Extrinsic geometric flows on foliated manifolds, II *

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Abstract

Extrinsic Geometric Flows (EGF) for a codimension-one foliation have been recently introduced by authors as deformations of Riemannian metrics subject to quantities expressed in terms of its second fundamental form. In the paper we introduce soliton solutions to EGF and study their geometry for totally umbilical foliations, foliations on surfaces, and when the EGF is generated by the extrinsic Ricci tensor.

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Introduction

The Hamilton's *Ricci Flow* and the *Mean Curvature Flow* are among the central themes in recent Riemannian geometry. The special *soliton* solutions of these flows motivate the general analysis of the singularity formation. The collection of surveys [**CY**] demonstrates increase of interest in geometric flows $\partial_t g_t = h(g_t)$ of different types. The *Extrinsic Geometric Flow (EGF)* has been recently introduced by authors [**RW0**] as leaf-wise deformations of Riemannian metrics on a manifold M equipped with a codimension-one *foliation* $\mathcal{F}$ subject to quantities expressed in terms of the second fundamental form of leaves.

In the paper we begin studying the geometry of soliton solutions to EGF. Throughout the work, (M^{n+1}, g) is a compact Riemannian space with a codimension one transversely oriented foliation $\mathcal{F}$, ∇ the Levi-Civita connection of g, N the positively oriented unit normal to $\mathcal{F}$, $A : X \in T\mathcal{F} \mapsto -\nabla_X N$ the Weingarten operator of the leaves, which we extend to a $(1,1)$-tensor on M by $A(N) = 0$.

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The definition $S(X_1,X_2)=S(\hat X_1,\hat X_2)$ $(X_i\in TM)$ of the $\mathcal{F}$-*truncated* $(0,2)$-*tensor field* S ($\hat{}$ denotes the $T\mathcal{F}$-component) will be helpful throughout the paper. Let $\hat g$ and $g^\perp$ be components of g along $T\mathcal{F}$ and $T\mathcal{F}^\perp$, respectively.

Let $b: T\mathcal{F}\times T\mathcal{F}\to\mathbb{R}$ be the second fundamental form of (the leaves of) $\mathcal{F}$ with respect to N, and $\hat b$ its extension to the $\mathcal{F}$-truncated symmetric $(0,2)$-tensor field on M. In other words, $\hat b(N,\cdot)=0$ and $\hat b$ is dual to the extended Weingarten operator A. Denote by $\hat b_j$ the symmetric $(0,2)$-tensor fields on M dual to powers A^j of extended Weingarten operator,

$$\hat b_0(X,Y)=\hat g(X,Y),\quad \hat b_j(X,Y)=\hat g(A^jX,Y)\quad (j>0,\quad X,Y\in TM). \tag{1}$$

The symmetric functions $\tau_j=\operatorname{Tr}A^j=\sum_{i=1}^n k_i^j$ $(j\ge 0)$, are called *power sums* of the principal curvatures $\{k_i\}$ of $\mathcal{F}$. They can be expressed using *elementary symmetric functions* $\sigma_j=\sum_{i_1<\ldots<i_j}k_{i_1}\cdot\ldots\cdot k_{i_j}$ $(0\le j\le n)$, called *mean curvatures.* Certainly, τ_{n+i} are not independent: using Newton formulae

$$\begin{aligned}&\tau_j-\tau_{j-1}\sigma_1+\ldots+(-1)^{j-1}\tau_1\sigma_{j-1}+(-1)^j j\,\sigma_j=0\quad(1\le j\le n),\\&\tau_j-\tau_{j-1}\sigma_1+\ldots+(-1)^n\,\tau_{j-n}\sigma_n=0\quad(j>n),\end{aligned}$$

one can express the functions τ_{n+i} $(i>0)$ as polynomials of $\vec\tau:=(\tau_1,\ldots,\tau_n)$.

Definition 1 (see [**RW0**])**.** We distinguish two types of EGF, depending of functions f_j $(0\le j<n)$ in use (at least one of them is not identically zero):

(a) $f_j=f_j(p)\in C^2(M)$ and (b) $f_j=f_j(\vec\tau)\in C^2(\mathbb{R}^n)$.

Given f_j of type either (a) or (b), a family g_t of Riemannian metrics on $(M,\mathcal{F})$ is called an *Extrinsic Geometric Flow (EGF)*, whenever

$$\partial_t g_t=h(b_t),\quad\text{where } h(b_t)=\sum\nolimits_{j=0}^{n-1}f_j\,\hat b_j^{\,t} \tag{2}$$

and $\hat b_j^{\,t}$ are $(0,2)$-tensors dual to powers A_t^j of extended Weingarten operator A_t.

In [**RW0**], we proved the local existence and uniqueness theorems and estimated the existence time of solutions for some cases. The key step of the solution procedure for the EGF of type (b) is studying a system of quasilinear PDE's for the symmetric functions $\tau_1,\ldots,\tau_n$. The EGF preserves *totally umbilical* (i.e., $A=\lambda\,\hat{\mathrm{id}}$), *totally geodesic* (i.e., $A=0$) and *Riemannian* foliations [**RW0**]. (A foliation $\mathcal{F}$ on M is called Riemannian if there is a bundle-like metric g. In this case, N-curves are geodesics, i.e., $\nabla_N N=0$.)

For generic setting of f_j's (while $f_0(0)=0$), the *fixed points* of EGF are totally geodesic $(A_t\equiv 0)$ foliations only. Several classes of foliations appear as fixed points of the flow $\partial_t g_t=f(\vec\tau)\,\hat g_t$ for special choices of f, for example:

– foliations of constant τ_i, when $f=\tau_i-c$ (minimal for $i=1$ and $c=0$),

– parabolic foliations, when $f=\sigma_n$,

– totally umbilical foliations, when $f = n\,\tau_2 - \tau_1^2 = \sum_{i<j}(k_i - k_j)^2$,
– totally geodesic foliations, when $f(0) = 0$, etc.

Let $\mathrm{Diff}(M)$ be the diffeomorphism group of M. Introduce the notation
– $\mathcal{D}(\mathcal{F})$ the subgroup of $\mathrm{Diff}(M)$ preserving $\mathcal{F}$;
– $\mathcal{D}(\mathcal{F}, N)$ the subgroup of $\mathrm{Diff}(\mathcal{F})$ preserving both, $\mathcal{F}$ and N.

Recall that Fibration Theorem of D. Tischler (see for example [**CC**]) states that the property of a closed manifold

(i) *M admits a codimension one C^1-foliation invariant by a transverse flow*

is equivalent to any of the following conditions:

(ii) *M fibers over the circle S^1;*

(iii) *M supports a closed 1-form of a class C^1 without singularities.*

Taking into account the Theorem of R. Sacksteder, see for example [**CC**], one obtains in a class C^2 the following equivalent to any of (i), (ii) or (iii) condition:

(iv) *M admits a codimension one foliation without holonomy.*

Definition 2. We say that a solution $g_t = \hat g_t \oplus g_t^\perp$ to (2) is a *self-similar Extrinsic Geometric Soliton (EGS)* on $(M, \mathcal{F}, N)$ if there exist a smooth function $\sigma(t) > 0$ ($\sigma(0) = 1$), and a family of diffeomorphisms $\phi_t \in \mathcal{D}(\mathcal{F})$ such that $\phi_0 = \mathrm{id}_M$,

$$\hat g_t = \sigma(t)\, \phi_t^* \,\hat g_0. \tag{3}$$

In addition to (3), one should require for any $p \in M$, $t \ge 0$ and $0 \le j < n$:

$$(a)\quad f_j(\phi_t(p)) = f_j(p), \qquad (b)\quad f_j(\vec\tau^{\,0} \circ \phi_t^{-1}(p)) = f_j(\vec\tau^{\,0}(p)). \tag{4}$$

The reason of using (4) follows form the proof of Theorem 1.

By $\mathcal{M}(M)$, we denote the space of smooth Riemannian metrics on M of finite volume with N being a unit normal to $\mathcal{F}$. Let $\mathbb{R}_+$ acts on $\mathcal{M}(M)$ by scalings along $T\mathcal{F}$. (By Remark 1 in what follows, the Weingarten operator A and the principal curvatures of $\mathcal{F}$ are invariant under uniform scaling of the metric on $\mathcal{F}$).

