h(t)), v-v^*(t)\right\rangle \geq 0, \text{ a.e., a.s..}$$

Proof. Noting that the forward equation of (5.3) is independent of the backward one, we easily know that it is uniquely solvable. It is straightforward to use the same arguments in Section 3 to obtain the desired results. We omit the detailed proof. □

From the results in [14], we easily have the following propositions.

Proposition 9. *For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Then (5.3) has a unique solution $(X^{t,x}(\cdot), Y^{t,x}(\cdot), Z^{t,x}(\cdot)) \in M^2(0,T;\mathbf{R}\times\mathbf{R}\times\mathbf{R})$.*

Proposition 10. *For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Let $\{u(t,x); 0\leq t\leq T, x\in\mathbf{R}\}$ be a random field such that $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable for each (t,x), $u\in C^{0,2}([0,T]\times\mathbf{R};\mathbf{R})$ a.s., and u satisfies SPDE (5.1). Then $u(t,x) = Y^{t,x}(t)$.*

Proposition 11. *For any given admissible control $v(\cdot)$, we assume (A1) and (A2) hold. Then $\{u(t,x) = Y^{t,x}(t); 0\leq t\leq T, x\in\mathbf{R}\}$ is a unique classical solution of SPDE (5.1).*

Set the Hamilton function

$$\begin{aligned}\bar{H}(t,x,u,\nabla u\sigma,v,p,q,k,h) &= l(t,x,u,\nabla u\sigma,v) - k\cdot g(t,x,u,\nabla u\sigma,v) \\ &\quad +q\cdot b(x,v) - p\cdot f(t,x,u,\nabla u\sigma,v) + h\cdot\sigma(x,v).\end{aligned}$$

Now we can state the maximum principle for the optimal control problem of SPDE (5.1).

Theorem 12. *(**Necessary maximum principle**) Suppose $u(t,x)$ is the optimal solution of SPDE (5.1) corresponding to the optimal control $v^*(\cdot)$ of (5.1). Then we have, for any $v\in\mathcal{U}$ and $t\in[0,T]$, $x\in\mathbf{R}$,*

$$\left\langle \bar{H}_v(t,x,u(t,x),(\nabla u\sigma)(t,x),v^*(t),p(t),q(t),k(t),h(t)), v-v^*(t)\right\rangle \geq 0, \text{ a.e., a.s.}$$

Proof. By virtue of Proposition 10, 11 and 12, the optimal control problem of SPDEs (5.1) can be transformed into the one of FBDSDEs (5.3). Hence, from Theorem 9, the desired result is easily obtained. □

Next we apply our sufficient maximum principle to get the following result.

Theorem 13. *(***Sufficient maximum principle***) For $\forall t\in[0,T]$, let $\hat{v}=\hat{v}(t)\in\mathcal{U}_{ad}$ with corresponding solution $\hat{u}(t,x)$ of (5.1) and let $\left(\hat{X}(t),\hat{Y}(t),\hat{Z}(t),\hat{v}(t)\right)$ be quaternion and $\left(\hat{p}(t),\hat{q}(t),\hat{k}(t),\hat{h}(t)\right)$ be a solution of the associated adjoint FBDSDEs (5.3) and (5.4), respectively. Assume that $\bar{H}\left(t,X,Y,Z,v,\hat{p}(t),\hat{q}(t),\hat{k}(t),\hat{h}(t)\right)$ is convex in (X,Y,Z,v), and $\gamma(Y)$ is convex in Y, moreover the following condition holds*

$$\begin{aligned}&\mathbf{E}\left[\bar{H}\left(t,\hat{X}(t),\hat{Y}(t),\hat{Z}(t),\hat{v}(t),\hat{p}(t),\hat{q}(t),\hat{k}(t),\hat{h}(t)\right)\right]\\&=\inf_{v\in\mathcal{U}}\mathbf{E}\left[\bar{H}\left(t,\hat{X}(t),\hat{Y}(t),\hat{Z}(t),v,\hat{p}(t),\hat{q}(t),\hat{k}(t),\hat{h}(t)\right)\right].\end{aligned}$$

Then $\hat{v}(t)$ is an optimal control for the problem (5.2).

