On the cardinality of sumsets in torsion-free groups

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Abstract

Let A, B be finite subsets of a torsion-free group G. We prove that for every positive integer k there is a $c(k)$ such that if $|B| \geq c(k)$ then the inequality $|AB| \geq |A| + |B| + k$ holds unless a left translate of A is contained in a cyclic subgroup. We obtain $c(k) < c_0 k^6$ for arbitrary torsion-free groups, and $c(k) < c_0 k^3$ for groups with the unique product property, where c_0 is an absolute constant. We give examples to show that $c(k)$ is at least quadratic in k.

1 Introduction

Let G be a torsion-free group written multiplicatively, and let $|\cdot|$ denote the cardinality of a finite set. A basic problem in Additive Combinatorics is to estimate the cardinality of $AB = \{ab : a \in A, b \in B\}$ of two finite sets $A, B \subset G$ in terms of $|A|$ and $|B|$. A basic notion is a progression with ratio $r \neq 1$ and length n, which is a set of the form $\{a, ar, \ldots, ar^{n-1}\}$ where a and r commute.

Let us review some related results if G is abelian. In this case we have the simple inequality

$$|AB| \geq |A| + |B| - 1,$$

with equality if and only if A and B are progressions with common ratio. Following Ruzsa [21], we call the minimal rank of a subgroup whose some coset contains A the dimension of A. According to Freiman [5], if the dimension of A is d, then

$$|A^2| \geq (d+1)|A| - \binom{d+1}{2}. \tag{1}$$

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This estimate is optimal. It follows that if $|A^2| \leq 3|A| - 4$, then A is contained in some coset of a cyclic group. Actually, even a progression of length $2|A|-3$ contains A according to the $(3k-4)$-theorem of Freiman [5]. More precise structural information on A is available if $|A^2| = 2|A| + n$ for $0 \leq n \leq |A|-4$ by Freiman [7], for example, A is contained in a progression of length $|A| + n + 1$.

The inequality (1) was generalized to a pair of sets by Ruzsa [20] who proved that if $|A| \geq |B|$, and the dimension of AB is d, then

$$|AB| \geq |A| + d|B| - \binom{d+1}{2}. \tag{2}$$

By requiring additionally that the smaller set B is d–dimensional, Gardner and Gronchi [8] proved a discrete version of the Brunn–Minkowski inequality which shows that

$$|AB| \geq |A| + (d-1)|B| + (|A| - d)^{(d-1)/d}(|B| - d)^{1/d} - \binom{d}{2}. \tag{3}$$

Additional lower bounds with stronger geometric requirements on the sets A and B have been also obtained by Matolcsi and Ruzsa [16] and Green and Tao [9].

In the non-abelian case the situation is much less understood. Kempermann [15] implies in the case of any torsion-free group G that

$$|AB| \geq |A| + |B| - 1. \tag{4}$$

Brailovsky and Freiman [1] characterized the extremal sets in the inequality (4) by showing that, if $\min\{|A|, |B|\} \geq 2$ then, up to appropriate left and right translations, both A and B are progressions with common ratio. In particular, A and B lie in a left and a right coset, respectively, of a cyclic subgroup.

The analogy with the abelian case was extended in Hamidoune, Lladó and Serra [14] to the inequality

$$|AB| \geq |A| + |B| + 1, \tag{5}$$

if $|B| \geq 4$, and A is not contained in some left coset of a cyclic subgroup.

These known facts are connected with the following conjecture of Freiman (personal communication), extending the $(3k-4)$-theorem above.

Conjecture 1 *Let A be a finite subset of a torsion-free group with $|A| \geq 4$. If*

$$|A^2| \leq 3|A| - 4$$

then A is covered by a progression of length at most $2|A| - 3$.

By using the so–called isoperimetric method, see Hamidoune [11], [12] or [13], we obtain the following results:

Theorem 2 *For any integer $k \geq 1$ there exists a $c(k)$ such that the following holds. If G is a torsio-free group, $A \subset G$ is not contained in a left coset of any cyclic subgroup, $B \subset G$ has more than $c(k)$ elements, then*

$$|AB| > |A| + |B| + k.$$

Remark Our current methods yield $c(k) \leq 32(k+3)^6$.