The EGF, (2), may be regarded as a dynamical system on the quotient space $\mathcal{M}(M)/(\mathcal{D}(\mathcal{F}) \times \mathbb{R}_+)$. The solutions to (2) of the form (3) correspond to *fixed points* of the above dynamical system.

Question. *Given $(M, \mathcal{F}, N)$, N is a transversal vector field to $\mathcal{F}$,* $\operatorname{codim} \mathcal{F} = 1$, *and $f_j \in C^2(\mathbb{R}^{n+1})$ $(0 \le j < n)$, do there exist complete EGS metrics on M? If they do exist, study their properties, classify them, etc.*

The above Question is closely related to the basic problem (first posed by H. Gluck for geodesic foliations) in the theory of foliations.

Problem (see [**Rov**] and the survey [**CW**]). *Given (P), a property of Riemannian submanifolds expressed in terms of the second fundamental form and its invariants, and a foliated manifold $(M, \mathcal{F})$, decide if there exist (P)-metrics on $(M, \mathcal{F})$, that is, Riemannian metrics such that all the leaves enjoy the property (P). If they do exist, study their properties, classify them etc.*

Some solutions to Problem for the 3-sphere with a Reeb foliation $\mathcal{F}_R$ are the following: (i) *there exist metrics for which $\mathcal{F}_R$ is totally umbilic* [**BG**], and (ii) *there is no metric making $\mathcal{F}_R$ minimal (or totally geodesic).*

In the paper we introduce and study EGS as generalized fixed points of the EGF (see Theorems 1, 2), and characterize them for some particular cases: totally umbilical foliations (Theorems 3, 4), foliations on surfaces (Corollaries 1, 2 and Theorem 5), for the EGF generated by extrinsic Ricci tensor (Theorem 6).

1 Preliminaries

It is well known that Ricci tensor Ric is preserved by diffeomorphisms $\phi \in \mathrm{Diff}(M)$ in the sense that $\phi^*(\mathrm{Ric}(g)) = \mathrm{Ric}(\phi^* g)$, hence Ric is an *intrisic geometric tensor*. In the same way, the second fundamental form b of a foliation is invariant under diffeomorphisms preserving the foliation, hence b is an *extrinsic geometric tensor* for the class of foliations. More precisely, we have the following

Proposition 1. *Let $(M_i, \mathcal{F}_i, g_i)$ $(i = 1, 2)$ be Riemannian manifolds with a codimension 1 foliations, Weingarten operators A_i and the second fundamental forms $b^{(i)}$. If $\phi : M_1 \to M_2$ is a diffeomorphism such that $\mathcal{F}_1 = \phi^{-1}(\mathcal{F}_2)$, $g_1^\perp = \phi^* g_2^\perp$ and $\hat g_1 = \sigma \cdot \phi^* \hat g_2$ (for some positive function $\sigma : M \to \mathbb{R}$) then*

$$b^{(1)} = \sigma \cdot \phi^*(b^{(2)}), \qquad A_1 = (\phi_*)^{-1} A_2\, \phi_*, \qquad \vec\tau^{(2)} \circ \phi = \vec\tau^{(1)}, \tag{5}$$

where $\vec\tau^{(i)}$ is the set of τ-s for A_i. The tensors $h^{(i)} = \sum_{j=0}^{n-1} f_j(\vec\tau^{(i)})\, \hat b_j^{(i)}$ of type (b), and $h^{(1)} = \sum_{j=0}^{n-1} f_j(p)\, \hat b_j^{(1)}$, $h^{(2)} = \sum_{j=0}^{n-1} f_j(\phi(p))\, \hat b_j^{(2)}$ of type (a) satisfy

$$h^{(1)}(g_1) = \sigma \cdot \phi_*(h^{(2)}(g_2)). \tag{6}$$

Proof follows from the direct computation. We include it here just for the convenience of a reader.

Notice that for any vector field $X \subset T\mathcal{F}_1$ tangent to $\mathcal{F}_1$, $\phi_* X = d\phi(X) \subset T\mathcal{F}_2$ is a vector field tangent to $\mathcal{F}_2$. For the pullback metric $g_1 = \sigma \cdot \phi^* \hat g_2 \oplus \phi^* g_2^\perp$ we have $g_1(X, Y) = \sigma \cdot g_2(\phi_* X, \phi_* Y)$, where $X, Y \subset T\mathcal{F}_1$. Let N_2 be the (local) unit normal field of $\mathcal{F}_2$. Then $N_1 = \phi_*^{-1} N_2$ is the unit normal of $\mathcal{F}_1$. In fact, $g_1(X, N_1) = g_2(\phi_* X, N_2) = 0$ $(X \in T\mathcal{F}_1)$, and

$$g_1(N_1, N_1) = \phi^*(g_2)(\phi_*^{-1} N_2, \phi_*^{-1} N_2) = g_2(\phi_* \phi_*^{-1} N_2, \phi_* \phi_*^{-1} N_2) = g_2(N_2, N_2).$$

For the second fundamental form $b_2 : T\mathcal{F}_2 \times T\mathcal{F}_2 \to \mathbb{R}$ of $\mathcal{F}_2$, we obtain a $(0,2)$-tensor field $\phi^* b^{(2)} : T\mathcal{F}_1 \times T\mathcal{F}_1 \to \mathbb{R}$. Denote $\overset{i}{\nabla}$ the Levi-Civita connection of (M_i, g_i). For any $X, Y \subset T\mathcal{F}_1$ we have

$$\begin{aligned}(\phi^* b^{(2)})(X, Y) &= b^{(2)}(\phi_* X, \phi_* Y) = -g_2(\overset{2}{\nabla}_{\phi_* X} N_2, \phi_* Y) = -g_2(\overset{2}{\nabla}_{\phi_* X} \phi_* N_1, \phi_* Y)\\ &= -g_2(\phi_* \overset{1}{\nabla}_X N_1, \phi_* Y) = -\phi_*(g_2)(\overset{1}{\nabla}_X N_1, Y) = \sigma \cdot b^{(1)}(X, Y).\end{aligned}$$

Hence, the second fundamental form of a foliation is an *extrinsic geometric tensor*. Consequently, Weingarten operators are related by $A_2 = \phi_\sharp A_1$. In fact, we have

$$\phi_*(A_1 X) = A_2(\phi_* X) \quad (X \subset T\mathcal{F}_1) \quad \Rightarrow \quad A_2 = \phi_* A_1 \phi_*^{-1} \overset{def}{=} \phi_\sharp A_1.$$

From the above and definition of $h^{(i)}$, compare with (4), it follows (6). $\square$

Next lemma is standard and we omit its proof.

Lemma 1. *Let $(M,\, g = \hat g \oplus g^\perp)$ be a Riemannian manifold with a codimension-1 foliation $\mathcal{F}$ and a unit normal N. Define a metric $\bar g = e^{2\varphi}\hat g \oplus g^\perp$, where $\varphi : M \to \mathbb{R}$ is a smooth function. Then*

$$\bar b = e^{2\phi}(b - N(\varphi)\,\hat g), \quad \bar A = A - N(\varphi)\,\widehat{\mathrm{id}}, \quad \bar\tau_i = e^{2\phi}(\tau_i - n(N(\varphi))^i).$$

Remark 1. If φ is constant, then by Lemma 1 we have
(i) $\bar b = e^{2\varphi}\, b$ (the second fundamental forms);
(ii) $\bar A = A$ (the Weingarten operators);
(iii) $\bar\tau_j = \tau_j$ (the power sums).

Vector fields infinitesimally represent diffeomorphisms. Let $\mathcal{X}(M)$ be Lie algebra of all vector fields on M with the bracket operation. Introduce the notation
– $\mathcal{X}(\mathcal{F})$ the set of vector fields on M preserving $\mathcal{F}$,
– $\mathcal{X}(\mathcal{F}, N)$ the set of vector fields on M preserving both, $\mathcal{F}$ and N.
By Jacobi identity, $\mathcal{X}(\mathcal{F})$ is a subalgebra of the Lie algebra $\mathcal{X}(M)$. Moreover, for any $X \in \mathcal{X}(\mathcal{F})$ (or $X \in \mathcal{X}(\mathcal{F}, N)$) there exists a family $\phi_t \in \mathcal{D}(\mathcal{F})$ (resp., $\phi_t \in \mathcal{D}(\mathcal{F}, N)$) such that $X = d\phi_t/dt$ at $t = 0$.

