Edge-Colouring Hypergraphs Properly (Covering with Matchings) or Polychromatically (Packing Covers)

David Pritchard and Thomas Rothvoß

October 1, 2010

Abstract

Let the *chromatic index* of a hypergraph be the smallest number of colours needed to colour the edges such that similarly-coloured edges are disjoint. Likewise, let the *cover index* be the maximum number of colours so that each colour class covers all vertices. Trivially the chromatic index is at least the maximum degree, and the cover index is at most the minimum degree. We survey some classes of structured hypergraphs and ask how far the trivial bounds are from tight. We are motivated by a large amount of recent work in this area for geometric settings.

1 Introduction

This paper discusses generalizations of *edge colouring* in graphs, and so we begin by reviewing this basic concept. Given a graph (V, E), a *k-edge-colouring* is an assignment $\chi : E \to \{1, \dots, k\}$ of k colours such that at each vertex, the incident edges all get distinct colours. Clearly no such colouring exists unless k is at least the maximum degree Δ; and clearly one exists for large enough k. Thus we're interested in the smallest k for which a k-edge-colouring exists — this number is traditionally called the *chromatic index* or less commonly the *edge-chromatic number*.

Classical results of Vizing and Shannon show that the chromatic index is pretty close to its trivial lower bound of Δ. For *simple graphs*, where any pair of vertices can be joined by at most one edge, Vizing proved the chromatic index is at most $\Delta + 1$. On the other hand for *multigraphs* where multiple edges may join the same pair of vertices, Shannon proved the chromatic index is at most $\lfloor \frac{3}{2}\Delta \rfloor$. Both of these bounds are tight, e.g. for all Δ there is a multigraph with maximum degree Δ where the chromatic index is exactly $\lfloor \frac{3}{2}\Delta \rfloor$; in fact the tight example is simply a triangle with the edges duplicated an appropriate number of times.

We can restate a k-edge-colouring as *a partition of the edge set into k "colour" classes, so that the subgraph corresponding to each class has degree at most 1 at each vertex.* Dually, it is just as interesting to seek subgraphs with degree *at least 1* at each vertex. As such, we call the *cover index*[1] of a graph the maximum k such that there is an assignment $\chi : E \to \{1, \dots, k\}$ so that at each vertex, at least one edge of each colour is incident. Analogous to before, the minimum degree δ is a trivial upper bound on the cover index. Again δ is close to a lower bound, since Gupta [26, §28.8] proved in 1978 the cover index is at least $\delta - 1$ on all simple graphs; and Gupta [17] also proved[2] the cover index is at least $\lfloor \frac{3\delta+1}{4} \rfloor$ in all multigraphs. Again, both bounds are tight, the latter even if there are only 3 vertices.

Here is a computational fact relating the chromatic index and the cover index. Holyer [18] showed that it is NP-complete to determine the chromatic index even if G is a simple cubic graph. For cubic simple graphs, the chromatic index is either 3 or 4; notice that the former is equivalent to saying that the edges can be partitioned into three perfect matchings. Similarly, we can see that for cubic simple graphs, the cover index is 3 if and only if the edges can be partitioned into three perfect matchings. Hence for cubic simple graphs it is NP-complete to compute the cover index.

[1] Also called the *edge cover chromatic index* [29], the *polychromatic number* [2], or the *edge cover packing number* [26, p. 479]

[2] This theorem was reproved in 1979 [5] but both papers [5, 17] seem to be "lost" for a while [29, 25]. In [2] a slick proof is given; again Gupta's work was mentioned in the journal version, but unnoticed in the conference version.

1.1 Multi-Hypergraphs

A hypergraph (sometimes called a *set system* or *range space*) is a generalization of a graph where edges are replaced by *hyperedges* which may contain more than two vertices. In a *multi-hypergraph* the same hyperedge can appear any number of times. **Attention:** we always work with multi-hypergraphs in this paper even though we usually do not say "multi" explicitly.

