arXiv:1009.6141v2 [math-ph] 2 Oct 2010

Dependence of Repeated Interaction Asymptotic States on Environment

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October 5, 2010

Abstract

Consider a quantum system $\mathcal{S}$ that interacts sequentially with a chain (environment) of identical probes $\mathcal{C} = \mathcal{P} + \mathcal{P} + \ldots$, with each interaction governed by a fixed interaction time τ and operator V. It is known how to construct the asymptotic state (large times) if the initial states of $\mathcal{P}$ belong to a class of so-called reference states. We generalize the analysis to a broader class of initial states, including the physically important situation of pure states. This is done by a simple modification to the effective dynamics generator.

Mathematics Subject Classification (2010): 82C10, 47N50
Keywords: repeated interaction asymptotic states, open quantum systems, Liouville operator, non-equilibrium dynamics.

1 Introduction and Main Results

1.1 Introduction

We first give a brief introduction to repeated interaction models. For a more complete overview, see [1]. In these models one has a chain of probes $\mathcal{C} = \mathcal{P} + \mathcal{P} + \ldots$ that interact sequentially with a fixed system of interest $\mathcal{S}$. We associate the (finite dimensional) Hilbert spaces $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{P}$ and the W^*-dynamical systems $(\mathcal{M}_\mathcal{S}, \alpha^t_\mathcal{S})$ and $(\mathcal{M}_\mathcal{P}, \alpha^t_\mathcal{P})$ to $\mathcal{S}$ and $\mathcal{P}$, respectively. $\mathcal{M}_\mathcal{S}$, $\mathcal{M}_\mathcal{P}$ are the von Neumann algebras of "observables" that act on $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{P}$ and $\alpha^t_\mathcal{S}$, $\alpha^t_\mathcal{P}$ are groups of *-automorphims that describe the Heisenberg dynamics. Let $\Omega_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Omega_\mathcal{P} \in \mathcal{H}_\mathcal{P}$ be reference states. These are vectors that are cyclic and separating and determine states on $\mathcal{M}_\mathcal{S}$ and $\mathcal{M}_\mathcal{P}$ that are invariant under $\alpha^t_\mathcal{S}$ and $\alpha^t_\mathcal{P}$.

The interaction dynamics of observables on $\mathcal{M} = \mathcal{M}_\mathcal{S} \otimes \mathcal{M}_\mathcal{P}$ is given by a fixed interaction time $\tau \in (0, \infty)$ and a self-adjoint interaction operator

$$V \in \mathcal{M}. \tag{1.1}$$

During the interval $[(m-1)\tau, m\tau)$, $\mathcal{S}$ interacts with the m-th probe while the remaining probes evolve under their own free dynamics given by $\alpha^\tau_\mathcal{P}$. Each interaction between $\mathcal{S}$ and a probe is governed by V and thus identical.

Let $L_\mathcal{S}$ and $L_\mathcal{P}$ be self-adjoint operators on $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{P}$ that generate the Heisenberg dynamics. That is,

$$\alpha^t_\mathcal{S}(A) = \mathrm{e}^{\mathrm{i}tL_\chi} A \mathrm{e}^{-\mathrm{i}\tau L_\chi}, \qquad \forall A \in \mathcal{M}_\chi, \tag{1.2}$$

and,

$$L_\chi \Omega_\chi = 0 \tag{1.3}$$

where $\chi = \{\mathcal{S}, \mathcal{P}\}$. These are called the standard Liouville operators, and will take the form

$$L_\chi = h_\chi \otimes \mathbb{1} - \mathbb{1} \otimes h_\chi \tag{1.4}$$

where h_χ are the free Hamiltonians.

The interacting dynamics between $\mathcal{S}$ and an individual $\mathcal{P}$ will include the free dynamics of $\mathcal{S}$ and $\mathcal{P}$ and the interaction operator V. That is, the self-adjoint operator

$$L = L_{\mathcal{S}} + L_{\mathcal{P}} + \lambda V, \qquad \lambda \in \mathbb{R}, \tag{1.5}$$

generates the automorphism group $\mathrm{e}^{\mathrm{i}tL} \cdot \mathrm{e}^{-\mathrm{i}tL}$ on $\mathcal{M}$. The total dynamics during the interval $[(m-1)\tau, m\tau)$ are generated by

$$\widetilde{L}_m = L_m + \sum_{n \neq m} L_{\mathcal{P},n} \tag{1.6}$$

where L_m is an operator on $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{P}} \otimes \dots$ that acts trivially on each probe except the m-th one where it acts as L. Similarly, $L_{\mathcal{P},n}$ is an operator on $\mathcal{H}$ that acts trivially on each probe except the n-th where it acts as $L_{\mathcal{P}}$.

