

On a Novel Class of Integrable ODEs Related to the Painlevé Equations

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*This paper is dedicated to Professor T. Bountis on the occasion of his 60th birthday
with appreciation of his important contributions to “Nonlinear Science”.*

Abstract

One of the authors has recently introduced the concept of conjugate Hamiltonian systems: the solution of the equation $h = H(p, q, t)$, where H is a given Hamiltonian containing t explicitly, yields the function $t = T(p, q, h)$, which defines a new Hamiltonian system with Hamiltonian T and independent variable h . By employing this construction and by using the fact that the classical Painlevé equations are Hamiltonian systems, it is straightforward to associate with each Painlevé equation two new integrable ODEs. Here, we investigate the conjugate Painlevé II equations. In particular, for these novel integrable ODEs, we present a Lax pair formulation, as well as a class of implicit solutions. We also construct conjugate equations associated with Painlevé I and Painlevé IV equations.

KEYWORDS: Painlevé equation, conjugate Painlevé equation, integrable nonlinear ODE, Lax pair

1 Introduction

The mathematical and physical significance of Painlevé equations [10] is well established [7]. In particular, regarding Painlevé II, we note the following:

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- (a) A new method for solving its initial value problem, the so-called isomonodromy method, was introduced in [4]; this method was imbedded within the framework of the Riemann-Hilbert formalism in [6]. Rigorous aspects of this formalism, including the proof that the solution possesses the so-called Painlevé property, were discussed in Fokas and Zhou [8].
- (b) There exist several Lax pairs for Painlevé II, including those presented in [4], in Jimbo and Miwa [11], and in Harnad, Tracy and Widom [9].
- (c) Painlevé II is a Hamiltonian system [4][14][13].
- (d) It is possible to construct certain particular explicit solutions of Painlevé II using certain “Bäcklund transformations” [15] [13] and [5].

The concept of conjugate Hamiltonian systems is introduced in [16]: The solution of the equation $h = H(p, q, t)$, where H is a given Hamiltonian which contains t explicitly, yields the function $t = T(p, q, h)$. The Hamiltonian system with Hamiltonian T and independent variable h is called *conjugate* to the Hamiltonian system with Hamiltonian H . The conjugate Hamiltonian system has the following properties:

1. If $p = p(t)$, $q = q(t)$ is a solution of the Hamiltonian system with Hamiltonian H , then $p = p(t(h))$, $q = q(t(h))$ ¹, is a solution of the conjugate Hamiltonian system, where $t = t(h)$ is the so-called t -function, the inverse function of the h -function².
2. A first integral of a Hamiltonian system, also provides a first integral of the associated conjugate Hamiltonian system.

The classical Painlevé equations are Hamiltonian systems, thus we can associate with each Painlevé equation a conjugate Hamiltonian system. The gauge freedom of a Hamiltonian implies that we can in fact associate an *infinite* family of integrable second-order nonlinear ODEs with a given Painlevé equation. Furthermore, by utilising the gauge freedom of the conjugate Hamiltonian, we can associate with any of the conjugate ODEs constructed, another infinite family of integrable ODEs, etc. Here, we only present the ODEs with the simplest form.

This paper is organized as follows: In section 2, we construct the conjugate Painlevé II and also derive an associated Lax pair. In section 3, we construct a class of explicit solutions of the conjugate Painlevé II. In section 4, we derive the conjugate ODEs corresponding to Painlevé I and IV. In section 5, we prove a general theorem for constructing Lax pairs for conjugate Painlevé equations. In section 6, we discuss further these results.

¹ We sometimes write $p(h), q(h)$ instead of $p(t(h)), q(t(h))$.

²We recall that the h -function, $h = h(t)$, is defined by $h(t) = H(p(t), q(t), t)$.

2 The conjugate Painlevé II equation

Let P_{II} denote the second Painlevé equation, namely

$$P_{II} : \frac{d^2q}{dt^2} = 2q^3 + tq + b - \frac{1}{2}, \quad t, q \in \mathbb{C}, \quad (2.1)$$

where b is an arbitrary complex constant. P_{II} possesses the Hamiltonian H , where

$$H(p, q, t) = \frac{1}{2}p^2 - \left(q^2 + \frac{t}{2}\right)p - bq. \quad (2.2)$$

Indeed, Hamilton's equations associated with H are

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = 2pq + b, \quad (2.3a)$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p - q^2 - \frac{t}{2}. \quad (2.3b)$$

Eliminating from equations (2.3) the variable p we find P_{II} :

$$\frac{d^2q}{dt^2} = 2pq + b - 2q\left(p - q^2 - \frac{t}{2}\right) - \frac{1}{2},$$

which is equation (2.1).