Remark 2. (i) The following conditions are equivalent, [**Wa**]:

$$X \in \mathcal{X}(\mathcal{F}) \iff [X, T\mathcal{F}] \subset T\mathcal{F}. \tag{7}$$

If $\phi_t \in \mathcal{D}(\mathcal{F}, N)$, then we have $\varphi_{t*}N = N \circ \varphi_t$. For $X \in \mathcal{X}(\mathcal{F}, N)$, the above yields $\mathcal{L}_X N = 0$. From this and (7) we conclude that

$$X \in \mathcal{X}(\mathcal{F}, N) \iff [X, T\mathcal{F}] \subset T\mathcal{F} \ \text{ and } \ [X, N] = 0. \tag{8}$$

(ii) Denote $\nabla^{\mathcal{F}} f$ the $\mathcal{F}$-gradient of a function f. Using (7), for any $Y \perp N$ we get

$$0 = g([X, Y], N) = -\mu g(\nabla_N N, Y) + Y(\mu).$$

By the above, a vector field $X = \mu\, N$ ($\mu \in C^1(M)$) preserves $\mathcal{F}$ if and only if

$$\nabla^{\mathcal{F}} \log\, \mu = -\nabla_N N.$$

In particular, if $\mathcal{F}$ is a Riemannian foliation, then $N \in \mathcal{X}(\mathcal{F}, N)$.

The *Lie derivative* of a $(0,p)$-tensor S w. r. to a vector field X is given by

$$(\mathcal{L}_X S)(Y_1,\ldots,Y_p) = X(S(Y_1,\ldots,Y_p)) - \sum\nolimits_{i=1}^p S(Y_1,\ldots,\mathcal{L}_X Y_i,\ldots,Y_p). \tag{9}$$

Lemma 2. *For any vector fields $X\in\mathcal{X}(\mathcal{F})$ and $Y_i\in\Gamma(T\mathcal{F})$, we have*

$$\begin{aligned}(\mathcal{L}_X\,\hat g)(Y_1,Y_2) &= \hat g(\nabla_{Y_1}X,Y_2)+\hat g(\nabla_{Y_2}X,Y_1),\\ (\mathcal{L}_X\,\hat g)(N,Y_i) &= \hat b_1(X,Y_i)-\hat g(\nabla_{X^\perp}N,Y_i),\quad (\mathcal{L}_X\,\hat g)(N,N)=0.\end{aligned}$$

Proof. Using $\nabla g=0$, we obtain

$$(\nabla_X\,\hat g)(N,N)=(\nabla_X\,\hat g)(Y_1,Y_2)=0,\quad (\nabla_X\,\hat g)(N,Y_i)=\hat b_1(X,Y_i)+\hat g(\nabla_{X^\perp}N,Y_i).$$

By above and (9), we have

$$\begin{aligned}(\mathcal{L}_X\hat g)(Y_1,Y_2) &= D_X(\hat g(Y_1,Y_2))-\hat g([X,Y_1],Y_2)-\hat g(Y_1,[X,Y_2])\\ &= \hat g(\nabla_{Y_1}X,Y_2)+\hat g(\nabla_{Y_2}X,Y_1),\\ (\mathcal{L}_X\,\hat g)(N,Y_i) &= -\hat g([X,N],Y_i)=b(X,Y_i)-\hat g(\nabla_{X^\perp}N,Y_i).\end{aligned}$$

Similarly, $(\mathcal{L}_X\,\hat g)(N,N)=0$. Notice that $(\mathcal{L}_N\,\hat g)(N,Y_i)=-\hat g(\nabla_N N,Y_i)$. □

2 Introducing extrinsic geometric solitons

Proposition 2. *Let g_t be a self-similar soliton (see Definition 2). Then*

$$b_t=\sigma(t)\,\phi_t^*\,b_0,\quad A_t=\phi_{t*}A_0\,\phi_{t*}^{-1},\quad \vec\tau^{\,t}=\vec\tau^{\,0}\circ\phi_t^{-1}, \tag{10}$$

$$h(b_t)=\sigma(t)\,\phi_t^*\,h(b_0). \tag{11}$$

Proof. Let g_t be the EGF of type (b). By (5) and Remark 1(i) we have

$$b_t=\sigma(t)\,b(\phi_t^*\,g_0)=\sigma(t)\,\phi_t^*\,b_0.$$

Similarly, for $X\in T\mathcal{F}$ we get

$$A_tX=A(\phi_t^*\,g_0)X=\phi_{t*}A_0\,\phi_{t*}^{-1}X.$$

Taking the trace of $A_t^j=\phi_{t*}A_0^j\,\phi_{t*}^{-1}$ we get the identity for τ's.

From the above, (4) and the definition of $h(b_t)$ in (2), equality (11) follows.

The case of the EGF of type (a) is similar. □

We are looking at what initial conditions give rise to self-similar EGS. Differentiating (3) yields

$$h(b_t)=\dot\sigma(t)\,\phi_t^*\hat g_0+\sigma(t)\phi_t^*(\mathcal{L}_{X(t)}\hat g_0), \tag{12}$$

where $X(t)\in\mathcal{X}(\mathcal{F})$ is a family of vector fields such that $X(t)=d\phi_t/dt$.

Since $\phi_0^*=\mathrm{id}$, one may omit the pull-back in (12),

$$h(b_0)=\dot\sigma(t)/\sigma(t)\,\hat g_0+\mathcal{L}_{X(t)}\hat g_0. \tag{13}$$

Motivated by the above, we accept the following

Definition 3. A pair (g, X) consisting of a metric $g = \hat{g} \oplus g^{\perp}$ on $(M, \mathcal{F})$, and a complete vector field $X \in \mathcal{X}(\mathcal{F})$ satisfying for some $\varepsilon \in \mathbb{R}$ the condition

$$h(b) = \varepsilon\,\hat{g} + \mathcal{L}_X \hat{g}, \quad (\text{where} \quad h(b) = \sum\nolimits_{0 \le j < n} f_j\,\hat{b}_j\,) \tag{14}$$

is called an *EGS structure*. We say that X is *the vector field the EGS is flowing along*. In addition, one should require the infinitesimal analogue of (4)

$$(a)\ \nabla_X f_j(p) = 0, \qquad (b)\ \nabla_X f_j(\vec{\tau}(p)) = 0, \qquad 0 \le j < n \tag{15}$$

for any $p \in M$. If $X = \nabla f$ for some $f \in C^1(M)$, we have a *gradient EGS*. In this case, $\frac{1}{2}\mathcal{L}_{\nabla f}\,\hat{g} = \widehat{\mathrm{Hess}}_g f$ (the truncated hessian), and

$$h(b) = \varepsilon\,\hat{g} + 2\,\widehat{\mathrm{Hess}}_g f. \tag{16}$$

If X in (14) is orthogonal to N, we have an *$\mathcal{F}$-EGS*, and if X is parallel to N, we have an *N-EGS*. For a *gradient $\mathcal{F}$-EGS* one has $N(f) = 0$. For a *gradient N-EGS*, the function f is constant along the leaves.

Remark 3. For f_j of type (b) we have $\nabla_X f_j(\vec{\tau}(p)) = \sum_{i=1}^n f_{j,\tau_i}(\vec{\tau}(p))\,\nabla_X \tau_i(p)$ at any $p \in M$. For general f_j, the gradients ∇f_j $(0 \le j < n)$ are linearly independent almost everywhere (in $\mathbb{R}^n$ of variables $\vec{\tau}$). In this case, if X belongs to the EGS structure then from (15)(b) it follows the constancy of τ's along X:

$$\nabla_X \tau_i(p) = 0, \quad p \in M, \quad 1 \le i \le n. \tag{17}$$

For the extrinsic Ricci flow (Section 4) we have only $\nabla_X \tau_1(p) = 0$ for all $p \in M$.

Next we observe that Definitions 2 and 3 are, in fact, equivalent.

Theorem 1. *(a) If g_t is a self-similar EGS on M, then there exists $X \in \mathcal{X}(\mathcal{F})$ such that $g = g_0$ solves (14) – (15). (b) Conversely, given $X \in \mathcal{X}(\mathcal{F})$ and a solution g to (14) – (15), there is a function $\sigma(t) > 0$ and a family $\phi_t \in \mathcal{D}(\mathcal{F})$ such that g_t, defined by (3) – (4) on $(M, \mathcal{F})$, is a solution to (2) with $g_0 = g$.*

Proof. (a) Recall that $\sigma(0) = 1$ and $\phi_0 = \mathrm{id}$. Let $Y(t) \in \mathcal{X}(\mathcal{F})$ be the family of vector fields generated by diffeomorphisms ϕ_t. Then we have

$$h(b_0) = \partial_t g_{t|\,t=0} = \sigma'(0)\hat{g}_0 + \mathcal{L}_{Y(0)}\hat{g}_0.$$

This implies that g_0 satisfies (14) with $\varepsilon = \sigma'(0)$ and $X = Y(0)$.