The repeated interaction dynamics after m interactions of an operator A on $\mathcal{H}$ is defined as

$$\alpha_{RI}^m(A) = U_{RI}^*(m) A U_{RI}(m) \tag{1.7}$$

where

$$U_{RI}(m) = \mathrm{e}^{-\mathrm{i}\tau\widetilde{L}_m} \dots \mathrm{e}^{-\mathrm{i}\tau\widetilde{L}_1}. \tag{1.8}$$

We are interested in the expected values of observables on $\mathcal{S}$ in the large number of interactions limit. That is, we are interested in

$$\omega_\infty(A_{\mathcal{S}}) := \lim_{m\to\infty} \langle \alpha_{RI}^m(A_{\mathcal{S}}) \rangle = \lim_{m\to\infty} \langle \Psi_0, \alpha_{RI}^m(A_{\mathcal{S}}) \Psi_0 \rangle, \qquad A_{\mathcal{S}} \in \mathcal{M}_{\mathcal{S}}, \tag{1.9}$$

where Ψ_0 is the initial state of the small system and the probes.

The system $\mathcal{S}$ feels an effective dynamics induced by the interaction with the chain $\mathcal{C}$. To determine this effective dynamics, previous work([1],[5],[7]) on the subject takes

$$\Psi_0 = \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{P}} \otimes \Omega_{\mathcal{P}} \otimes \dots. \tag{1.10}$$

That is, they take the small system and the probes to be initially in reference states. This is restrictive, however, since many physically relevant states are not reference states, the prime example being pure states. Thus one would like to extend the formalism to account for such states as well. We show how to do this in Section 1.2.

Returning to intial states given by (1.10), if we let $\mathcal{J}$ and Δ denote the modular conjugation and modular operator associated to $(\mathcal{M}_{\mathcal{S}} \otimes \mathcal{M}_{\mathcal{P}}, \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{P}})$([3],[4]), then the effective dynamics is given by

$$T_\lambda = P \mathrm{e}^{\mathrm{i}\tau K_\lambda} P, \tag{1.11}$$

where,

$$P = \mathbb{1}_{\mathcal{S}} \otimes |\Omega_{\mathcal{P}}\rangle\langle\Omega_{\mathcal{P}}| \tag{1.12}$$

and

$$K_\lambda = L - \lambda \mathcal{J} \Delta^{1/2} V \Delta^{-1/2} \mathcal{J}. \tag{1.13}$$

K_λ is a C-Liouville operator([8],[9]) such that $\mathrm{e}^{\mathrm{i}tK_\lambda}$ implements the same dynamics as $\mathrm{e}^{\mathrm{i}tL}$ and

$$K_\lambda \Omega_{\mathcal{S}} \otimes \Omega_{\mathcal{P}} = 0. \tag{1.14}$$

T_λ is identified as an operator acting on $\mathcal{H}_{\mathcal{S}}$ only. If the spectrum of $T_\lambda, \lambda \neq 0$ on the complex unit circle consists solely of the simple eigenvalue $\{1\}$ with corresponding eigenvector $\Omega_{\mathcal{S}}$, then we have

$$\lim_{m\to\infty} T_\lambda^m = |\Omega_{\mathcal{S}}\rangle\langle\Omega_{\mathcal{S}}^*|, \tag{1.15}$$

where $\Omega_{\mathcal{S}}^*$ is the unique invariant vector of T_λ^* normalized as $\langle \Omega_{\mathcal{S}}^*, \Omega_{\mathcal{S}} \rangle = 1$. The asymptotic state of observables on $\mathcal{S}$ is then

$$\omega_\infty(A_{\mathcal{S}}) = \langle \Omega_{\mathcal{S}}^*, A_{\mathcal{S}} \Omega_{\mathcal{S}} \rangle. \tag{1.16}$$

1.2 Main Results

We now show how to enlarge the above formalism to allow for a broader range of initial states for $\mathcal{P}$. Since $\Omega_{\mathcal{P}}$ is cyclic and separating, one can generate any other state for $\mathcal{P}$ by acting on $\Omega_{\mathcal{P}}$ with an element of $\mathcal{M}'_{\mathcal{P}}$ (commutant). That is, for any $\Psi_{\mathcal{P}} \in \mathcal{H}_{\mathcal{P}}$, $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$ for some $(\mathbb{1} \otimes b) \in \mathcal{M}'_{\mathcal{P}}$. It turns out that modifying the incoming state of the probes results in a new term in the effective dynamics generator. This is the main result and is stated in the following theorem:

Theorem 1. *For $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$ where b is invertible or $[b, h_{\mathcal{P}}] = 0$ the discrete dynamics generator is given by*

$$T_{\lambda}^{(b)} = \langle \Omega_{\mathcal{P}}, (\mathbb{1} \otimes b^* b) \mathrm{e}^{\mathrm{i}\tau K_{\lambda}} \Omega_{\mathcal{P}} \rangle. \tag{1.17}$$

The proof is found in section 2.