Generalizing the cover index, for a hypergraph $(V, \mathcal{E})$, let its *covering decomposition number*, denoted $cd(V, \mathcal{E})$, be the maximum k for which $\mathcal{E}$ can be partitioned into k sets $\mathcal{E} = \biguplus_{i=1}^{k} \mathcal{E}_i$ such that each $\mathcal{E}_i$ covers all of V (i.e., such that $\bigcup\{E \mid E \in \mathcal{E}_i\} = V$ for all i). Likewise, generalizing the chromatic index, let its *packing decomposition number*, denoted $pd(V, \mathcal{E})$ be the minimum k for which $\mathcal{E}$ can be partitioned into k sets $\mathcal{E} = \biguplus_{i=1}^{k} \mathcal{E}_i$ such that each $\mathcal{E}_i$ is a family of disjoint sets. We define the degree analogously to before — the degree of vertex v is the number of hyperedges containing v — and the minimum (resp. maximum) degree is denoted δ (resp. Δ).

Then, for a family $\mathcal{F}$ of hypergraphs, let its *cover-decomposition function* $\mathsf{C}(\mathcal{F}, \delta)$ be

$$\mathsf{C}(\mathcal{F}, \delta) := \min\{cd(V, \mathcal{E}) \mid (V, \mathcal{E}) \in \mathcal{F} \ \& \ \forall v \in V, \deg_{\mathcal{E}}(v) \geq \delta\},$$

i.e. the minimum covering decomposition number for hypergraphs in $\mathcal{F}$ of min-degree at least δ. Similarly define

$$\mathsf{P}(\mathcal{F}, \Delta) := \max\{pd(V, \mathcal{E}) \mid (V, \mathcal{E}) \in \mathcal{F} \ \& \ \forall v \in V, \deg_{\mathcal{E}}(v) \leq \Delta\}.$$

If the family of graphs is clear from context we will simply write $\mathsf{C}(\delta)$ and $\mathsf{P}(\Delta)$.

For example, let Graphs be the family of hypergraphs where every edge has size at most 2. Then Gupta's and Shannon's theorems are $\mathsf{C}(\text{Graphs}, \delta) = \lfloor \frac{3\delta+1}{4} \rfloor$ and $\mathsf{P}(\text{Graphs}, \Delta) = \lfloor \frac{3}{2}\Delta \rfloor$.

We remark that the statement $cd(H) > 1$ is equivalent to saying the transpose hypergraph H^t has "Property B," a well-studied notion. (The points of H^t are the sets of H and vice-versa, with point-set incidence preserved).

We always have the trivial bounds $\mathsf{C}(\delta) \leq \delta$ and $\mathsf{P}(\Delta) \geq \Delta$ (assuming a nonempty set of hypergraphs meet the degree constraint). We seek to determine, in natural settings, how near to tight are the trivial bounds? For example, when is $\mathsf{C}(\delta) = \Omega(\delta)$ or $\mathsf{P}(\Delta) = O(\Delta)$?

1.2 Overview of Results

In Section 2 we show that for hypergraphs with edges of size at most r we have $\mathsf{C}(\delta) = \max\{1, \Theta(\delta/\log r)\}$ and $\mathsf{P}(\Delta) = \Theta(r\Delta)$. To prove this we weave together several results from the literature including the Lovász Local Lemma and iterated rounding methods.

In Section 3 we consider some structured hypergraphs. In interval hypergraphs we observe $\mathsf{C}(\delta) = \delta$ and $\mathsf{P}(\Delta) = \Delta$. Then, we consider hypergraphs corresponding to paths in trees, and their transposes, obtaining asymptotically tight bounds on C and P for both.

1.3 Geometric Settings and Other Related Work

In geometric literature, a hypergraph family is said to be *cover-decomposable* if there is any finite t with $\mathsf{C}(t) > 1$. One of the motivating questions for cover-decomposability in geometry is the following problem of Pach [22] which has remained open for 30 years. For a planar set A, consider the family of hypergraphs whose ground set is $\mathbb{R}^2$, and whose edges are the translates of A. Then Pach conjectured that for every *convex* A, this family is cover-decomposable. One state-of-the-art partial result, due to Gibson & Varadarajan [15], is that $\mathsf{C}(\delta) = \Omega(\delta)$ when A is an open convex *polygon*. We refer the reader to a current survey [23] for details, but mention here a couple more results we think are relevant.