Note that, in Theorem 2, the assumption on A not being contained in a left coset of a cyclic group is crucial. For example, if A is a progression of length at most $k+2$ with ratio $r \neq 1$, and B is the union of two r-progressions of arbitrary length, then $|AB| \leq |A| + |B| + k$.

The value of the lower bound $c(k)$ can be improved for unique product groups. Recall that a group G has the unique product property if, for every pair of finite sets $A, B \subset G$, there is an element $g \in AB$ which can be uniquely expressed as a product of an element of A and an element of B. In this case, G is torsion-free. We note that every right linearly orderable group has the unique product property, and any residually finite word hyperbolic group has a finite index unique product subgroup, according to T. Delzant [4]. On the other hand, it was first shown by Rips and Segev [18] that not all torsion-free groups have the unique product property, and S.D. Promislow [17] even provided an explicit construction for such an example. In addition, unique product groups are discussed in A. Strojnowski [22], S.M. Hair [10], and W. Carter [3]. For unique product groups, the bound on $c(k)$ in Theorem 2 can be reduced to a cubic polynomial on k; namely, Lemma 12 yields that

$$c(k) \leq 4(2k+3)^3 \quad \text{if } G \text{ is a unique product group.} \tag{6}$$

We note that it can be deduced with the help of (3) that in abelian torsion-free groups, the optimal order of $c(k)$ is quadratic. The $c(k)$ in Theorem 2 is at least of quadratic order also for non-abelian unique product groups, as the following example shows.

We consider the Klein bottle group $G_0 = \langle u, v | u^{-1}vu = v^{-1} \rangle$, and hence $vu = uv^{-1}$ and $v^{-1}u = uv$. Since $\langle v \rangle$ is a normal subgroup with factor isomorphic to $\mathbb{Z}$, G_0 is a non-abelian unique product group. Let $A = \{1, u, v\}$ and $B = \{u^i v^j : i, j = 0, 1, ..., m-1\}$ for $m \geq 1$. Then $|B| = m^2$ and $AB = \cup_{i,j=0,1,...,m-1} \{u^i v^j, u^{i+1} v^j, u^i v^{j+(-1)^i}\}$, thus

$$|AB| = m^2 + 2m = |A| + |B| + 2|B|^{\frac{1}{2}} - 3.$$

In line with Conjecture 1, we conjecture that, if $A = B$, then the lower bound on $|B|$ in Theorem 2 can be replaced by a bound linear in k. We construct an example in the group G_0 above to indicate, what to expect in Theorem 2 in this case.

For $m \geq 1$, let $A = P \cup vuQ$ where $P = \{u^i : i = 0, 1, ..., 2m\}$ and $Q = \{u^{2i} : i = 0, 1, ..., m-1\}$, and hence $|A| = 3m + 1$. Since v commutes with u^2, we have $(vuQ)(vuQ) = (vuv)uQ^2 = u^2 Q^2 \subset P^2$. Moreover, denoting by P_0 and P_1 the set of even and odd powers of u in P, respectively, we have $PvuQ = vuP_0Q \cup v^{-1}uP_1Q \subset vuQP \cup v^{-1}uP_1Q$. It follows that

$$|A^2| = |P^2| + |QP| + |P_1Q| = 10m - 1 = \tfrac{10}{3}|A| - \tfrac{13}{3}.$$

We note that the above example seems to match a conjecture of Freiman [6], which would yield that A is the union of two progressions provided that $|A^2| < \frac{10}{3}|A| - 5$.

In the direction of Conjecture 1 for a torsion-free group G, our results yield the following.

Corollary 3 *If A is a subset of a torsion-free group with $|A| \geq 6^6$, and $|A^2| = 2|A| + n$ for $0 \leq n \leq 2^{-5/6}|A|^{1/6} - 3$, then A is contained in a progression of length $|A| + n + 1$.*

Remark In unique product groups, the conditions are $|A| \geq 6^3$ and $0 \leq n \leq 2^{-5/3}|A|^{1/3} - \frac{3}{2}$.

If A is a finite subset of a torsion-free group G, then Corollary 3 provides strong structural information when $|A^2|$ is very close to $2|A|$. This has been made possible in part by the known structural properties in abelian groups.