We will now give two examples demonstrating the new results. We consider the case where both the small system $\mathcal{S}$ and the probes $\mathcal{P}$ are two-level systems. The von Neumann algebras of observables for the small system and the probes are ($\chi = \{\mathcal{S}, \mathcal{P}\}$)

$$\mathcal{M}_{\chi} = M_2(\mathbb{C}) \otimes \mathbb{1}, \tag{1.18}$$

which act on the Hilbert spaces

$$\mathcal{H}_{\chi} = \mathbb{C}^2 \otimes \mathbb{C}^2, \tag{1.19}$$

respectively. We denote by $\{\phi_1, \phi_2\}$ the basis of $\mathbb{C}^2$. Then $\mathcal{H}_{\mathcal{S}}$ has for a basis $\phi_i \otimes \phi_j := \phi_{ij}$, $i, j = 1, 2$. To avoid confusion, we denote the basis of $\mathcal{H}_{\mathcal{P}}$ as ψ_{ij}, $i, j = 1, 2$.

The free evolution of the small system and the probes is given by

$$\alpha_{\chi}^{t}(A \otimes \mathbb{1}) = \mathrm{e}^{\mathrm{i}th_{\chi}} A \mathrm{e}^{-\mathrm{i}th_{\chi}} \otimes \mathbb{1} \tag{1.20}$$

where

$$h_{\chi} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.21}$$

We take the reference states of the small system and the probe to be the trace state. That is,

$$\Omega_{\mathcal{S}} = \frac{1}{\sqrt{2}}(\phi_{11} + \phi_{22}) \tag{1.22}$$

and

$$\Omega_{\mathcal{P}} = \frac{1}{\sqrt{2}}(\psi_{11} + \psi_{22}) \tag{1.23}$$

Then for the small system and the probes the Liouvillian is

$$L_{\chi} = h_{\chi} \otimes \mathbb{1} - \mathbb{1} \otimes h_{\chi}, \tag{1.24}$$

and the associated modular conjugate and modular operator are

$$\mathcal{J}_{\chi}(\alpha \otimes \beta) = \overline{\alpha} \otimes \overline{\beta}, \qquad \Delta_{\chi} \mathbb{1} \otimes \mathbb{1}. \tag{1.25}$$

Finally, let a and a^* denote the annihilation and creation operators, respectively. That is,

$$a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad a^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{1.26}$$

The interaction between the small system and each probe is governed by a fixed $\tau \in \mathbb{R}^+$ and a self-adjoint $V \in \mathcal{M}_{\mathcal{S}} \otimes \mathcal{M}_{\mathcal{P}}$ where

$$V = a \otimes \mathbb{1}_{\mathcal{S}} \otimes a^* \otimes \mathbb{1}_{\mathcal{P}} + a^* \otimes \mathbb{1}_{\mathcal{S}} \otimes a \otimes \mathbb{1}_{\mathcal{P}}, \tag{1.27}$$

the so-called Jaynes-Cummings interaction [6].

We would now like to consider two different incoming states for the probe. The first will be the state given by the density matrix

$$\rho^{(1)} = p\Big|\frac{1}{\sqrt{2}}(\phi_1+\phi_2)\Big\rangle\Big\langle\frac{1}{\sqrt{2}}(\phi_1+\phi_2)\Big| + (1-p)\Big|\frac{1}{\sqrt{2}}(\phi_1-\phi_2)\Big\rangle\Big\langle\frac{1}{\sqrt{2}}(\phi_1-\phi_2)\Big|, \quad p\in[0,1] \tag{1.28}$$

In the GNS construction this state is

$$\Psi_{\mathcal{P}} = \sqrt{2(p-p^2)}\psi_{11} + \frac{2p-1}{\sqrt{2}}\psi_{12} + \frac{1}{\sqrt{2}}\psi_{22}. \tag{1.29}$$

The operator

$$\mathbb{1}\otimes b = \mathbb{1}\otimes\begin{bmatrix} 2\sqrt{p-p^2} & 0 \\ 2p-1 & 1\end{bmatrix} \quad \in \mathcal{M}_{\mathcal{P}}' \tag{1.30}$$

satisfies $\Psi_{\mathcal{P}} = (\mathbb{1}\otimes b)\Omega_{\mathcal{P}}$. This operator is invertible so we have by Theorem 1

$$T_\lambda^{(b)} = \langle\Omega_{\mathcal{P}}, (\mathbb{1}\otimes b^*b)\mathrm{e}^{\mathrm{i}\tau(K_0+\lambda W)}\Omega_{\mathcal{P}}\rangle. \tag{1.31}$$

where

$$K_0 = L_{\mathcal{S}} + L_{\mathcal{P}}, \qquad W = V - \mathcal{J}V\mathcal{J} \tag{1.32}$$