Let Hpl denote the family of hypergraphs such that the ground set is a finite set in $\mathbb{R}^2$, where a hyperedge may exist only if it corresponds to the subset lying in some halfspace. Recently Smorodinsky and Yuditsky [27] proved $\mathsf{C}(\text{Hpl}^t, \delta) = \lceil \delta/2 \rceil$ and $\mathsf{C}(\text{Hpl}, \delta) \geq \lceil \delta/3 \rceil$. They note the latter is not tight, since Fulek [14] showed $\mathsf{C}(\text{Hpl}, 3) = 2$.

There is recent work on a so-called *approximate integer decomposition property* [11] which can be phrased in a setting similar to what is presented here. It was used by Chekuri, Mydlarz & Shepherd [10] to get an approximation algorithm for packing paths in capacitated trees. The idea, roughly, is to take an LP relaxation's optimal solution and scale it up by a factor of k to make it integral, obtaining a solution violating the packing bounds by a factor of k; then show it can be decomposed into $4k$ individual feasible packings. Hence, once of these solutions has value at least $k/4k = 1/4$ times optimal, giving a 4-approximation.

Pálvölgyi [24, Question 9.5] raises a nice combinatorial question about cover-decomposability in "shift chains" which would shed some light on Pach's conjecture. Another question from [24] is whether one can bound $\min\{\delta \mid C(\delta) = 3\}$ in terms of $\min\{\delta \mid C(\delta) = 2\}$, when the family is required to be closed under deleting and parallelizing hyperedges.

We finished these notes and posted them to the arXiv after seeing an independent posting from Bollobás and Scott [9] on a similar theme. In particular they determine the asymptotics of C for hypergraphs of bounded edge size (our Theorem 1). In addition, they show $\{\min t \mid \mathsf{C}(t) = 2\} = 4$ both for the family of hypergraphs with maximum edge size 3, and for maximum edge size 4; and they establish differences in the behaviour of C between simple hypergraphs and multihypergraphs with bounded edge size.

2 Hypergraphs of Bounded Edge Size

Let $\textsc{Hyp}(r)$ denote the family of hypergraphs with all edges of size at most r. Then we have the following.

Proposition 1. $\mathsf{C}(\textsc{Hyp}(r), \delta) \geq \Omega(\delta/\log(r\delta))$.

The proof uses the Lovász Local Lemma. It extends a simple example (Theorem 5.2.1) in *The Probabilistic Method* [3], and was also essentially observed by Mani-Levitska and Pach [21].

Proof. Given any hypergraph $H = (V, \mathcal{E})$ where every edge has size at most r and such that each $v \in V$ is covered at least δ times, we must show for $t = \Omega(\delta/\log(r\delta))$ that $cd(\mathcal{E}) \geq t$, i.e. that $\mathcal{E}$ can be decomposed into t disjoint edge covers.

Consider the following randomized experiment: for each set (hyperedge) $E \in \mathcal{E}$, assign a random colour between 1 and t to E. If we can show that with positive probability, every vertex is incident with a hyperedge of each colour, then it will follow that $cd(V, \mathcal{E}) \geq t$. In order to get this approach to go through, it will be helpful that we reduce the degree of every vertex to *exactly* δ (this will reduce the so-called dependence degree); but actually $\deg(v) = \delta$ is without loss of generality since otherwise we can repeatedly "shrink": replace a set E containing v by $E\backslash\{v\}$ until the degree of v drops to δ.

For each vertex v define the *bad event* $\mathfrak{E}_v$ to be the event that v is not incident with a hyperedge of each colour. The probability of $\mathfrak{E}_v$ is at most $t(1-\frac{1}{t})^\delta$, by using a union bound. Now let us determine the dependence degree. The event $\mathfrak{E}_v$ only depends on the colours of the hyperedges containing v; therefore the events $\mathfrak{E}_v$ and $\mathfrak{E}_{v'}$ are independent unless v, v' are in a common hyperedge. Thus the dependence degree is at most $(r-1)\delta$. It follows by LLL that as long as

$$((r-1)\delta + 1)t(1-\tfrac{1}{t})^\delta \leq 1/\mathrm{e}$$

then with positive probability, no bad events happen. Using a little asymptotic analysis we can verify that for some $t = O(\delta/(\log r\delta))$, the Local Lemma goes through and we are done. □