(b) Conversely, suppose that g_0 satisfies (14). Denote $\sigma(t) := 1+\varepsilon\,t$, hence $\varepsilon = \sigma'(t)$. Define a 1-parameter family of vector fields on M by $Y(t) := (1/\sigma(t))X$. Let $\phi_t \in \mathcal{D}(\mathcal{F})$ be diffeomorphisms generated by the family $Y(t)$, where $\phi_0 = \mathrm{id}_M$. A smooth 1-parameter family of metrics on M defined by $g_t := \sigma(t)\psi_t^*\hat{g}_0 \oplus \psi_t^*(g_0^{\perp})$ is of the form (3). Moreover,

$$\partial_t g_t = \sigma'(t)\psi_t^*(\hat{g}_0) + \sigma(t)\psi_t^*(\mathcal{L}_{Y(t)}\hat{g}_0) = \psi_t^*(\varepsilon\hat{g}_0 + \mathcal{L}_X\hat{g}_0) = \psi_t^*(h(b_0)).$$

Since X and $Y(t)$ generate the same family of diffeomorphisms up to re-parametrization, by (4) we have $f_j(\vec{\tau}^0 \circ \psi_t^{-1}) = f_j(\vec{\tau}^0)$ for $t \geq 0$. Hence, by definition, $h(b_t) = \sum_{j=0}^{n-1} f_j(\vec{\tau}^t)\hat{b}_j^t$, see $(10)_3$. Using (11) of Proposition 2, we obtain

$$\begin{aligned}\psi_t^*(h(b_0)) &= \psi_t^*\big(\sum\nolimits_{j=0}^{n-1} f_j(\vec{\tau}^0)\hat{b}_j^0\big) = \sum\nolimits_{j=0}^{n-1} f_j(\vec{\tau}^0 \circ \psi_t^{-1})\, \psi_t^*\hat{b}_j^0 \\ &= \sum\nolimits_{j=0}^{n-1} f_j(\vec{\tau}^t)\, \sigma^{-1}(t)\, \hat{b}_j^t = \sigma^{-1}(t)\, h(b_t).\end{aligned}$$

Hence $h(b_t) = \sigma(t)\, \psi_t^*(h(b_0))$, and we conclude that g_t is a solution to (2). □

Theorem 2 (Canonical form)**.** *Let a self-similar EGS g_t be unique among soliton solutions to (2) with initial metric g_0. Then there is a 1-parameter family $\psi_t \in \mathcal{D}(\mathcal{F})$ and a constant $\varepsilon \in \{-1, 0, 1\}$ such that*

$$\hat{g}_t = (1 + \varepsilon\, t)\, \psi_t^* \hat{g}_0. \tag{18}$$

The cases $\varepsilon = -1, 0, 1$ in (18) correspond to *shrinking, steady*, or *expanding* EGS.

Proof is similar to the one of [**CK2**, Proposition 1.3]. For convenience of a reader we provide it here in the case $\ddot{\sigma}(0) \neq 0$. From (13) it follows

$$h(b_0) = (\log \sigma(t))'\, \hat{g}_0 + \mathcal{L}_{X(t)}\hat{g}_0, \tag{19}$$

where $X(t) \in \mathcal{X}(\mathcal{F})$ is a family of vector fields such that $X(t) = d\phi_t/dt$. Differentiating (19) with respect to t gives

$$(\log \sigma(t))''\hat{g}_0 + \mathcal{L}_{\dot{X}(t)}\hat{g}_0 = 0. \tag{20}$$

Let $Y_0 = -\dot{X}_1/(\log \sigma)''(0)$. We then have $\mathcal{L}_{Y_0}\hat{g}_0 = \hat{g}_0$. Substituting this into (13), we have for all t

$$h(b_0) = \mathcal{L}_{\dot{\sigma}(t) Y_0 + (\log \sigma(t))''\, X(t)}\hat{g}_0.$$

Put $X_0 = \dot{\sigma}(0)Y_0 + \sigma(0)\, X(0)$. Then $h(b_0) = \mathcal{L}_{X_0}\hat{g}_0$. Let $\psi_t \in \mathcal{D}(\mathcal{F})$ be a group of diffeomorphisms generated by X_0. We will check that

$$\hat{\tilde{g}}_t = \psi_t^* \hat{g}_0. \tag{21}$$

is the EGF with the initial conditions g_0 (and $h(b_0)$), and that it is a steady (i.e., $\sigma(t) = 0$ for all t) EGS. Indeed, differentiating (21), we have by (11),

$$\partial_t \tilde{g}_t = \psi_t^* \mathcal{L}_{X_0}\hat{g}_0 = \psi_t^*(h(b_0)) = h(\psi_t^* b_0) = h(\tilde{b}_t).$$

Thus $g_t = \tilde{g}_t$, by uniqueness assumption for EGS solutions to our flow with initial metric g_0. By replacing ϕ_t by ψ_t we have $\sigma(t) \equiv 1$ in (3). □

Remark 4. Equation (14) yields a rather strong condition on the EGS structure (g, X). For example, contracting (14) with g (tracing) and using the identity $\operatorname{Tr}\mathcal{L}_X\hat{g} = 2\operatorname{div}_{\mathcal{F}} X$ (see Lemma 2) yields

$$\operatorname{Tr}_g h(b) = n\,\varepsilon + 2\operatorname{div}_{\mathcal{F}} X. \tag{22}$$

For a gradient EGS, (22) means

$$\operatorname{Tr}_g h(b) = n\,\varepsilon + 2\,\Delta_{\mathcal{F}} f. \tag{23}$$

Proposition 3. *(a) Let (g, X) be the EGS structure on $(M, \mathcal{F})$. If a leaf L is compact, then*

$$n\,\varepsilon = \int_L \operatorname{Tr}_g h(b)\, d\operatorname{vol}_{g,L}/\operatorname{vol}(L, g). \tag{24}$$

(b) Let M be compact and either $X \perp N$ and $\nabla_N N = 0$ or $X||N$ and τ_1=0. Then

$$n\,\varepsilon = \int_M \operatorname{Tr}_g h(b)\, d\operatorname{vol}_g/\operatorname{vol}(M, g). \tag{25}$$

In particular, if $\mathcal{F}$ is Riemannian with minimal leaves, then (25) holds.

Proof. Integrating (22) and using the Divergence Theorem we obtain (24). From $\operatorname{div}(fN) = f\operatorname{div} N + N(f) = -\tau_1 f + N(f)$, by the Divergence theorem, $\int_M \operatorname{div}(fN)\, d\operatorname{vol} = 0$, we obtain

$$\int_M N(f)\, d\operatorname{vol} = \int_M \tau_1 f\, d\operatorname{vol}.$$

Next, we have $g(\nabla_N X, N) = Ng(X, N) - g(X, \nabla_N N)$, and by the above,

$$\int_M g(\nabla_N X, N)\, d\operatorname{vol}_g = \int_M \big(\tau_1\, g(X, N) - g(X, \nabla_N N)\big) d\operatorname{vol}_g.$$

Therefore, in case (b), integrating of (22) implies (25). If $\mathcal{F}$ is Riemannian with minimal leaves, then $\tau_1 = 0$ and $\nabla_N N = 0$. □

Proposition 4. *Equation (14) for EGS with $X = \mu\, N$ ($\mu : M \to \mathbb{R}_+$) reads as*

$$h(b) = \varepsilon\hat{g} - 2\,\mu\,\hat{b}_1.$$

For Riemannian foliations we certainly have $\mu \equiv 1$.

Proof. From (14) and Lemma 2 (for $X = \mu\, N$) we obtain

$$(\mathcal{L}_{\mu N}\,\hat{g})(Y_1, Y_2) = -2\,\mu\,\hat{b}_1(Y_1, Y_2), \quad Y_i \perp N. \qquad \square,$$

Example 1. Let $h(b) = \hat{b}_1$. Any metric making $(M, \mathcal{F})$ a Riemannian foliation, is a steady EGS with $X = N$. Indeed, for a bundle-like metric g, by Lemma 2, N preserves $\mathcal{F}$, and we certainly have $h(b) = \frac{1}{2}\,\mathcal{L}_N\hat{g}$.