Now, to determine the asymptotic state of the system we must determine

$$\lim_{n\to\infty}(T_\lambda^{(b)})^n = \lim_{n\to\infty}\Big(\langle\Omega_{\mathcal{P}}, (\mathbb{1}\otimes b^*b)\mathrm{e}^{\mathrm{i}\tau(K_0+\lambda W)}\Omega_{\mathcal{P}}\rangle\Big)^n. \tag{1.33}$$

First, we use the Dyson series expansion to approximate $\mathrm{e}^{\mathrm{i}\tau(K_0+\lambda W)}$:

$$\mathrm{e}^{\mathrm{i}\tau(K_0+\lambda W)} = \mathrm{e}^{\mathrm{i}\tau K_0} + \mathrm{i}\lambda\int_0^\tau \mathrm{e}^{\mathrm{i}(\tau-t)K_0}W\mathrm{e}^{\mathrm{i}tK_0}\mathrm{d}t - \lambda^2\int_0^\tau\int_0^t \mathrm{e}^{\mathrm{i}(\tau-t)K_0}W\mathrm{e}^{\mathrm{i}(t-s)K_0}W\mathrm{e}^{\mathrm{i}sK_0}\mathrm{d}s\mathrm{d}t + O(\lambda^3). \tag{1.34}$$

Next, we substitute this series into (1.17) to get

$$\begin{aligned} T_\lambda^{(b)} &= \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} + \frac{\mathrm{i}\lambda\tau(2p-1)}{2}\mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}}\left(\mathrm{e}^{-2\mathrm{i}\tau}(a\otimes\mathbb{1} - \mathbb{1}\otimes a^*) + \mathrm{e}^{2\mathrm{i}\tau}(a^*\otimes\mathbb{1} - \mathbb{1}\otimes a)\right) \\ &\quad - \frac{\lambda^2\tau^2}{4}\mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}}\left(aa^*\otimes\mathbb{1} - 2a\otimes a - 2a^*\otimes a^* + \mathbb{1}\otimes aa^* + \mathbb{1}\otimes a^*a\right) + O(\lambda^3). \end{aligned} \tag{1.35}$$

Using perturbation theory [2] we find that $T_\lambda^{(b)}$ has 4 distinct eigenvalues. These are 1, $e_+(\lambda)$, $e_-(\lambda) = \overline{e_+(\lambda)}$ and $e_1(\lambda)$ where

$$e_+(\lambda) = \mathrm{e}^{2\mathrm{i}\tau}\left[1 - \frac{\lambda^2\tau^2}{2}\left(\frac{\mathrm{e}^{2\mathrm{i}\tau} - 4p + 4p^2}{\mathrm{e}^{2\mathrm{i}\tau}-1}\right)\right] + O(\lambda^3), \tag{1.36}$$

$$e_1(\lambda) = 1 - \frac{\lambda^2\tau^2}{2}(1+4p-4p^2) + O(\lambda^{5/2}). \tag{1.37}$$

Since $\lim_{n\to\infty}(T_\lambda^{(b)})^n = |\Omega_{\mathcal{S}}\rangle\langle\Omega_{\mathcal{S}}^*|$ for $\lambda\neq 0$, we now need only compute the eigenvector associated to eigenvalue 1 of T_λ^*. This is found easily to be $\Omega_{\mathcal{S}}^* = \Omega_{\mathcal{S}} + O(\lambda^2)$. The asymptotic state is then

$$\omega_\infty^{(1)}(A_{\mathcal{S}}) = \langle\Omega_{\mathcal{S}}, A_{\mathcal{S}}\Omega_{\mathcal{S}}\rangle + O(\lambda^2), \qquad A_{\mathcal{S}}\in\mathcal{M}_{\mathcal{S}}. \tag{1.38}$$

The second incoming state will be given by the density matrix

$$\rho^{(2)} = \mathfrak{p}|\phi_1\rangle\langle\phi_1| + (1-\mathfrak{p})|\phi_2\rangle\langle\phi_2|, \quad \mathfrak{p}\in(0,1). \tag{1.39}$$

In the GNS construction this state is

$$\Psi_{\mathcal{P}} = \frac{1}{\sqrt{2\mathfrak{p}^2 - 2\mathfrak{p} + 1}} \left(\mathfrak{p}\psi_{11} + (1-\mathfrak{p})\psi_{22}\right). \tag{1.40}$$

The operator

$$(\mathbb{1} \otimes b) = \mathbb{1} \otimes \sqrt{\frac{2}{2\mathfrak{p}^2 - 2\mathfrak{p} + 1}} \begin{bmatrix} \mathfrak{p} & 0 \\ 0 & 1-\mathfrak{p} \end{bmatrix} \tag{1.41}$$

satisfies $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$. This operator is diagonal so by Theorem 1 we again use (1.31) - (1.33).