In 3-uniform hypergraphs, the above reasoning shows $\mathsf{C}(\textsc{Hyp}(3), 7) \geq 2$, since $(2 \cdot 7 + 1)2(\frac{1}{2})^7 < 1/\mathrm{e}$. Compare this with the Fano plane which gives $\mathsf{C}(\textsc{Hyp}(3), 3) = 1$: if its seven sets are partitioned into two parts, one part has only three sets, and it is not hard to verify the only covers consisting of three sets are pencils through a point and therefore preclude the remaining sets from forming a cover. Bollobás and Scott [9] prove the sharp bound $\mathsf{C}(\textsc{Hyp}(3), 4) = 2$.

Proposition 1 can be improved to the following:

Proposition 2. $\mathsf{C}(\textsc{Hyp}(r), \delta) \geq \Omega(\delta/\log r)$.

Proof. If $\delta \le 3r$ then we are done by Proposition 1, so assume $\delta > 3r$. As in the proof of Proposition 1, we assume each vertex has degree exactly δ by shrinking edges if necessary.

A *fractional c-factor* is vector $x \in [0,1]^{\mathcal{E}}$ such that for each vertex v, the sum of the incident weights is c, i.e. $\sum_{E:v\in E} x_E = c$. Note that the constant function $x \equiv 2r/\delta$ is a fractional $2r$-factor. Then, we use an *iterated LP rounding* result of Karp et al. [19]; namely, the fact that the hyperedges have size at most r implies that this fractional factor can be converted to an integral $\widehat{x} \in \{0,1\}^E$ such that the subgraph $(V, \widehat{x})$ has degrees in $2r \pm (r-1)$. Deleting these edges and iterating, (and shrinking again to make the new graph $\delta - (3r-1)$-regular), we obtain $\lfloor \delta/(3r-1) \rfloor$ disjoint subhypergraphs, each with minimum degree $r+1$.

By Proposition 1, each of these graphs of min degree $r+1$ can be decomposed into $\Omega(r/\log r)$ covers. Therefore we get $\Omega(\delta/(3r-1)) \cdot \Omega(r/\log r) = \Omega(\delta/\log r)$ covers, as needed. □

A matching upper bound exists. Known results on Property B immediately imply that $\min\{\delta \mid \mathsf{C}(\textsc{Hyp}(r), \delta) > 1\}$ is a function of order $\Omega(\log r)$, but here is the more general matching bound.

Theorem 1. *We have* $\mathsf{C}(\textsc{Hyp}(r), \delta) = \max\{1, \Theta(\delta/\log r)\}$.

The proof is adapted from an example [6] illustrating a lack of a decomposition theorem for edge-connectivity in hypergraphs. We remark that an integrality gap construction of Vazirani for set cover [28, Ex. 13.4] improves the upper bound constant by a factor of 2, when $\delta \ge r$.

Proof. For $1 \le \delta \le O(\log r)$ clearly $cd \ge 1$ by taking the entire graph as a cover. Given Proposition 2 it remains only to show that $\mathsf{C}(\textsc{Hyp}(r), \delta) = O(\delta/\log r)$ for $\delta = \Omega(\log r)$.

For a positive integer k, consider the hypergraph $\binom{[2k-1]}{k}^t$: it has $2k-1$ edges, and $\binom{2k-1}{k}$ vertices such that for every set of k edges, some vertex appears in exactly those edges. Now, the minimum size of a cover in this hypergraph is k, since for any set of $k-1$ or fewer edges, they miss a vertex. This shows $\mathsf{C}(\binom{2k-2}{k-1}, k) = 1$ since the hypergraph is $\binom{2k-2}{k-1}$-uniform and k-regular. Moreover, take μ parallel copies of every edge to get a hypergraph exhibiting $\mathsf{C}(\binom{2k-2}{k-1}, \mu k) \le \lfloor \mu(2k-1)/k \rfloor$. Rearranging, the result follows. □

What about packing decomposition in $\textsc{Hyp}(r)$? It is illustrative to notice that *pd* is (strong-)edge-colouring but can also be viewed as (strong-)vertex-colouring the transpose, and that the parameters r, Δ are exchanged when taking the transpose.