Proof. Noting above assumptions, by Theorem 8, it is fairly to get desired result. □

Remark 1 In [12], Bernt Øksendal proved a sufficient maximum principle for the optimal control of system described by a quasilinear stochastic heat equation, that is

$$\begin{aligned}dY(t,x)&=&\left\{\begin{array}{l}=[LY(t,x)+b(t,x,Y(t,x),v(t))]\,dt\\+\sigma(t,x,Y(t,x),u(t))\,d\overrightarrow{W}_t;\end{array}\right.\\(t,x)&\in&[0,T]\times G.\end{aligned} \tag{5.6}$$

$$Y(0,x)=\xi(x);\qquad x\in\overline{G} \tag{5.7}$$

$$Y(t,x)=\eta(t,x);\qquad (t,x)\in(0,T)\times\partial G. \tag{5.8}$$

Here G is an open set in $\mathbf{R}^n$ with C^1 boundary ∂G and

$$L\phi(x)=\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}\phi+\sum_{i=1}^{n}b_i(x)\frac{\partial}{\partial x_i}\phi,\qquad \phi\in C^2(\mathbf{R}^n)$$

where $a(x)=[a_{ij}(x)]_{1\le i,j\le n}$ is a given symmetric definite symmetric $n\times n$ matrix with entries $a_{ij}(x)\in C^2(G)\cap C\left(\overline{G}\right)$ for all $i,j=1,2,\cdots,n$ and $b_i(x)\in C^2(G)\cap C\left(\overline{G}\right)$ for all $i,j=1,2,\cdots,n$. For more detail, see [12]. It is worth to pointing out that our method to get the sufficient maximum principle is completely different from his, and the most important thing is that in our SPDEs, all the coefficients contain the control variables, while in [12], the coefficients a and b do not satisfy it (for more details see Theorem 2.1-Theorem 2.3 in [12]).

6 Applications

Theoretically, the maximum principles presented in Section 3 and Section 4 characterizes the optimal control through some necessary and sufficient conditions. However, it is not immediately feasible to implement such principles directly, partially due to the difficulty of computing fully coupled forward-backward doubly stochastic system. In this section, we give two special examples and show how to explicitly solve them using our maximum principle.

6.1 Example 1

We provide a concrete example of forward-backward doubly stochastic LQ problems and give the explicit optimal control and validate our major theoretical results in Theorem 6. (Necessary maximum principle). First let the control domain be $\mathcal{U} = [-1, 1]$. Consider the following linear forward-backward doubly stochastic control system. We assume that $l = d = 1$.

$$\begin{cases} \mathrm{d}y(t) = (z(t) - Z(t) + v(t))\,\mathrm{d}\overrightarrow{W}_t - z(t)\,\mathrm{d}\overleftarrow{B}_t, \\ \mathrm{d}Y(t) = -(z(t) + Z(t) + v(t))\,\mathrm{d}\overleftarrow{B}_t + Z(t)\,\mathrm{d}\overrightarrow{W}_t, \\ y(0) = 0, \quad Y(T) = 0, \quad t \in [0, T], \end{cases} \tag{6.1}$$

where $T > 0$ is a given constant and the cost function is

$$\begin{aligned} J(v(\cdot)) &= \frac{1}{2}\mathbf{E}\int_0^T \left(y^2(t) + Y^2(t) + z^2(t) + Z^2(t) + v^2(t)\right)\mathrm{d}t \\ &\quad + \frac{1}{2}\mathbf{E}Y^2(0) + \frac{1}{2}\mathbf{E}y^2(T). \end{aligned} \tag{6.2}$$

Note that (6.1) are linear control system. According to the existence and uniqueness of (6.1), it is straightforward to know the optimal control is $u(\cdot) \equiv 0$, with the optimal state trajectory $(y(t), Y(t), z(t), Z(t)) \equiv 0$, $t \in [0, T]$. Notice that the adjoint equation associated with the optimal quadruple $(y(t), Y(t), z(t), Z(t)) \equiv 0$ are

$$\begin{cases} \mathrm{d}p(t) = -Y(t)\,\mathrm{d}t + (-k(t) - h(t) - Z(t))\,\mathrm{d}\overrightarrow{W}_t - k(t)\,\mathrm{d}\overleftarrow{B}_t, \\ \mathrm{d}q(t) = -y(t)\,\mathrm{d}t + (-k(t) - h(t) - z(t))\,\mathrm{d}\overleftarrow{B}_t + h(t)\,\mathrm{d}\overrightarrow{W}_t, \\ p(0) = 0, \quad q(T) = 0, \quad t \in [0, T]. \end{cases} \tag{6.3}$$

Obviously, $(p(t), q(t), k(t), h(t)) \equiv 0$ is the unique solution of (6.3). Instantly, we give the

Hamiltonian function is

$$
\begin{aligned}
& H\left(t, y\left(t\right), Y\left(t\right), z\left(t\right), Z\left(t\right), v, p\left(t\right), q\left(t\right), k\left(t\right), h\left(t\right)\right) \\
= & \frac{1}{2}\left(y^2\left(t\right)+Y^2\left(t\right)+z^2\left(t\right)+Z^2\left(t\right)+v^2\right) \\
& -k\left(t\right)\left(z\left(t\right)+Z\left(t\right)+v\right) \\
& +h\left(t\right)\left(z\left(t\right)-Z\left(t\right)+v\right) \\
= & \frac{1}{2}v^2.
\end{aligned}
$$