3 Totally umbilical extrinsic geometric solitons

For a totally umbilical foliation $\mathcal{F}$ with the normal curvature λ, we have $\tau_j = n\lambda^j$. Hence $h(b) = \psi(\lambda)\,\hat g$ holds, and the EGF of type (b) is given by

$$\partial_t g_t = \psi(\lambda_t)\,\hat g_t, \quad \text{where} \quad \psi(\lambda) := \sum\nolimits_{j=0}^{n-1} f_j(n\lambda, n\lambda^2, \ldots, n\lambda^n)\,\lambda^j \in C^2. \tag{26}$$

In this case, λ_t obeys the PDE, see [**RW0**],

$$\partial_t\lambda_t + (1/2)N(\psi(\lambda_t)) = 0. \tag{27}$$

If (g, X) is the EGS structure with totally umbilical metric, by (17) we have

$$\nabla_X\lambda = 0.$$

Recall that the EGF preserve total umbilicity of $\mathcal{F}$.

Proposition 5. [**RW0**] *Let g_t $(0 \le t < \varepsilon)$ be the EGF (2) on $(M, \mathcal{F})$ ($\mathcal{F}$ is a transversally oriented codimension-one foliation). If $\mathcal{F}$ is totally umbilical for g_0, then $\mathcal{F}$ is totally umbilical for any g_t, and (26) is satisfied.*

Remark 5. Let $\mathcal{F}$ be a codimension-1 totally umbilical foliation with the metric

$$g_t = e^{2\,\phi_t}\hat g_0 \oplus g_0^\perp, \tag{28}$$

where $\phi_t : M \to \mathbb{R}$ $(\phi_0 = 0)$ are smooth functions. We claim that

$$2\,\partial_t\phi_t = \psi\big(\lambda - N(\phi_t)\big). \tag{29}$$

Indeed, by Lemma 1, $A_t = A_0 - N(\phi_t)\,\widehat{\rm id}$, hence, $\lambda_t = \lambda - N(\phi_t)$. By Lemma 1, we also have $b_t = e^{2\,\phi_t}(\lambda - N(\phi_t))\,\hat g_0 = (\lambda - N(\phi_t))\,\hat g_t$. Similarly, $b_t^j = (\lambda - N(\phi_t))^j\,\hat g_t$. Differentiating (28) yields $\psi(\lambda_t)\hat g_t = h(b_t) = \partial_t g_t = (2\partial_t\phi_t)\,\hat g_t$. Hence, $2\,\partial_t\phi_t = \psi(\lambda_t, t)$ that gives us the non-linear PDE (29).

One can solve explicitly for ϕ_t only the particular case $\psi = c_1\lambda + c_2$ of the problem $\partial_t g_t = \psi(\lambda_t)\,\hat g_t$, when (29) becomes linear of the form

$$2\,\partial_t\phi_t + N(\phi_t) = c_1\,\lambda_t + c_2. \tag{30}$$

In general, the non-linear equation (29) is difficult to solve, and we apply the EGF in two steps, see also [**RW0**]: first we find λ_t from (27), then g_t from (26).

Example 2.

(a) Let $h(b) = \varepsilon\hat g$. Then $(g_0,\, X)$ with $X = 0$ is the EGS structure on $(M, \mathcal{F})$. Notice that the metric $g_t = e^{\varepsilon\, t}\,\hat g_0 \oplus g_0^\perp$ satisfies PDE $\partial_t g_t = \varepsilon\,\hat g_t$. By Remark 1, $\hat b_j^t = e^{\varepsilon\, t}\,\hat b_j^0$ and $\vec\tau^{\,t} = \vec\tau^{\,0}$, hence $h(b_t) = e^{\varepsilon\, t}\,h(b_0) = e^{\varepsilon\, t}\,\varepsilon\,\hat g_0 = \varepsilon\,\hat g_t$.

(b) Let $\mathcal{F}$ be a totally umbilical foliation on (M, g_0) with $\lambda = \text{const}$. Then the functions $\phi_t = \frac{1}{2}\,\psi(\lambda)\,t$ $(t \in \mathbb{R})$ are constant on M and generate Riemannian metrics (28) $\mathcal{F}$-homothetic to g_0. (Indeed, $\mathcal{F}$ is totally umbilical with respect to each of g_t and its normal curvatures λ_t are constant, equal to λ).

We have $h(b_0) = \psi(\lambda)\hat{g}_0$. Hence g_0 is the EGS with $X = 0$ and $\varepsilon = \psi(\lambda)$. Moreover, $g_t = e^{\psi(\lambda)\,t}\,\hat{g}_0 \oplus g_0^\perp$ is a self-similar EGS with $\phi_t = \mathrm{id}_M$ and $\sigma(t) = e^{\psi(\lambda)\,t}$. For a specific case of a totally geodesic foliation (i.e., $\lambda = 0$, if such g_0 exists on $(M, \mathcal{F})$), the family $g_t = e^{f_0(0)\,t}\,\hat{g}_0 \oplus g_0^\perp$ is also a self-similar EGS.

(c) Let (g, X) be a EGS structure on $(M, \mathcal{F})$. If $\mathcal{F}$ is totally umbilical with the normal curvature λ, then X is a leaf-wise conformally Killing field: $\mathcal{L}_X \hat{g} = (\psi(\lambda) - \varepsilon)\,\hat{g}$. If $\mathcal{F}$ is totally geodesic (hence $\psi = f_0(0)$), then X is an *infinitesimal homothety along leaves* with the factor $C = f_0(0) - \varepsilon$: $\mathcal{L}_X \hat{g} = C\,\hat{g}$. In particular, X is a *leaf-wise Killing field* when $f_0(0) = \varepsilon$. This happens, when M is a surface of revolution in $M^{n+1}(c)$ foliated by parallels, see Example 6 in what follows.

Definition 4 (see [**Nir**]). Denote the torus $\mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$ by T^{n+1}. For $v \in \mathbb{R}^{n+1}$, let $R_v^t(x) := x + tv$ be the flow on T^{n+1} induced by a "constant" vector field X_v. We say $v \in \mathbb{R}^{n+1}$ is *Diophantine* if there is $s > 0$ such that $\inf\limits_{u \in \mathbb{Z}^{n+1} \setminus \{0\}} |\langle u, v\rangle| \cdot \|u\|^s > 0$, where $\langle\ ,\ \rangle$ and $\|\cdot\|$ are the Euclidean inner product and norm in $\mathbb{R}^{n+1}$. When v is Diophantine, we call R_v a *Diophantine linear flow*.

Theorem 3. *Let $\mathcal{F}$ be a codimension-one totally umbilical foliation on a torus (T^{n+1}, g) $(n > 0)$, and $h(b) = \sum_{0 \le j < n} f_j\,\hat{b}_j$, see (2). Suppose that X is a unit vector field on $T\mathcal{F}$ with the conditions*

$$\nabla_X X \perp T\mathcal{F}, \qquad R(X, Y)Y \in T\mathcal{F} \qquad (Y \in T\mathcal{F}). \tag{31}$$

If X-flow is conjugate to a Diophantine liner flow R_v, then there exists $f : \mathrm{T}^{n+1} \to \mathbb{R}$ such that (g, fX) is a EGS structure.

Proof. For a totally umbilical foliation with the normal curvature λ, the Weingarten operator is conformal: $A = \lambda\,\hat{\mathrm{id}}$. By $(31)_2$ and the Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = R(X, Y)N^\top,$$

we have $\lambda = \mathit{const}$ along X-curves. By $(31)_1$, the X-curves are $\mathcal{F}$-geodesics.