Theorem 2. $\mathsf{P}(\textsc{Hyp}(r), \Delta) = \Theta(r\Delta)$ *for* $r \ge 1, \Delta \ge 2$.

Proof. The following *greedy* algorithm shows $\mathsf{P}(\textsc{Hyp}(r), \Delta) \le 1 + r(\Delta - 1)$: iteratively pick any edge e not yet coloured, and assign it any colour distinct from the colours already assigned to the other edges intersecting e. Each edge intersects at most $r(\Delta-1)$ other edges, whence the bound.

For the lower bound we use an example from [1, §3]. It suffices to consider the case that Δ is even. Take a hypergraph with $r+1$ vertex "groups" of size $\Delta/2$ and include a hyperedge containing every pair of groups. Its transpose shows $\mathsf{P}(\textsc{Hyp}(r), \Delta) \ge (r+1)\Delta/2$. □

In fact the bound from the greedy algorithm is exactly tight for many values of Δ, r: an $(v,k,2)$-*Steiner system* is a partition of the edges of K_v into k-cliques; and when it exists, it shows that $\mathsf{P}(\frac{v-1}{k-1}, k) = v$, matching the greedy bound. Whenever $v \ge k^2$, a $(v', k', 2)$-Steiner system exists [16] with $v' \approx v, k' \approx k$ up to a factor of 2.

3 Path, Tree, and Shore Hypergraphs

In a *path* or *interval* hypergraph, the ground set corresponds to the edges of a path, and hyperedges correspond to edges contained in some sub-paths. (Equivalently the ground set is $[n]$ and every hyperedge is an interval.) For this family $\textsc{Intervals}$ of hypergraphs, one can show that $\mathsf{C}(\delta) = \delta$ and $\mathsf{P}(\Delta) = \Delta$, i.e. the trivial bounds are exactly tight. The same holds for the transpose family $\textsc{Intervals}^t$. More generally, whenever the incidence matrix of the hypergraph is totally unimodular, by results on the *integer decomposition* *property* (IDP), we have $cd = \delta$ and $pd = \Delta$. (In fact the definition of IDP [8] quickly leads to the following: for every hypergraph H, let χ_H be its incidence matrix with columns as vertices and sets as rows, then the polyhedron $\{x \mid \chi_H x \geq 1, x \geq 0\}$ has the IDP if and only if every multisubhypergraph of H has $cd = \delta$.)

A geometric generalization of $\mathsf{C}(\textsc{Intervals}^t)$ to intervals in 2 or more dimensions (i.e., rectangles) appears in [4].

Another generalization of paths is for the ground set to be the edges of a tree, with hyperedges corresponding to sub-paths. Let $\textsc{PathsInTrees}$ denote this family of hypergraphs. We think the following theorem is folklore in the literature on path colouring in trees (e.g. [13]) but are not aware of a reference.

Theorem 3. $\mathsf{P}(\textsc{PathsInTrees}, \Delta) = \lfloor \frac{3}{2}\Delta \rfloor$.

Proof. The upper bound means to show the following: given a tree and a family of subpaths, if every edge is used by at most Δ paths, show we can $\lfloor \frac{3}{2}\Delta \rfloor$-colour the paths so that similarly-coloured paths are always disjoint. On the one hand, if the tree is a star, this follows immediately from Shannon's theorem. On the other hand, suppose the tree has some edge uv such that neither u nor v is a leaf. If we contract the part tree lying on either side of uv, we get two smaller instances, which can be $\lfloor \frac{3}{2}\Delta \rfloor$-coloured by induction. In both smaller colourings, all paths containing uv get distinct colours, so we can permute the colours in one of these smaller colourings so they agree on the paths through uv. Then it is easy to see the colourings combine to give a valid colouring of the whole original instance.

The lower bound follows from the same construction as in Shannon's theorem. $\square$

By extending iterated rounding methods of Könemann et al. [20] we can prove the following theorem; we defer its lengthy proof to the end of the section.

Theorem 4. $\mathsf{C}(\textsc{PathsInTrees}, \delta) \geq 1 + \lfloor (\delta - 1)/7 \rfloor$.

As for the transposes,

Theorem 5. $\mathsf{C}(\textsc{PathsInTrees}^t, \delta) \geq \lceil \delta/2 \rceil$.