It is clear that, for any $v \in \mathcal{U}$, we always have

$$
\mathbf{E}\left\langle H_v\left(t, y\left(t\right), Y\left(t\right), z\left(t\right), Z\left(t\right), u\left(t\right), p\left(t\right), q\left(t\right), k\left(t\right), h\left(t\right)\right), v-u\left(t\right)\right\rangle = 0.
$$

6.2 Example 2

In this subsection we will provide a special optimal control of SPDEs by Theorem 14. (Sufficient maximum principle). We now introduce some notations. For any random variable F of the form $F = f\left(W\left(h_1\right), \ldots W\left(h_n\right); B\left(k_1\right), \ldots B\left(k_p\right)\right)$ with $f \in C_b^{\infty}\left(R^{n+p}\right)$, $h_1, \ldots h_n \in L^2\left(\left[0,T\right], R^d\right)$, $k_1, \ldots k_p \in L^2\left(\left[0,T\right], R^l\right)$, where

$$
W\left(h_i\right) = \int_0^T h_i\left(t\right) \mathrm{d}W_t, \quad B\left(h_i\right) = \int_0^T k_i\left(t\right) \mathrm{d}B_t,
$$

we let

$$
D_t F = \sum_{i=1}^n f_i^{'}\left(W\left(h_1\right), \ldots W\left(h_n\right); B\left(k_1\right), \ldots B\left(k_p\right)\right) h_i\left(t\right), \quad 0 \leq t \leq T.
$$

For such an F, we define its 1,2-norm as:

$$
\left\|F\right\|_{1,2} = \left(\mathbf{E}\left[F^2 + \int_0^T \left|D_t F\right|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}}.
$$

S denotes the set of random variable of the above form. We define the Sobolev space:

$$
\mathbb{D}^{1,2} = \overline{S}^{\left\|\cdot\right\|_{1,2}}.
$$

The "derivation operator" $D_{\cdot}$ extends as an operator from $\mathbb{D}^{1,2}$ into $L^2\left(\Omega; L^2\left(\left[0,T\right], R^n\right)\right)$.

Now we modify the stochastic reaction-diffusion equation considered in [12] which can be described the density of a population at time $t \in [0,T]$ and at the point $x \in R$ as follows.

$$
\left\{
\begin{array}{l}
u\left(t,x\right) = x + \int_t^T \left[v^2\left(s\right) \triangle u\left(s,x\right) + u\left(s,x\right) + \nabla u\left(s,x\right) v\left(s\right)\right] \mathrm{d}s \\
\qquad + \int_t^T u\left(s,x\right) \mathrm{d}\overleftarrow{B}_s, \quad 0 \leq t \leq T,
\end{array}
\right.
\tag{6.4}
$$

and $x \in R$, $v \in \mathcal{U}_{ad}$. The two Brownian motions W and B are one-dimensional. Suppose we want to minimize the following performance criterion

$$J(v) = \mathbf{E}\left[\int_0^T \frac{v^\gamma(s)}{\gamma} \mathrm{d}s + u(0,x)\right],$$

where $\gamma \geq 1$. In this case the Hamiltonian gets the form

$$\begin{aligned} & H(t, X, Y, Z, v, p, q, k, h) \\ = & \frac{v^\gamma}{\gamma} - k(Y+Z) - pY + hv. \end{aligned}$$

Obviously, it is convex in (Y, Z, v). The corresponding FBDSDEs are

$$\begin{cases} X^{t,x}(s) = x + \int_t^s v(r)\,\mathrm{d}\overrightarrow{W}_r, \\ Y^{t,x}(s) = X^{t,x}(T) + \int_s^T (Y^{t,x}(r) + Z^{t,x}(r))\,\mathrm{d}r \\ \qquad + \int_s^T Y(r)\,\mathrm{d}\overleftarrow{B}_r - \int_s^T Z^{t,x}(r)\,\mathrm{d}\overrightarrow{W}_r, \quad 0 \leq t \leq s \leq T, \end{cases} \tag{6.5}$$

It is easy to obtain the solutions of (6.5) are

$$Y^{t,x}(s) = \mathbf{E}\left[X^{t,x}(T)\exp\{W_T - W_t + B_T - B_s\}\middle|\mathcal{F}_s\right]. \tag{6.6}$$

Besides, the adjoint processes are

$$\begin{cases} \mathrm{d}p(s) = (p(s) + k(s))\,\mathrm{d}s + p(t)\,\mathrm{d}\overrightarrow{W}_t - k(s)\,\mathrm{d}\overleftarrow{B}_s, \\ \mathrm{d}q(s) = h(s)\,\mathrm{d}\overrightarrow{W}_s, \\ p(t) = -1, \quad q(T) = -p(T), \qquad t \leq s \leq T. \end{cases} \tag{6.7}$$