Let $(g, \tilde{X})$ be a EGS structure with $\tilde{X} = fX$. For a totally umbilical metric g, the EGS equations (14) reduce to the PDE, see also (22),

$$\psi(\lambda) = \varepsilon + (2/n)\,\mathrm{div}_{\mathcal{F}}\,\tilde{X}. \tag{32}$$

From above and the known identity $\mathrm{div}_{\mathcal{F}}\, fX = f\,\mathrm{div}_{\mathcal{F}}\,X + X(f)$ it follows that $\mathrm{div}_{\mathcal{F}}\,\tilde{X} = X(f)$. We are looking for solution of PDE, see (34),

$$\psi(\lambda) - \varepsilon = (2/n)\,X(f). \tag{33}$$

Since X-flow is conjugate to a Diophantine linear flow, by Kolmogorov theorem, see [**Nir**], the above PDE has a solution $(f, \varepsilon) \in C^\infty(\mathrm{T}^{n+1}) \times \mathbb{R}$. $\square$

Remark 6. The EGF of type (b) for a foliation on a surface (M^2, g) is given by

$$\partial_t g_t = \psi(\lambda)\, \hat{g}_t,$$

where $\lambda = \tau_1$ is the geodesic curvature of the leaves (curves), and $\psi = f_0 \in C^2(\mathbb{R})$. The EGS equations (14) on $(M^2, \mathcal{F})$ reduce to the PDE, see (32) with $n = 1$,

$$\psi(\lambda) = \varepsilon + 2\ \mathrm{div}_{\mathcal{F}} X. \tag{34}$$

Assuming $\psi' \neq 0$ almost everywhere, by (17) we also obtain

$$X(\lambda) = 0. \tag{35}$$

Example 3. Consider biregular foliated coordinates (x_0, x_1) on M^2 (see [**CC**], Section 5.1). Then the coordinate vectors are directed along N and $\mathcal{F}$, respectively. The metric in biregular foliated coordinates has the form $g = g_{00}\, dx_0^2 + g_{11}\, dx_1^2$. Let $X = X^0\partial_0 + X^1\partial_1$. Recall that $h(b) = \psi(\lambda)\, g_{11}$ and, by [**RW0**, Lemma 4] with $n = 1$, $\lambda = -\frac{1}{2\sqrt{g_{00}}}(\log g_{11})_{,0}$. Using $g_{01} = 0$, we obtain,

$$\mathrm{div}_{\mathcal{F}} X = g(\nabla_{\partial_1} X, \partial_1) = [\,\partial_1(X^1) + X^0\Gamma^1_{01} + X^1\Gamma^1_{11}]\, g_{11},$$

where $\Gamma^1_{01} = (1/2)(\log g_{11})_{,0}$ and $\Gamma^1_{11} = (1/2)(\log g_{11})_{,1}$. By the above, the EGS equation (34) has the form

$$\psi(\lambda) - \varepsilon = 2\,\partial_1(X^1) g_{11} + X^0 g_{11,0} + X^1 g_{11,1}. \tag{36}$$

Finally, from $[X, \partial_1] \perp \partial_0$, see (7), and

$$[X, \partial_1] = [X^0\partial_0, \partial_1] + [X^1\partial_1, \partial_1] = -\partial_1(X^0)\partial_0 - \partial_1(X^1)\partial_1$$

we obtain $\partial_1(X^0) = 0$. If either (i) $X^1 = 0$ and $X^0 \neq 0$ or (ii) $X^0 = 0$ and $X^1 \neq 0$, then (35) reads, respectively, as

$$(i)\ (\,{g_{00}}^{-1/2}(\,\log g_{11})_{,0}\,)_{,0} = 0, \qquad (ii)\ (\,{g_{00}}^{-1/2}(\,\log g_{11})_{,0}\,)_{,1} = 0.$$

From Theorem 3 it follows

Corollary 1. *Let a unit vector field X on a torus (T^2, g) defines a foliation $\mathcal{F}$ by curves of constant geodesic curvature λ. If X-flow is conjugated to a Diophantine liner flow R_v, then there exists $f : \mathrm{T}^2 \to \mathbb{R}$ such that (g, fX) is a EGS structure.*

Proof. If $\tilde{X} = fX$, we have $\mathrm{div}_{\mathcal{F}} \tilde{X} = X(f)$. We are looking for solution of

$$\psi(\lambda) - \varepsilon = 2\, X(f), \tag{37}$$

see (34). Since X-flow is conjugated to a Diophantine linear flow, by Kolmogorov theorem, see [**Nir**], the above PDE has a solution $(f, \varepsilon) \in C^\infty(\mathrm{T}^2) \times \mathbb{R}$. □

Notice that if $\psi \in C^2(\mathbb{R})$ then the following function belongs to C^1:

$$\mu = \begin{cases} -\frac{n}{2}(\psi(\lambda) - \psi(0))/\lambda, & \lambda \neq 0, \\ -\frac{n}{2}\psi'(0), & \lambda = 0. \end{cases} \tag{38}$$

Theorem 4. *Let $\mathcal{F}$ be a totally umbilical foliation on (M, g), $\psi \in C^2(\mathbb{R})$ is given in (26) and satisfies $\psi' \neq 0$. Then the following properties are equivalent:*

(1) the normal curvature λ of $\mathcal{F}$ satisfies $N(\lambda) = 0$;

(2) $(g, \mu N)$, where μ is given in (38), is the EGS structure with $\varepsilon = \psi(0)$.

Proof. (1) $\Rightarrow$ (2): The EGS equations (for a totally umbilical metric g and the vector field $X = \mu\, N$) are, see Proposition 4 or (32),

$$\psi(\lambda) - \varepsilon = -(2/n)\,\mu\,\lambda, \qquad X(\lambda) = 0. \tag{39}$$

For $X = \mu N$ and μ given in (38), the above (39) are satisfied. Hence, by Definition 3, $(g, \mu N)$ is the EGS structure.

(2) $\Rightarrow$ (1): Using Definition 3, (35) and $\psi' \neq 0$, we obtain the equality $\mu N(\lambda) = 0$ with μ given in (38). Denote $\Omega = int\{p \in M : \mu = 0\}$ – an open set. Indeed, $N(\lambda) = 0$ on $M \setminus \Omega$. By (38), we have $\psi(\lambda) = \psi(0)$ and hence $N(\psi(\lambda)) = \psi'(\lambda)N(\lambda) = 0$ on Ω. Since $\psi' \neq 0$, we have $N(\lambda) = 0$ on Ω. From the above we conclude that $N(\lambda) = 0$ on M. □

From Theorem 4 it follows

Corollary 2. *Let $\mathcal{F}$ be a foliation (by curves) on a surface (M^2, g) and $\psi \in C^2(\mathbb{R})$ is given in (26) and satisfies $\psi' \neq 0$. Then the following properties are equivalent:*

(1) the geodesic curvature λ of $\mathcal{F}$ satisfies $N(\lambda) = 0$;

(2) $(g, \mu N)$, μ is given in (38) with $n = 1$, is the EGS structure with $\varepsilon = \psi(0)$.

Theorem 5. *Let (g, X) be a EGS structure for a foliation $\mathcal{F}$ (by curves) on a closed surface M^2, and $\psi \in C^2(\mathbb{R})$ is given in (26) and satisfies $\psi' \neq 0$. If $X \,||\, N$ then X-curves are closed and define a fibration $\pi : M^2 \to S^1$, and $\mathcal{F}$ is the suspension of a diffeomorphism $f : S^1 \to S^1$. Moreover, if $\psi(\lambda) = -2\lambda + c$ then (g, X) is the EGS structure (with $\varepsilon = c$) for any metric $g \in \mathcal{M}$ satisfying $N(\lambda) = 0$, otherwise $\lambda = 0$ (i.e., $\mathcal{F}$ is a geodesic foliation).*

Proof. Assume an opposite, then the foliation $\mathcal{F}_N$ has a limit cycle. Since M is compact, there is a domain $\Omega \subset M^2$ bounded by closed N-curves (which are limit cycles). By (35), $\lambda = const$ along N-curves. Since there are limiting leaves, $\lambda = const$ on Ω. From the relation $\operatorname{div} N = -\lambda$ and the Divergence Theorem

$$\int_\Omega \operatorname{div} N\, d\,\mathrm{vol} = \int_{\partial\Omega} \langle N,\, \nu\rangle\, d\,\omega,$$

where ν is the outer normal to the boundary $\partial\Omega$ (hence $\nu \perp N$), we conclude that $\lambda = 0$ on Ω, hence $\mathcal{F}_N$ is a Riemannian foliation, see [**Rov**], – a contradiction.