Proof. It is enough to show that, given a tree and a family of subpaths of length at least $2k+1$, we can $(k+1)$-colour the edges so that every path is polychromatic. Root the tree arbitrarily and give all edges in level i colour $i \bmod (k+1)$. Since a path of length $2k+1$ hits at least $k+1$ consecutive levels, we are done. $\square$

Theorem 6. $\mathsf{P}(\textsc{PathsInTrees}^t, \Delta) = +\infty$ *for all* $\Delta \geq 2$.

Proof. Take a tree which is a star with n tips and let the hyperedges be all possible paths of length 2 in the tree; consider the transpose of its incidence hypergraph. It has $\binom{n}{2}$ vertices and n edges each of size $n-1$, with all vertices of degree $2 \leq \Delta$. (The resulting hypergraph is the same as $\binom{[n]}{2}^t$.)

To compute pd we need to partition the edge set into as few matchings as possible. But every pair of edges intersect, so $pd \geq n$. Since n was arbitrary, we are done. $\square$

Here are two other combinatorial settings related to connectivity. For a finite multigraph $G = (V, E)$ define the *shore system* of G to be the hypergraph with ground set $S = \mathcal{P}(V) \backslash \{\varnothing, \mathcal{P}\}$ and for each $\{u, v\} \in E$, the hyperedge $\{U \in S \mid |\{u, v\} \cap U| = 1\}$. Then since every $2t$-edge connected graph contains t disjoint spanning trees (this is a corollary of the Nash-Williams/Tutte theorem), the family $\mathcal{S}$ of all shore systems satisfies $\mathsf{C}(\delta) \geq \lfloor \delta/2 \rfloor$.

For the analogue of the above family in directed graphs, it is conjectured [7] that some finite δ has $\mathsf{C}(\delta) > 1$, but this is still open. (I.e., they conjecture that δ-arc-strongly-connected digraphs contain two arc-disjoint spanning strongly-connected subdigraphs.)

3.1 Proof of Theorem 4

Proof. We are given a tree (V, E). If a collection of paths is such that every $e \in E$ lies in at least c paths, call that collection a *c-cover*. Given a multiset $\mathcal{P}$ of paths in the tree which is a $(7k+1)$-cover, we need to show the paths can k-coloured so that each colour class is a 1-cover. We do this by induction on k, where the base case $k = 0$ is trivial. Following the previous conventions, let $d(e)$, the degree of edge $e \in E$, denote the number of paths which contain e (so $d(e) \geq 7k+1$ for all e).

Consider the following polyhedron, with a variable x_P for each path P; parameters a and b are set to $a = 1$ and $b = 7k - 6$.

$$\forall P \in \mathcal{P} : \quad 0 \leq x_P \leq 1 \tag{1}$$

$$\forall e \in E : \quad \sum_{P : e \in P} x_P \geq 2 + \left\lceil \frac{a}{a+b+6} \cdot d(e) \right\rceil \tag{2}$$

$$\forall e \in E : \quad \sum_{P : e \in P} (1 - x_P) \geq 2 + \left\lceil \frac{b}{a+b+6} \cdot d(e) \right\rceil . \tag{3}$$

We will use this LP to find an *integral* x which is the characteristic vector of a 1-cover and such that $1 - x$ is the characteristic vector of a $(7k-6)$-cover. Then we will be done by induction.

For starters, observe that the polyhedron is nonempty: the point x with all components set to $\frac{a+3}{a+b+6}$ is feasible, since $\mathcal{P}$ is an $(a+b+6)$-cover. In our iterated LP rounding algorithm, in each iteration we (i) find an extreme point solution x^*, (ii) for each integral x^*_P fix its value forever, and then (iii) do a *relaxation* step in which we discard a constraint. The algorithm terminates since only a finite number of variables can be fixed and only a finite number of constraints can be discarded.