Substituting the Dyson series approximation (1.34) into (1.17) gives

$$\begin{aligned} T_\lambda^{(b)} = \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} - \frac{\lambda^2\tau^2}{2(1-2\mathfrak{p}-2\mathfrak{p}^2)} \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \Big(& (\mathfrak{p}-1)^2 (aa^* \otimes \mathbb{1} + \mathbb{1} \otimes aa^* - 2a \otimes a) \\ & + \mathfrak{p}^2 (a^*a \otimes \mathbb{1} + \mathbb{1} \otimes a^*a - 2a^* \otimes a)\Big) + O(\lambda^3). \end{aligned} \tag{1.42}$$

The matrix representation of $T_\lambda^{(b)}$ to second order in the ϕ_{ij} basis is triangular and it has 4 distinct eigenvalues: 1, $e_+(\lambda)$, $e_-(\lambda) = \overline{e_+(\lambda)}$ and $e_1(\lambda)$ where

$$e_+(\lambda) = \mathrm{e}^{2\mathrm{i}\tau} \left(1 - \frac{\lambda^2\tau^2}{2}\right) + O(\lambda^3), \tag{1.43}$$

$$e_1(\lambda) = 1 - \lambda^2\tau^2 + O(\lambda^3). \tag{1.44}$$

The eigenvector associated with eigenvalue 1 of T_λ^* can also be found explicitly to be

$$\Omega_{\mathcal{S}}^* = \sqrt{2} \left(\frac{\mathfrak{p}^2}{1 - 2\mathfrak{p} + 2\mathfrak{p}^2} \phi_{11} + \frac{(\mathfrak{p}-1)^2}{1-2\mathfrak{p}+2\mathfrak{p}^2} \phi_{22} \right) + O(\lambda^2) \tag{1.45}$$

and so the asymptotic state is

$$\omega_\infty^{(2)}(A_{\mathcal{S}}) = \langle \Omega_{\mathcal{S}}^*, A_{\mathcal{S}} \Omega_{\mathcal{S}} \rangle + O(\lambda^2), \qquad A_{\mathcal{S}} \in \mathcal{M}_{\mathcal{S}}. \tag{1.46}$$

As a check, we can set $p = \mathfrak{p} = \frac{1}{2}$ in the above examples. In doing so we end up with $\omega_\infty^{(1)}(A_{\mathcal{S}}) = \omega_\infty^{(2)}(A_{\mathcal{S}})$ in both examples as one would expect since (1.28) and (1.39) coincide.

2 Proof of Theorem 1

The proof will be split into two cases: one where b is invertible and the other where $[b, h_{\mathcal{P}}] = 0$.

2.1 Case 1

We begin with a general result:

Lemma 1. *Suppose* $\mathcal{H}$ *is a Hilbert space with* $\dim \mathcal{H} < \infty$, $\mathcal{M}$ *is a Von Neumann algebra over* $\mathcal{H}$ *and* $\Omega \in \mathcal{H}$ *is cyclic and separating. Let* $B \in \mathcal{M}$ *and* $B' \in \mathcal{M}'$ *(commutant). Then we have*
(a) $\mathrm{Ker}B = \{0\} \Leftrightarrow \Psi = B\Omega$ *is cyclic and separating for* $\mathcal{M}$.
(b) $\mathrm{Ker}B' = \{0\} \Leftrightarrow \Psi = B'\Omega$ *is cyclic and separating for* $\mathcal{M}'$.

Proof. (a) "$\Rightarrow$" Assume $\mathrm{Ker}B = \{0\}$ and consider $M' \in \mathcal{M}'$ such that $M'\Psi = 0$. Then $0 = M'\Psi = M'B\Omega = BM'\Omega$, so $M'\Omega \in \mathrm{Ker}B$ and hence $M'\Omega = 0$. Since Ω is cyclic for $\mathcal{M}$ we have $M' = 0$. Therefore Ψ is cyclic for $\mathcal{M}$

To show that Ψ is separating, we note that $\mathcal{M}'\Psi = \mathcal{M}'B\Omega = B\mathcal{M}'\Omega = B\mathcal{H} = \mathrm{Im}(B)$. But the kernel of B is trivial, so $\mathrm{Im}(B) = \mathcal{H}$ and hence Ψ is separating for $\mathcal{M}$.

"$\Leftarrow$" Assume Ψ is cyclic for $\mathcal{M}$. Then $\mathcal{H} = \mathcal{M}'\Psi = \mathcal{M}'B\Omega = B\mathcal{M}'\Omega = B\mathcal{H} = \mathrm{Im}(B)$, and therefore $\mathrm{Ker}B = \{0\}$.