Now if G is abelian and $|A^2| < K|A|$ for some $K > 3$ then still strong structural properties have been established by Freiman [5] using multidimensional progressions, see the monograph of Tao and Vu [24] or the survey by Ruzsa [21] for recent developments. But if G is any torsion-free group and $K \geq \frac{10}{3}$, then A may not be contained in an abelian subgroup. Actually, it is still not completely understood, what to expect, in spite the results about some specific groups (see Breuillard and Green [2], or Tao's blog [23]).

2 Atoms and fragments

For this section, we fix a torsion-free group G.

For $n \geq 1$ and a finite non-empty set $C \subset G$, the n-th isoperimetric number of C is defined to be

$$\kappa_n(C) = \min\{|XC| - |X| : X \subset G \text{ and } |X| \geq n\}.$$

A finite set $V \subset G$ is an n-fragment for C, if $|V| \geq n$ and $|VC| - |V| = \kappa_n(C)$. In addition an n-fragment of minimal cardinality is an n-atom for C.

Naturally, if U is an n-atom for C, then xU is also an n-atom for Cy for any $x, y \in G$. In what follows, we present simple statements about atoms. For the sake of completeness, we verify even the known ones, except for the following crucial property of atoms, due to Hamidoune [11]: If U is an n–atom and F is an n–fragment for a finite nonempty subset $C \subset G$, then

$$\text{either } U \subset F \text{ or } |U \cap F| \leq n - 1.$$

This property has the following useful consequence.

Corollary 4 *For a torsion-free group G and $n \geq 1$, if U is an n-atom for $C \subset G$ and $g \in G \backslash 1$, then $|U \cap gU| \leq n - 1$.*

For right translations we have a weaker result.

Lemma 5 *For a torsion-free group G and $n \geq 2$, if U is an n-atom for $C \subset G$ and $g \in G \backslash 1$, then*

$$|U \cap Ug| \leq \frac{n-2}{n-1}\,|U| + \frac{1}{n-1} \leq \frac{n-1}{n}\,|U|.$$

Remark In particular, if $n = 2$, then $|U \cap Ug| \leq 1$.
Proof: Let us partition U into the maximal left g-progressions $U_1, \ldots, U_m$, where $U_i = \{h_i, h_i g, \ldots, h_i g^{\alpha_i}\}$, $i = 1, \ldots, m$. In particular, $U_i \cap U_j g = \emptyset$ for $i \neq j$. We may assume that $|U_1| \geq |U_i|$, $i = 2, \ldots, m$ and that $h_1 = 1$.

It follows by Corollary 4 that

$$|U_1 \cap gU_1| \leq n - 1,$$

thus $|U_1| \leq n$. In addition, for $i \geq 2$, we have

$$|U_1 \cap h_i^{-1} U_i| \leq n - 1,$$

thus $|U_i| \leq n - 1$. Therefore $|U| \leq m(n-1) + 1$ and

$$|U \cap Ug| = |U| - m \leq \frac{n-2}{n-1}|U| + \frac{1}{n-1} \leq \frac{n-1}{n}|U|,$$

as claimed. Q.E.D.

The minimality of the cardinality of atoms directly yields (see [11] or [14])

Lemma 6 *If U is an n-atom for $C \subset G$, $n \geq 1$, in a torsion-free group G, and $|U| > n$, then any element in UC can be represented in at least two ways as a product of an element of U and an element of C.*

We deduce two rough, but useful estimates about 2-atoms which can be found in [11] as well.

Lemma 7 *If U is a 2-atom for $C \subset G$, $|C| \geq 3$, in a torsion-free group G, then $|U| \leq |C| - 1$.*

Proof: We may assume that $|U| > 2$ and $1 \in C$, and hence $U \subset UC$. According to Lemma 6, for any $u \in U$, there are $v_u \in U$ and $c_u \in C \backslash 1$ such that $u = v_u c_u$. If $c_u = c_w$ for $u \neq w \in U$, then $\{v_w, w\} \subset U \cap v_w v_u^{-1} U$, contradicting Corollary 4. Therefore $u \mapsto c_u$ is an injective map from U into $C \backslash 1$. Q.E.D.

For any non-empty $C \subset G$, let $C^{-1} = \{g^{-1} : g \in C\}$.