The key now is to establish what sort of relaxation step is guaranteed to be possible. The setup in any given iteration is that some of the x_P variables have been fixed to 0 and 1, and some of the constraints (2) and (3) have been discarded in previous iterations. We want to show the following:

Claim 3. *In each iteration, some constraint of the form* (2) *or* (3) *(that has not already been discarded) involves at most 3 nonfixed variables.*

Proof. Let x^* be the most recent extreme point solution. As is typical in iterated LP-based algorithms, take a maximal linearly independent family of the tight constraints for x^*. Observe that for each e, the two corresponding constraints (2) and (3) are linearly dependent, hence at most one is in the family. So the nonfixed part of x^* is a vector strictly between 0 and 1, such that it is the unique solution to a family of integral exact-capacity constraints on edges. Lemma 4 from [20] shows that, under this hypothesis, some exact-capacity constraint has at most three nonintegral variables, as needed. □

Now we discard the constraint whose existence is guaranteed by the Claim. Say it is (2) for some specific e; the other case is the same. The sum $\sum_{P:e \in P} x^*_P$ consists of some variables fixed at 0 or 1, plus at most 3 nonfixed fractional variables. We also know $\sum_{P:e\in P} x^*_e = 2 + \lceil \frac{a}{a+b+6} \cdot d(e) \rceil$ since the constraint is tight. The fractional (nonfixed) variables in the sum add up to an integer which is less than 3. Hence the variables fixed to 1 ensure that throughout the rest of the algorithm,

$$\sum_{P : e \in P} x^*_e \geq \left\lceil \frac{a}{a+b+6} \cdot d(e) \right\rceil \tag{4}$$

will hold.

Hence for each constraint (2), either it is satisfied at termination, or the weaker (4) holds. But in either case the final x is a 1-cover. Similarly, $(1-x)$ is a $(k-6)$-cover at termination. This completes the proof. □

4 Closing Comments

We took Δ as the "natural lower bound" for the chromatic index/covering decomposition number of a hypergraph. However, a larger natural lower bound for the chromatic index is the largest size of any pairwise intersecting set of hyperedges; denote this by ω'. For other structured hypergraphs, can we get good upper bounds on the chromatic index in terms of ω'? Dumitrescu and Jiang [12] do this for some geometrically defined hypergraphs. For ordinary multigraphs, let T be $\max_{u,v,w} \#E(G[\{u,v,w\}])$, the largest number of edges in any induced 3-vertex subgraph. Then $\omega' = \max\{\Delta, \mathsf{T}\}$, and since Gupta [17, Thm. 4D] showed

$$pd \leq \max\{\lfloor \frac{5\Delta+2}{4} \rfloor, \mathsf{T}\}$$

we see that $\omega' \leq pd \leq \lfloor \frac{5\omega'+2}{4} \rfloor$, an improved ratio of dependence compared to Shannon's theorem $\Delta \leq pd \leq \lfloor \frac{3}{2}\Delta \rfloor$. (Note $pd \leq \lfloor \frac{5\omega'+2}{4} \rfloor$ is tight for all ω', by considering suitable parallelizations of a 5-cycle.)

4.1 Acknowledgments

We thank Jessica McDonald and Dömötör Pálvölgyi for helpful discussions on these topics.

References

[1] G. Agnarsson, Á. S. Egilsson, and M. M. Halldórsson. Vertex coloring acyclic digraphs and their corresponding hypergraphs. *Discrete Applied Mathematics*, 156:1918 1928, 2008.

[2] N. Alon, R. Berke, K. Buchin, M. Buchin, P. Csorba, S. Shannigrahi, B. Speckmann, and P. Zumstein. Polychromatic colorings of plane graphs. *Discrete & Computational Geometry*, 42(3):421–442, 2009. Preliminary version appeared in *Proc. 24th SOCG*, pages 338–345, 2008.

[3] N. Alon and J. H. Spencer. *The Probabilistic Method (3rd Edition).* Wiley, New York, 2008.

[4] G. Aloupis, J. Cardinal, S. Collette, S. Imahori, M. Korman, S. Langerman, O. Schwartz, S. Smorodinsky, and P. Taslakian. Colorful strips. In *Proc. 9th LATIN*, pages 2–13, 2010. To appear.

[5] L. D. Andersen. Lower bounds on the cover-index of a graph. *Discrete Mathematics*, 25(3):199 – 210, 1979.

[6] J. Bang-Jensen and S. Thomassé. Highly connected hypergraphs containing no two edge-disjoint spanning connected subhypergraphs. *Discrete Appl. Math.*, 131:555–559, 2003.