(b) The proof is exactly the same as (a), except change B to B', M' to M and $\mathcal{M}'$ to $\mathcal{M}$ (and interchange the words 'cyclic' and 'separating'). $\square$

In other words, states $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$, where $(\mathbb{1} \otimes b^{-1})$ exists, are again cyclic and separating. For the rest of this section we will only consider such $\Psi_{\mathcal{P}}$.

For $\Psi_{\mathcal{P}}$ to be a reference state on $\mathcal{M}_{\mathcal{P}}$ it would have to determine a state on $\mathcal{M}_{\mathcal{P}}$ that is invariant under $\alpha^t_{\mathcal{P}}$. The following lemma shows how to construct this state.

Lemma 2. *For $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$, the Liouvillian $L^{(b)}_{\mathcal{P}} = h_{\mathcal{P}} \otimes \mathbb{1} - \mathbb{1} \otimes b h_{\mathcal{P}} b^{-1}$ satisfies:*

$$L^{(b)}_{\mathcal{P}} \Psi_{\mathcal{P}} = 0. \tag{2.1}$$

Proof. The proof is straight forward.

$$L^{(b)}_{\mathcal{P}} \Psi_{\mathcal{P}} = \left[(h_{\mathcal{P}} \otimes \mathbb{1}) - (\mathbb{1} \otimes b h_{\mathcal{P}} b^{-1})\right] (\mathbb{1} \otimes b)\Omega_{\mathcal{P}} = (\mathbb{1} \otimes b) \left[(h_{\mathcal{P}} \otimes \mathbb{1}) - (\mathbb{1} \otimes h_{\mathcal{P}})\right] \Omega_{\mathcal{P}} = 0$$

where the last equality holds since $L_{\mathcal{P}}\Omega_{\mathcal{P}} = 0$ by assumption. $\square$

Putting the above two lemmas together, we get the following theorem:

Theorem 2. *Given a reference state $\Omega_{\mathcal{P}}$ one can generate new reference states of the form $\Psi_{\mathcal{P}} = (\mathbb{1} \otimes b)\Omega_{\mathcal{P}}$, b invertible, by a suitable modification of the Liouvillian given by (2.1). The modular data, $(\mathcal{J}_b, \Delta^{1/2}_b)$ associated to the new reference states can be determined from $(\mathcal{J}, \Delta^{1/2})$, the modular data associated to $\Omega_{\mathcal{P}}$ by:*

$$\mathcal{J}_b = \left[(\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1} \otimes b^{-1})\right] \left[(\mathbb{1} \otimes (b^*)^{-1})\Delta^{1/2}(\overline{b^* b} \otimes \mathbb{1})\Delta^{1/2}(\mathbb{1} \otimes b^{-1})\right]^{-1/2}, \tag{2.2}$$

and,

$$\Delta^{1/2}_b = \left[(\mathbb{1} \otimes (b^*)^{-1})\Delta^{1/2}(\overline{b^* b} \otimes \mathbb{1})\Delta^{1/2}(\mathbb{1} \otimes b^{-1})\right]^{1/2}. \tag{2.3}$$

In particular,

$$\mathcal{J}_b \Delta^{1/2}_b = (\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1} \otimes b^{-1}). \tag{2.4}$$

Proof. We only need to show how to construct the modified modular data. First we must find S_b such that $S_b(M \otimes \mathbb{1})\Psi_{\mathcal{P}} = (M^* \otimes \mathbb{1})\Psi_{\mathcal{P}}\ \forall M \otimes \mathbb{1} \in \mathcal{M}_{\mathcal{P}}$:

$$\begin{aligned}
(M^* \otimes \mathbb{1})\Psi_{\mathcal{P}} &= (M^* \otimes \mathbb{1})(\mathbb{1} \otimes b)\Omega_{\mathcal{P}} \\
&= (\mathbb{1} \otimes b)(M^* \otimes \mathbb{1})\Omega_{\mathcal{P}} \\
&= (\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(M \otimes \mathbb{1})\Omega_{\mathcal{P}} \\
&= (\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1} \otimes b^{-1})(\mathbb{1} \otimes b)(M \otimes \mathbb{1})\Omega_{\mathcal{P}} \\
&= (\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1} \otimes b^{-1})(M \otimes \mathbb{1})\Psi_{\mathcal{P}} \\
&:= S_B(M \otimes \mathbb{1})\Psi_{\mathcal{P}}
\end{aligned}$$