Lemma 8 *If U is a 2-atom for $C \subset G$, $|C| \geq 3$, in a torsion-free group G, and $|UC| \leq |U| + |C| + k$, then $|U| \leq k + 3$.*

Proof: Let V be a 2-atom for U^{-1} with $1 \in V$, thus

$$|VU^{-1}| - |V| - |U^{-1}| \le |C^{-1}U^{-1}| - |C^{-1}| - |U^{-1}| = |UC| - |U| - |C| \le k.$$

If $V = \{1, g\}$ with $g \neq 1$, then Lemma 5 yields

$$2|U| - 1 \le |UV^{-1}| = |VU^{-1}| \le k + 2 + |U|,$$

which in turn implies $|U| \le k+3$. If $|V| \ge 3$, then Lemma 5 and Lemma 7 yield

$$|U| + (|U| - 1) + (|U| - 2) \le |UV^{-1}| = |VU^{-1}| \le k + |V| + |U| \le k + 2|U| - 1,$$

which in turn implies $|U| \le k+2$. Q.E.D.

Now we extend Lemma 8 to n-atoms, which extension is the only novel result of this section.

Proposition 9 *If U is an n-atom for $C \subset G$, $|C| \ge 3$ and $n \ge 3$, in a torsion-free group G, and $|UC| \le |U| + |C| + k$, then $|U| \le n(2k+3)$.*

Proof: Let V be a 2-atom for U^{-1}, hence

$$|VU^{-1}| - |V| - |U^{-1}| \le |C^{-1}U^{-1}| - |C^{-1}| - |U^{-1}| \le k.$$

It follows by Lemma 8 that $|V| \le k+3$. Moreover, by Lemma 5, we have $|UV^{-1}| \ge 2|U| - \frac{n-1}{n}|U|$. Hence,

$$\tfrac{n+1}{n}\,|U| \le |UV^{-1}| = |VU^{-1}| \le |U| + |V| + k \le |U| + 2k + 3,$$

thus $|U| \le n(2k+3)$. Q.E.D.

All these statements about atoms would readily follow from the following conjecture of Y.O. Hamidoune [13].

Conjecture 10 *Any n-atom in a torsion-free group has cardinality n.*

We recall that a group G has the unique product property if for any finite non-empty sets $A, B \subset G$, there is a $g \in AB$ that can be represented in a unique way in the form ab with $a \in A$ and $b \in B$. In this case G is torsion-free. It follows by Lemma 6 that unique product groups satisfy Conjecture 10.

3 Small product sets

The proof of Theorem 2 together with an estimate of $c(k)$ will follow from the following Lemma and the estimations on the size of atoms in the previous section.

Lemma 11 *Let G be a torsion-free group. Suppose that $A \subset G$ with $|A| = 3$ is not contained in a left coset of any cyclic subgroup of G. For $d \geq 3$ and any finite set $B \subset G$ of cardinality greater than $4d^3$, we have*

$$|AB| > |B| + d.$$

Proof: We suppose that $|AB| \leq |B| + d$, and seek a contradiction. We may assume that $A = \{1, u, v\}$, where $\langle u, v\rangle$ is not cyclic.

For $g \in G$, we write $B_g = B \setminus g^{-1}B = \{x \in B \mid gx \notin B\}$. Since $|B \cup uB| \leq |B| + d$, we see that $|B_u| \leq d$. Similarly, $|B_{u^{-1}}| \leq d$, as $|u^{-1}B \cup B| = |B \cup uB| \leq |B| + d$, and, of course, $|B_v|, |B_{v^{-1}}| \leq d$ also hold. Since B is finite, for any $x \in B$ the coset $\langle u\rangle x$ must contain an element of B_u, hence the elements of B belong to at most d cosets of $\langle u\rangle$, and similarly for $\langle v\rangle$. Therefore there exists an $x_0 \in B$ such that

$$|B \cap \langle u\rangle x_0 \cap \langle v\rangle x_0| \geq |B|/d^2 > 4d.$$

In order to simplify notation, by replacing B with Bx_0^{-1}, we may assume without loss of generality that $x_0 = 1$. Let $Z = \langle u\rangle \cap \langle v\rangle$, and $B_0 = B \cap Z$. We have $|B_0| > 4d$. Elements of Z are powers of both u and v, hence Z is contained in the center of $H = \langle u, v\rangle$. As $Z \neq \{1\}$ and A does not generate a cyclic group, we deduce that u and v do not commute.