[7] J. Bang-Jensen and A. Yeo. Decomposing k-arc-strong tournaments into strong spanning subdigraphs. *Combinatorica*, 24(3):331–349, 2004.

[8] S. Baum and J. L. E. Trotter. Integer rounding for polymatroid and branching optimization problems. *SIAM Journal on Algebraic and Discrete Methods*, 2(4):416–425, 1981.

[9] B. Bollobás and A. Scott. Covering numbers for graphs and hypergraphs. *CoRR*, abs/1009.5893, 2010.

[10] C. Chekuri, M. Mydlarz, and F. B. Shepherd. Multicommodity demand flow in a tree and packing integer programs. *ACM Trans. Algorithms*, 3(3):27, 2007. Preliminary version appeared in *Proc. 30th ICALP*, pages 410–425, 2003.

[11] C. Chekuri and F. B. Shepherd. Approximate integer decompositions for undirected network design problems. *SIAM Journal on Discrete Mathematics*, 23(1):163–177, 2008.

[12] A. Dumitrescu and M. Jiang. Coloring translates and homothets of a convex body. *CoRR*, abs/1008.1360, 2010.

[13] T. Erlebach. Approximation algorithms for edge-disjoint paths and unsplittable flow. In E. Bampis, K. Jansen, and C. Kenyon, editors, *Efficient Approximation and Online Algorithms*, pages 97–134. Springer, 2006.

[14] R. Fulek. Coloring geometric hypergraph defined by an arrangement of half-planes. In *Proc. 22nd CCCG*, 2010. To appear.

[15] M. Gibson and K. Varadarajan. Decomposing coverings and the planar sensor cover problem. In *Proc. 50th FOCS*, pages 159–168, Washington, DC, USA, 2009. IEEE Computer Society.

[16] D. Gordon. La Jolla covering repository. http://www.ccrwest.org/cover/steiner.html.

[17] R. P. Gupta. On the chromatic index and the cover index of a mulltigraph. In Y. Alavi and D. R. Lick, editors, *Theory and Applications of Graphs: Int. Conf. Kalamazoo, May 11-15, 1976*, volume 642 of *Lecture Notes in Mathematics*, pages 204–215. Springer Verlag, 1978.

[18] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10(4):718–720, 1981.

[19] R. M. Karp, F. T. Leighton, R. L. Rivest, C. D. Thompson, U. V. Vazirani, and V. V. Vazirani. Global wire routing in two-dimensional arrays. *Algorithmica*, 2:113–129, 1987.

[20] J. Könemann, O. Parekh, and D. Pritchard. Max-weight integral multicommodity flow in spiders and high-capacity trees. In *Proc. 6th WAOA*, pages 1–14, 2008.

[21] P. Mani-Levitska and J. Pach. Decomposition problems for multiple coverings with unit balls. Manuscript, 1987.

[22] J. Pach. Decomposition of multiple packing and covering. In *Kolloquium über Diskrete Geometrie*, pages 169–178, Salzburg, 1980. Institut für Mathematik der Universität Salzburg.

[23] J. Pach, D. Pálvölgyi, and G. Tóth. Survey on the decomposition of multiple coverings. In preparation. See also Chapter 2 in D. Pálvölgyi's thesis, arXiv:1009.4641.

[24] D. Pálvölgyi. *Decomposition of Geometric Set Systems and Graphs*. PhD thesis, École Polytechnique Fédérale de Lausanne, 2010. arXiv:1009.4641.

[25] D. Pritchard. Edge cover colouring versus minimum degree in multigraphs. arXiv:0906.1778v1, 2009.

[26] A. Schrijver. *Combinatorial optimization*. Springer, New York, 2003.

[27] S. Smorodinsky and Y. Yuditsky. Polychromatic coloring for half-planes. In *Proc. 12th SWAT*, 2010. To appear. Preliminary version at arXiv:1006.3191.

[28] V. V. Vazirani. *Approximation Algorithms*. Springer, 2001.

[29] C. Xu and G. Liu. A note on the edge cover chromatic index of multigraphs. *Discrete Mathematics*, 308(24):6564–6568, 2008.