By the polar decomposition of S_b:

$$\begin{aligned}
\Delta^{1/2}_b &= (S^*_b S_b)^{1/2} = \left[(\mathbb{1} \otimes (b^*)^{-1})\Delta^{1/2}\mathcal{J}(\mathbb{1} \otimes b^*)(\mathbb{1} \otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1} \otimes b^{-1})\right]^{1/2} \\
&= \left[(\mathbb{1} \otimes (b^*)^{-1})\Delta^{1/2}(\overline{b^* b} \otimes \mathbb{1})\Delta^{1/2}(\mathbb{1} \otimes b^{-1})\right]^{1/2}
\end{aligned}$$

Then, using the fact that $\Delta_b^{-1/2}$ exists and $S_b = \mathcal{J}_b\Delta_b^{1/2}$ we get:

$$\mathcal{J}_b = S_b\Delta_b^{-1/2} = \Big[(\mathbb{1}\otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1}\otimes b^{-1})\Big]\Big[(\mathbb{1}\otimes (b^*)^{-1})\Delta^{1/2}(\overline{b^*b}\otimes\mathbb{1})\Delta^{1/2}(\mathbb{1}\otimes b^{-1})\Big]^{-1/2}$$

□

Now, since we have a reference state $\Psi_{\mathcal{P}}$ and its associated modular data we can now construct the effective dynamics generator [1]

$$T_\lambda^{(b)} = P_b \mathrm{e}^{\mathrm{i}\tau K_\lambda^{(b)}} P_b, \tag{2.5}$$

where

$$P_b = \mathbb{1}\otimes|\Psi_{\mathcal{P}}\rangle\langle\Psi_{\mathcal{P}}|. \tag{2.6}$$

We can however simplify $T_\lambda^{(b)}$ slightly since $K_\lambda^{(b)} = (\mathbb{1}\otimes b)K_\lambda(\mathbb{1}\otimes b^{-1})$ where K_λ is the C-Liouville operator associated with $\Omega_{\mathcal{P}}$. This is easily seen using Theorem 2:

$$\begin{aligned}
K_\lambda^{(b)} &= L_{\mathcal{S}} + L_{\mathcal{P}}^{(b)} + \lambda(V - \mathcal{J}_b\Delta_b^{1/2}V\mathcal{J}_b\Delta_b^{1/2}) && (2.7)\\
&= L_{\mathcal{S}} + (\mathbb{1}\otimes b)L_{\mathcal{P}}(\mathbb{1}\otimes b^{-1}) + \lambda V - \lambda(\mathbb{1}\otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1}\otimes b^{-1})V(\mathbb{1}\otimes b)\mathcal{J}\Delta^{1/2}(\mathbb{1}\otimes b^{-1}) && (2.8)\\
&= (\mathbb{1}\otimes b)(L_{\mathcal{S}} + L_{\mathcal{P}} + \lambda(V - \mathcal{J}\Delta^{1/2}V\mathcal{J}\Delta^{1/2}))(\mathbb{1}\otimes b^{-1}) && (2.9)\\
&= (\mathbb{1}\otimes b)K_\lambda(\mathbb{1}\otimes b^{-1}). && (2.10)
\end{aligned}$$

Thus $\mathrm{e}^{\mathrm{i}\tau K_\lambda^{(b)}} = (\mathbb{1}\otimes b)\mathrm{e}^{\mathrm{i}\tau K_\lambda}(\mathbb{1}\otimes b^{-1})$ and

$$T_\lambda^{(b)} = P_b(\mathbb{1}\otimes b)\mathrm{e}^{\mathrm{i}\tau K_\lambda}(\mathbb{1}\otimes b^{-1})P_b \simeq \langle\Omega_{\mathcal{P}}, (\mathbb{1}\otimes b^*b)\mathrm{e}^{\mathrm{i}\tau K_\lambda}\Omega_{\mathcal{P}}\rangle, \tag{2.11}$$

as desired.

2.2 Case 2

The second case is more direct. We once again consider $\Psi_{\mathcal{P}} = (\mathbb{1}\otimes b)\Omega_{\mathcal{P}}$ but this time assume that $[b, h_{\mathcal{P}}] = 0$ i.e., that b be diagonal in the energy eigenbasis. To construct the discrete dynamics generator we simply plug in the modified initial state $\Psi_0 = \Omega_{\mathcal{S}}\otimes(\mathbb{1}\otimes b)\Omega_{\mathcal{P}}\otimes(\mathbb{1}\otimes b)\Omega_{\mathcal{P}}\otimes\ldots$:

$$\begin{aligned}
\langle\alpha_{RI}^n(A_{\mathcal{S}})\rangle &= \langle\Psi_0,\ \mathrm{e}^{\mathrm{i}\tau\widetilde{L_1}}\ldots\mathrm{e}^{\mathrm{i}\tau\widetilde{L_n}}A_S\mathrm{e}^{-\mathrm{i}\tau\widetilde{L_n}}\ldots\mathrm{e}^{-\mathrm{i}\tau\widetilde{L_1}}\Psi_0\rangle\\
&= \langle\Psi_0,\ (U_n^{+*})\mathrm{e}^{\mathrm{i}\tau L_1}\ldots\mathrm{e}^{\mathrm{i}\tau L_n}(U_n^{-*})A_S(U_n^{-})\mathrm{e}^{-\mathrm{i}\tau L_n}\ldots\mathrm{e}^{-\mathrm{i}\tau L_1}(U_n^{+})\Psi_0\rangle\\
&= \langle(U_n^{+})\Psi_0,\ \mathrm{e}^{\mathrm{i}\tau L_1}\ldots\mathrm{e}^{\mathrm{i}\tau L_n}A_S\mathrm{e}^{-\mathrm{i}\tau L_n}\ldots\mathrm{e}^{-\mathrm{i}\tau L_1}(U_n^{+})\Psi_0\rangle\\
\text{for}\quad U_n^- &= exp(-\mathrm{i}\tau\sum_{j=1}^{n}(n-j)L_{\mathcal{P},j}) \text{ and } U_n^+ = exp(-\mathrm{i}\tau\sum_{j=2}^{n}(j-1)L_{\mathcal{P},j})
\end{aligned}$$

Consider: $U_n^+\Psi_0 = exp(-\mathrm{i}\tau\sum_{j=2}^n(j-1)L_{\mathcal{P},j})(\Omega_{\mathcal{S}}\otimes(\mathbb{1}\otimes b)\Omega_{\mathcal{P}}\otimes(\mathbb{1}\otimes b)\Omega_{\mathcal{P}}\otimes\ldots)$. The j-th factor in the chain is:

$$\mathrm{e}^{-\mathrm{i}\tau(j-1)L_{\mathcal{P},j}}(\mathbb{1}\otimes b)\Omega_{\mathcal{P}} = (\mathbb{1}\otimes b)\Omega_{\mathcal{P}}$$

Let $C_j' = (\mathbb{1}\otimes b)\in\mathcal{M}_{\mathcal{P}}'$ act on the j-th factor of the chain. Then $U_n^+\Psi_0 = C_1'\ldots C_n'\Omega := C_1'\ldots C_n'(\Omega_{\mathcal{S}}\otimes\Omega_{\mathcal{P}}\otimes\Omega_{\mathcal{P}}\otimes\ldots)$ and it follows that

$$\begin{aligned}
\langle\alpha_{RI}^n(A_{\mathcal{S}})\rangle &= \langle C_1'\ldots C_n'\Omega,\ \mathrm{e}^{\mathrm{i}\tau L_1}\ldots\mathrm{e}^{\mathrm{i}\tau L_n}A_S\mathrm{e}^{-\mathrm{i}\tau L_n}\ldots\mathrm{e}^{-\mathrm{i}\tau L_1}C_1'\ldots C_n'\Omega\rangle\\
&= \langle C_1'\ldots C_n'\Omega,\ C_1'\ldots C_n'\mathrm{e}^{\mathrm{i}\tau K_1}\ldots\mathrm{e}^{\mathrm{i}\tau K_n}A_S\Omega\rangle\\
&= \langle\Omega,\ C_1'^{*}C_1'\mathrm{e}^{\mathrm{i}\tau K_1}\ldots C_n'^{*}C_n'\mathrm{e}^{\mathrm{i}\tau K_n}A_S\Omega\rangle
\end{aligned}$$

Let $P^{(j)} = \mathbb{1}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{P}} \otimes \ldots \otimes |\Omega_{\mathcal{P}}\rangle\langle\Omega_{\mathcal{P}}| \otimes \ldots$ act non-trivially only on the j-th probe. Then:

$$\langle A_S \rangle_n = \langle \Omega,\ P^{(1)}(\mathbb{1} \otimes b^* b)\mathrm{e}^{\mathrm{i}\tau K_1} P^{(1)} \ldots P^{(n)}(\mathbb{1} \otimes b^* b)\mathrm{e}^{\mathrm{i}\tau K_n} P^{(n)} A_S \Omega\rangle. \tag{2.12}$$

So the effective dynamics generator is

$$T_\lambda^{(b)} = P(\mathbb{1} \times b^* b)\mathrm{e}^{\mathrm{i}\tau K_\lambda} P \simeq \langle \Omega_{\mathcal{P}}, (\mathbb{1} \otimes b^* b)\mathrm{e}^{\mathrm{i}\tau K_\lambda} \Omega_{\mathcal{P}}\rangle. \tag{2.13}$$

Acknowledgements

This work was completed under the supervision of Marco Merkli during the summer of 2009 during which time the author was an undergraduate student researcher holding an NSERC USRA. The author is grateful to both NSERC for their financial support and to Marco Merkli for his guidance.

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