We are going to show that $BZ \supseteq H$. Take an element $g \in H$, and let us choose a word of shortest length $a_n a_{n-1} \cdots a_2 a_1$, where each a_i is one of u, u^{-1}, v, v^{-1}, in the coset gZ. Then the cosets Z, a_1Z, a_2a_1Z, $\ldots$, $a_n a_{n-1} \cdots a_1 Z$ are pairwise disjoint. To any $x \in B_0$, we assign the sequence $S_x = \{a_i a_{i-1} \cdots a_1 x\}_{i=0,1,\ldots,n}$, which sequences are pairwise disjoint as x runs through B_0. If $a_n a_{n-1} \cdots a_1 x \notin B$, then there is a smallest $i \in \{1, \ldots, n\}$ such that $a_i \cdots a_1 x \notin B$. It follows that S_x has an element in $B_{a_i} \subset B_u \cup B_{u^{-1}} \cup B_v \cup B_{v^{-1}}$, namely, x if $i = 1$, and $a_{i-1} \cdots a_1 x$ if $i \geq 2$. Since $|B_0| > 4d \geq |B_u| + |B_{u^{-1}}| + |B_v| + |B_{v^{-1}}|$, and $S_x \cap S_y = \emptyset$ for $x \neq y$ in B_0, there exists an $x \in B_0$ such that $a_n a_{n-1} \cdots a_1 x \in B$. We conclude that $g \in a_n a_{n-1} \cdots a_1 x Z \subset BZ$.

Now the index of the central subgroup Z in H is finite (bounded by $|B|$), so the center has finite index in H. According to a classical theorem of Schur (see, e.g., Robinson [19, Theorem 10.1.4]), this implies that the commutator subgroup of H is finite. If the commutator subgroup is 1, then H is abelian, and if the commutator subgroup is non-trivial, then we have some torsion elements. In any case, we have contradicted the assumptions on A and G, and hence proved the lemma. Q.E.D.

Proof of Theorem 2: Without loss of generality we may assume that $1 \in A$. Then $\langle A \rangle$ is not cyclic by our assumption. Let $B \subset G$ be a finite set with $|B| > 32(k+3)^6$. If B is contained in some right coset of a cyclic subgroup H, then A intersects at least two left cosets of H. Let A_1 be one of these intersections. Then using (4) we get

$$|AB| = |A_1B| + |(A \backslash A_1)B| \geq |A| + 2|B| - 2 > |A| + |B| + k.$$

Therefore we may assume that B is not contained in a right coset of any cyclic subgroup.

If $|A| \leq k+3$, then let $A_0 = A$, and if $|A| > k+3$, then let A_0 be a $(k+3)$–atom for B with $1 \in A_0$. By definition, $|AB| \geq |A| + |A_0B| - |A_0|$. Proposition 9 gives that either $|A_0B| - |A_0| > |B| + k$, or $|A_0| \leq (k+3)(2k+3)$. If $\langle A_0 \rangle$ is not cyclic, then choose $u, v \in A_0 \setminus 1$ such that $\langle u, v \rangle$ is not cyclic. For $\widetilde{A}_0 = \{1, u, v\} \subset A_0$, we have, by Lemma 11, that

$$|AB| \geq |A| + |A_0B| - |A_0| \geq |A| + |\widetilde{A}_0B| - |A_0| > |A| + (|B| + 2(k+3)^2) - |A_0| > |A| + |B| + k.$$

Finally if $\langle A_0 \rangle$ is cyclic, then $A_0 \neq A$, so $|A_0| \geq k+3$, and B intersects at least two right cosets of $\langle A_0 \rangle$. Let B_1 be one of these intersections. We have by (4)

$$|AB| \geq |A| + |A_0B| - |A_0| = |A| + |A_0B_1| + |A_0(B \backslash B_1)| - |A_0| \geq |A| + |B| + |A_0| - 2 > |A| + |B| + k,$$

completing the argument. Q.E.D.

If G is a unique product group, then the argument above, just using Conjecture 10 in place of Proposition 9, leads to

Lemma 12 *Let G be a unique product group, $A, B \subset G$ finite subsets, and $k \geq 1$. Suppose that A is not contained in a left coset of any cyclic subgroup, and $|B| > 4(2k+3)^3$, then*

$$|AB| > |A| + |B| + k.$$

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