By the classification theorem for foliations on closed surfaces, see [**G**], all the X-curves are closed and define a fibration $\pi : M^2 \to S^1$. By (39) with $\mu = 1$ we have the following. If $\psi(\lambda) \neq -2\lambda + c$ then $\lambda = const$ on M. Hence, using the integral formula $\int_M \lambda\, d\,\mathrm{vol} = 0$, we conclude that $\lambda = 0$. In this case, by Lemma 1, any $\mathcal{F}$-geodesic metric on $(M, \mathcal{F}, N)$ has the form $\bar g = (\pi^{-1} \circ \sigma)\hat g \oplus g^\perp$, where $\sigma : S^1 \to \mathbb{R}$ is an arbitrary smooth function. □

Example 4 (Non-Riemannian EGS on double-twisted products)**.** Let $M = M_1 \times M_2$ be the product (with the metric $\tilde g = g_1 \oplus g_2$ and Levi-Civita connection $\tilde\nabla$) of a circle $M_1 = S^1$ with the canonical metric g_1 and a compact Riemannian manifold (M_2, g_2). Let $f_i : M \to \mathbb{R}$ be a positive differentiable function, $\pi_i : M \to M_i$ the canonical projection, $\pi_i^\perp : TM \to \ker \pi_{3-i}$ the vector bundle projection, for $i = 1, 2$. The metric of a *double-twisted product* $M_1 \times_{(f_1 \times f_2)} M_2$ is

$$g(X, Y) = f_1 g_1(\pi_{1*}X, \pi_{1*}Y) + f_2 g_2(\pi_{2*}X, \pi_{2*}Y), \quad X, Y \in TM,$$

i.e., $g = f_1 g_1 \oplus f_2 g_2$. The Levi-Civita connection ∇ of g obeys the relation [**PR**]

$$\nabla_X Y = \tilde\nabla_X Y + \sum\nolimits_i [g(\pi_{i*}X, \pi_{i*}Y)U_i - g(X, U_i)\pi_{i*}Y - g(Y, U_i)\pi_{i*}X].$$

Both foliations $M_1 \times \{p_2\}$ and $\{p_1\} \times M_2$ are totally umbilical with the mean curvature vectors $H_1 = \pi_2^\perp U_1$ and $H_2 = \pi_1^\perp U_2$, respectively, where $U_i = -\nabla(\log f_i)$. This property characterizes the double-twisted product, see [**PR**].

Suppose that $\dim M_1 = 1$, then the EGF preserves the above double-twisted product structure. The mean curvature of the foliation $\mathcal{F} := \{p_1\} \times M_2$ is constant along the fibers $M_1 \times \{p_2\}$ (i.e., N-curves) if and only if $\pi_1^\perp U_2$ is a function of M_2. In this case, due to Theorem 4, $\mathcal{F}$ admits the EGS structure with $X \,||\, N$.

Example 5. Notice that the EGF preserves *rotational symmetric metrics*

$$g = dx_0^2 + \varphi^2(x_0)\, ds_n^2 \quad \text{where} \quad ds_n^2 \text{ is a metric of curvature } 1, \tag{40}$$

see definition in [**Pet**]. The n-parallels $\{x_0 = c\}$ form a Riemannian totally umbilical foliation $\mathcal{F}$ with $N = \partial_0$. In this case, the EGF of type (b) has the form (26) discussed in Example 2. Any leaf-wise Killing field $X \perp N = \partial_0$ provides the EGS structure on M^{n+1} with the rotational symmetric metric g.

Assuming $\hat g_t = \varphi_t^2\, \hat g_0$, from (26) we obtain $\lambda_t = -(\varphi_t)_{,0}/\varphi_t^2$ and

$$\partial_t \varphi_t = \frac{1}{2}\, \psi(\lambda_t)\, \varphi_t \quad \Rightarrow \quad \varphi_t = \varphi_0 \exp\Big(\frac{1}{2}\int_0^t \psi(\lambda_t)\, dt\Big).$$

In particular case of $\psi(\lambda) = \lambda$, we get the linear PDE $\partial_t \lambda + \frac{1}{2} N(\lambda) = 0$ representing the "unidirectional wave motion" along any N-curve $\gamma(s)$,

$$\lambda_t(s) = \lambda_0(s - t/2). \tag{41}$$

If, in addition, $\lambda_0 = C \in \mathbb{R}$, then $\lambda_t = C$ and $\varphi_t = \varphi_0 \exp\big(\frac{1}{2}\, t\, \psi(C)\big)$.

Rotational symmetric metrics with $\lambda = \text{const}$ exist on hyperbolic space $\mathbb{H}^{n+1}$ with horosphere foliation. On the Poincaré $(n+1)$-ball B the leaves of such Riemannian totally umbilical foliations are Euclidean n-spheres tangent ∂B (Fig. 1). The normal field N and any leaf-wise Killing field $X \perp N = \partial_0$ provide the $\mathcal{F}$-EGS structures with g as above. Notice that there are no totally umbilical foliations with $\lambda = \text{const} \neq 0$ on compact (or of finite volume, see [**RW1**]) manifolds.

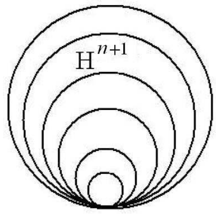

Figure 1: Horosphere foliation.

Some of rotational symmetric metrics come from hypersurfaces of revolution in space forms. Evolving them by EGF yields deformations of hypersurfaces of revolution foliated by n-parallels.

Example 6 (EGS on hypersurfaces of revolution)**.** Revolving the graph of $x_1 = f(x_0)$ about the x_0-axis of $\mathbb{R}^{n+1}$, we get the hypersurface $M^n:\ f^2(x_0) = \sum_{i=1}^n x_i^2$ foliated by $(n-1)$-spheres $\{x_0 = c\}$ (parallels) with the induced metric

$$g = (1 + f'(x_0)^2)\, dx_0^2 + f^2(x_0) \sum\nolimits_{i=1}^{n} dx_i^2. \tag{42}$$

(i) Revolving a line $\gamma_0 : x_1 = \tan\beta\, x_0$ about the x_0-axis, we build the cone $C_0 : (\tan\beta\, x_0)^2 = \sum_{i=1}^n x_i^2$, with the metric $g_0 = dx_0^2 + (x_0 \sin\beta)^2 \sum_{i=1}^n dx_i^2$. Hence $\varphi_0 = x_0 \sin\beta$ and $\lambda_0(x_0) = -2/x_0$. Applying the EGF $\partial_t g_t = \lambda_t \hat g_t$, by (41) we obtain $\lambda_t(x_0) = -\frac{2}{x_0 - t/2}$. The rotational symmetric metric $g_t = dx_0^2 + (x_0 - t/2)^2 \sin^2\beta \sum_{i=1}^n dx_i^2$ appears on the same cone translated across the x_0-axis, $C_t : (x_0 - t/2)^2 \tan^2\beta = \sum_{i=1}^n x_i^2$. Any leaf-wise Killing field $X \perp N$ provides the EGS structure on M^n with the induced metric g.

(ii) Let us find a curve $y = f(x) > 0$ such that the metric (42) on the surface of revolution $M^n : \sum_{i=1}^n x_i^2 = f^2(x_0)$ has $\lambda_0 = \mathrm{const} = 1$. Using $\lambda_0 = \frac{1}{f(x_0)} \sin\phi$, where $\tan\phi = f'(x_0)$, we get ODE $\frac{|f'|}{f\sqrt{1+(f')^2}} = 1 \Rightarrow \frac{dx_1}{dx_0} = \frac{x_1}{\sqrt{4+x_1^2}}$. The solution is

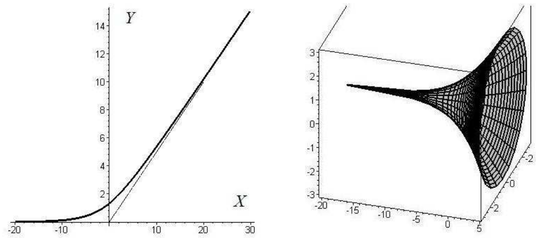

Figure 2: a) Graph of γ. b) Hypersurface of revolution (of γ) with $\lambda = const.$

$\gamma :\ x_0 = \log \frac{\sqrt{4+x_1^2}-2}{\sqrt{4+x_1^2}+2} + \sqrt{4 + x_1^2} + C$, where $C \in \mathbb{R}$. The hypersurface $M^n \subset \mathbb{R}^{n+1}$

looks like a pseudosphere (for $n = 2$ see Fig. 2 and [**RW0**]), but for $x_0 \to \infty$ it is asymptotic to the cone $(x_0 + C)^2 = \sum_{i=1}^n x_i^2$. The 2-dimensional sectional curvature is $K(\partial_0, \partial_1) = -\frac{1}{(x_1^2+2)^2} < 0$, and $\lim\limits_{x_1\to\pm\infty} K = 0$.