The solutions of (6.7) are

$$\begin{aligned} p(s) &= \mathbf{E}\left[-\exp\{W_s + W_t + B_s - B_t\}\middle|\mathcal{F}_s\right], \\ q(s) &= \mathbf{E}\left[-p(T)\middle|\mathcal{F}_s^W\right], \\ h(s) &= D_s q(s), \quad \text{a.e., } 0 \leq t \leq s \leq T. \end{aligned} \tag{6.8}$$

The function

$$\begin{aligned} v &\to H(t, X, Y, Z, v, p, q, k, h) \\ &= \frac{v^\gamma}{\gamma} - kY - pY + hv. \end{aligned}$$

is minimum when

$$v(t) = (h(t))^{\frac{1}{\gamma-1}}, \quad 0 \leq t \leq T.$$

where, $h(t)$ are given by (6.8).

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References

[1] A. Bensoussan, Lectures on Stochastic Control, Lecture Notes in Mathematics, Vol. 972, Nonlinear Filtering and Stochastic Control, Proceeding, Cortona, 1981.

[2] A. Bensoussan, Stochastic maximum principle for distributed parameter system. *J. Franklin Inst.* 315 (1983), pp. 387–406.

[3] A. Bensoussan, Stochastic Control of Partially Observable Systems. Cambridge University Press 1992.

[4] J. M. Bismut, An introductory approach to duality in optimal stochastic control. SIAM Rev., 20 (1978), pp. 62–78.

[5] U. G. Haussmann, General necessary conditions for optimal control of stochastic system, Maht. Programm. Stud., 6 (1976), pp. 34–48.

[6] U. G. Haussmann, A stochastic maximum principle for optimal control of diffusions. *Pitman Research Notes in Mathematics* 151 (1987).

[7] S. Ji and X.Y. Zhou, A maximum principle for stochastic optimal control with terminal state constraints, and its applications. Communications in Information and Systems 6 (4) (2006) pp. 321–338.

[8] H.J. Kushner, Necessary conditions for continuous parameter stochastic optimization problems, SIAM J. Control, 10 (1972), pp. 550–565.

[9] R. E. Mortensen, Stochastic optimal control with noisy observations. *Int. J. Control* 4 (1966), pp. 455–464.

[10] D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands, Probab. Theory Related Fields, 78 (1988), pp. 535–581.

[11] M. Nisio, Optimal control for stochastic partial differential equations and viscosity solutions of bellman equations Nagoya Math. J. Vol. 123 (1991), pp. 13-37.

[12] B. Øksendal, Optimal Control of Stochastic Partial Differential Equations. Stochastic Anal. Appl. **23** (2005), pp. 165–179.

[13] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. Systems Control Letters. 14 (1990). pp. 55–61.

[14] E. Pardoux and S. Peng, Backward doubly stochastic differential equations and systems of quasilinear parabolic SPDEs, Probab. Theory Related Fields, 98 (1994), pp. 209–227.

[15] S. Peng, A general stochastic maximum principle for optimal control problems, SIAM J. Control, 28 (1990), pp. 966–979.

[16] S. Peng, Backward stochastic differential equations and application to optimal control. Applied Mathematics and Optimization 27 (4) (1993), pp. 125-144.

[17] S. Peng and Y. Shi, A Type of Time-Symmetric Forward-Backward Stochastic Differential Equations. C. R. Acad. Sci. Paris, Ser. I 336 (9) (2003), pp. 773-778.

[18] S. Peng and Z. Wu, Fully Coupled Forward-Backward Stochastic Differential Equations and Applications to Optimal Control. SIAM J. Control Optim. 37 (1999), pp. 825-843.

[19] L.S. Pontryagin, V.G. Boltyanskti, R.V. Gamkrelidze, E.F. Mischenko, The Mathematical Theory of Optimal Control Processes. Interscience, John Wiley, New York (1962).

[20] J. Shi and Z. Wu, The maximum principle for fully coupled forward-backward stochastic control system. Acta Automatica Sinica 32 (2) (2006), pp. 161-169.

[21] Z. Wu, Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems. Systems Sci. Math. Sci. 11 (3) (1998), pp. 249-259.

[22] W. Xu, Stochastic maximum principle for optimal control problem of forward and backward system. J. Australian Mathematical Society B37 (1995), pp. 172-185.

[23] J. Yong and X.Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer, New York 1999.

[24] Q. Zhang and H. Zhao, Stationary solutions of SPDEs and infinite horizon BDSDEs, J. Funct. Anal., 252 (2007), pp. 171C219.

[25] L. Zhang and Y.Shi, Maximum Principle for Forward-Backward Doubly Stochastic Control Systems and Applications. ESAIM, preprinted.