As for horosphere foliation on $\mathbb{H}^{n+1}$ (Example 5), the normal N to parallels and any leaf-wise Killing field $X \perp N$ compose EGS structures on (M^n, g).

4 Extrinsic Ricci flow

The extrinsic Riemannian curvature tensor $\mathrm{Rm}^{\mathrm{ex}}$ of $\mathcal{F}$ is, roughly speaking, the difference of the curvature tensors of M and of the leaves (see, for example, [**Rov**]). More precisely, due to the Gauss formula, we have

$$\mathrm{Rm}^{\mathrm{ex}}(Z, X)Y = g(AX, Y)AZ - g(AZ, Y)AX \quad \text{for} \quad X, Y, Z \in T\mathcal{F}.$$

The *extrinsic Ricci flow* is defined by

$$\partial_t g_t = -2\,\mathrm{Ric}^{\mathrm{ex}}_t, \tag{43}$$

where $\mathrm{Ric}^{\mathrm{ex}}(X, Y) = \mathrm{Tr}\,\mathrm{Rm}^{\mathrm{ex}}(\cdot, X)Y = \tau_1 \hat b_1 - \hat b_2$ is the *extrinsic Ricci tensor*. For $n = 2$, we have $\mathrm{Ric}^{\mathrm{ex}} = \sigma_2\,\hat g$. Hence, $-2\,\mathrm{Ric}^{\mathrm{ex}}$ relates to $h(b)$ of (2) with $f_1 = -2\,\tau_1$, $f_2 = 2$ (others $f_j = 0$).

Example 7. Foliations satisfying $\mathrm{Ric}^{\mathrm{ex}} = 0$, the fixed points of (43), have the property $A(A - \tau_1\,\mathrm{id}) = 0$, or, $k_j(k_j - \tau_1) = 0$ $(1 \le j \le n)$ for the eigenvalues k_j of A. Since $\tau_1 = \sum_j k_j$, from above it follows $k_j = 0$ for all j. Hence, *extrinsic Ricci flat foliations are totally geodesic foliations only.*

In order to extend the set of solutions we define the *normalized EGF* by

$$\partial_t g_t = h(b_t) - (\rho_t/n)\,\hat g_t \quad \text{with} \quad \rho_t = \int_M \mathrm{Tr}\,A_h\,d\,\mathrm{vol}_t / \mathrm{Vol}(M, g_t). \tag{44}$$

Definition 5. We call g_t on a compact $(M, \mathcal{F})$ a *normalized extrinsic Ricci flow* if

$$\partial_t g_t = -2\,\mathrm{Ric}^{\mathrm{ex}}_t + (\rho^{\mathrm{ex}}_t/n)\,\hat g_t, \quad \rho^{\mathrm{ex}}_t = -2\int_M \mathrm{Ric}_t(N, N)\,d\,\mathrm{vol}_t / \mathrm{Vol}(M, g_t). \tag{45}$$

A metric g on $(M, \mathcal{F})$ is *extrinsic Einstein* if $\mathrm{Ric}^{\mathrm{ex}} = \rho/(2\,n) \cdot \hat g$ for some $\rho \in \mathbb{R}$. Follow Definition 3, an *extrinsic Ricci soliton structure* is a pair (g, X) of a metric g on $(M, \mathcal{F})$, and a complete field $X \in \mathcal{X}(\mathcal{F})$ satisfying for some $\varepsilon \in \mathbb{R}$

$$-\,2\,\mathrm{Ric}^{\mathrm{ex}} = \varepsilon\,\hat g + \mathcal{L}_X \hat g. \tag{46}$$

Remark 7. To explain $\rho_t^{\rm ex}$ in (45), we find the *extrinsic scalar curvature*: $R^{\rm ex} = \operatorname{Tr}\operatorname{Ric}^{\rm ex} = \operatorname{Tr}(\tau_1 A - A^2) = \tau_1^2 - \tau_2 = 2\,\sigma_2$. By the integral formula $\int_M (2\,\sigma_2 - \operatorname{Ric}(N,N))\,d\operatorname{vol} = 0$ (see [**RW1**]) we find

$$\int_M R^{\rm ex}\,d\operatorname{vol} = \int_M \operatorname{Ric}(N,N)\,d\operatorname{vol}.$$

Substituting this into (44) instead of $\int_M \operatorname{Tr} A_h\,d\operatorname{vol}$, we obtain $\rho_t^{\rm ex}$ of (45). Hence, extrinsic Einstein foliations are fixed points of the flow (45).

Codimension one totally umbilical foliations with $\lambda = const$ are extrinsic Einstein foliations. An extrinsic Ricci soliton structure with $X = 0$ has clearly extrinsic Einstein metric.

Remark 8. A codimension one foliation (M, g) will be called *CPC* (*constant principal curvatures*) if the principal curvatures of leaves are constant.

(a) (Non-) normalized EGF preserve CPC property of foliations, see [**RW0**]. Let such flow on $(M, \mathcal{F})$ starts from a CPC metric. From $N(\tau_i) = 0$ and $N(f_m(\vec{\tau}, t)) = 0$ for any m we conclude that τ_i do not depend on t.

(b) Let (G, g) be a compact Lie group with a left invariant metric g. Suppose that the corresponding Lie algebra has a codimension one subspace V such that $[V, V] \subset V$. Then V determines a CPC foliation on (G, g).

Theorem 6. *Let (g, X) be an extrinsic Ricci soliton structure on $(M^n, \mathcal{F})$ $(n > 2)$, and X a leaf-wise conformal Killling field (i.e., $\mathcal{L}_X \hat{g} = \mu\,\hat{g}$).*

(i) Then there are at most two distinct principal curvatures at any point $p \in M$.

(ii) Moreover, if μ is constant along the leaves, then $\mathcal{F}$ is CPC foliation.

Proof. (i) Since $(\operatorname{Ric}^{\rm ex})^\sharp = -A(A - \tau_1 \hat{\rm id})$ and $\mathcal{L}_X \hat{g} = \mu\,\hat{g}$, we obtain the equality $A(A - \tau_1\,{\rm id}) = r\,\hat{\rm id}$ with $r = \frac{1}{2}(\varepsilon + \mu)$, that yields equalities for the principal curvatures k_j,

$$k_j(k_j - \tau_1) = r \quad \forall j.$$

Hence each k_j is a root of a quadratic polynomial $P_2(k) = k^2 - \tau_1 k - r$. The roots are real if and only if $\tau_1^2 + 4\,r \ge 0$. In the case $r > -\tau_1^2/4$ we have two distinct roots $\bar{k}_{1,2} = (\tau_1 \pm \sqrt{\tau_1^2 + 4\,r})/2$. Let $n_1 \in (0, n)$ eigenvalues of A are equal to $\bar{k}_1$ and others to $\bar{k}_2$. From $\tau_1 = n_1 \bar{k}_1 + n_2 \bar{k}_2$ and $n = n_1 + n_2$ we obtain

$$n_2 - n_1 = (n-2)\,\tau_1/(\tau_1^2 + 4\,r)^{1/2} \ \in \mathbb{Z}. \tag{47}$$

If $n_2 = n_1$ then $\tau_1 = 0$ and $k_{1,2} = \pm\sqrt{r}$, otherwise, $\tau_1^2 = \frac{4\,r}{q^2-1}$ for $q = \frac{n-2}{n_2-n_1}$.

(ii) Assume that μ is constant along the leaves, then $r = const$ on any leaf. In view of $n > 2$, a continuous function $\tau_1 : M \to \mathbb{R}$ has values in a discrete set, hence it is constant. Then all k_j's (from both sets) are constant on M. For n even, (47) admits a particular solution: $\tau_1 = 0$, and $n/2$ principal curvatures k_j equal to $\sqrt{r}$, others to $-\sqrt{r}$. $\square$

By the above, extrinsic Einstein foliations satisfy the equality $A(A - \tau_1\,{\rm id}) = r\,{\rm id}$, where $r = \rho/(2n)$. Hence, from Theorem 6 we obtain

Corollary 3. *Let $\mathcal{F}$ be a foliation with extrinsic Einstein metric g. Then*
(i) If M is compact then $\mathcal{F}$ is a fixed point of the normalized extrinsic Ricci flow.
(ii) If $n > 2$ then $\mathcal{F}$ is CPC foliation with ≤ 2 distinct principal curvatures.

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