If we eliminate from equations (2.3) the variable q , we find an other second order integrable ODE, which appears in the list of Ince [10] as *XXXIV*, and which we denote by \widetilde{P}_{II} ³:

$$\widetilde{P}_{II} : \frac{d^2p}{dt^2} = \frac{1}{2p}\left(\frac{dp}{dt}\right)^2 - \frac{b^2}{2p} + 2p^2 - tp, \quad t, p \in \mathbb{C}. \quad (2.4)$$

Indeed,

$$\begin{aligned} \frac{d^2p}{dt^2} &= 2q(2pq + b) + 2p\left(p - q^2 - \frac{t}{2}\right) \\ &= 2p^2 - tp + 2pq^2 + 2bq. \end{aligned}$$

Replacing in this equation q from equation (2.3a), we find equation (2.4).

Remark 2.1 *It has been shown in [5] that there exists a one-to-one correspondence between solutions of P_{II} and \widetilde{P}_{II} . This result follows directly from their Hamiltonian structure (2.3).*

³Either \widetilde{P}_{II} or P_{II} could have been chosen as the second Painlevé equation. The historical choice of P_{II} is due to Painlevé himself [10].

Notations Prime, “ ’ ”, denotes derivative with respect to h .

Proposition 2.2 *The conjugate equations of P_{II} and of \widetilde{P}_{II} , i.e., the conjugate equations of equations (2.1) and (2.4), are the following ODEs:*

$$CP_{II} : \frac{d^2q}{dh^2} = (q' + 1) \left(\frac{1 - 2b - bq'}{h + bq} + 8q \left(\frac{-q' - 1}{2h + 2bq} \right)^{\frac{1}{2}} \right), \quad h, q \in \mathbb{C}, \quad (2.5a)$$

$$\widetilde{CP}_{II} : \frac{d^2p}{dh^2} = 4 + \frac{8h}{p^2} - \frac{4b^2}{p^3}, \quad h, p \in \mathbb{C}. \quad (2.5b)$$

Equations (2.5) possess the Hamiltonian function T , where

$$T(p, q, h) = (p - 2q^2) - \frac{2h + 2bq}{p}. \quad (2.6)$$

Moreover, equations (2.5) admit the following Lax pair:

$$\frac{\partial \psi}{\partial \lambda} = \left\{ \frac{1}{2\lambda} \begin{pmatrix} b & 0 \\ -p & -b \end{pmatrix} + \begin{pmatrix} q & \frac{1}{p}(2h + 2bq) \\ \frac{1}{2} & -q \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \psi, \quad (2.7a)$$

$$\frac{\partial \psi}{\partial h} = \left\{ -\frac{2}{p} \begin{pmatrix} -q & 0 \\ -\frac{1}{2} & q \end{pmatrix} - \lambda \frac{2}{p} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right\} \psi, \quad \lambda \in \mathbb{C}, \quad (2.7b)$$

where ψ is a 2×2 matrix-valued function of λ and h .

Proof. The solution of the equation

$$h = \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - bq, \quad (2.8)$$

yields

$$t = T(p, q, h), \quad (2.9)$$

where the function T denotes the RHS of equation (2.6). The associated Hamilton's equations are:

$$\frac{dp}{dh} = \frac{\partial T}{\partial q} = -4q - \frac{2b}{p}, \quad (2.10a)$$

$$\frac{dq}{dh} = -\frac{\partial T}{\partial p} = -1 - \frac{2h + 2bq}{p^2}. \quad (2.10b)$$

Eliminating from equations (2.10) the function q we find,

$$\frac{d^2p}{dh^2} = -4 \left(-1 - \frac{2h + 2bq}{p^2} \right) + \frac{2b}{p^2} \left(-4q - \frac{2b}{p} \right),$$

which is equation (2.5b). Similarly, eliminating from equations (2.10) the function p we find equation (2.5a).

The Lax pair (2.7) can be verified directly. Indeed, under the assumption that $p_\lambda = 0$ and $q_\lambda = 0$, i.e, the compatibility equation

$$\frac{\partial^2 \psi}{\partial h \partial \lambda} = \frac{\partial^2 \psi}{\partial \lambda \partial h},$$

is equivalent to equations (2.10), and so is equivalent to equations (2.5).

Alternatively, equations (2.7) can be derived from the following Lax pair of P_{II} and \widetilde{P}_{II} ⁴:

$$\frac{\partial \psi}{\partial \lambda} = \left\{ \frac{1}{2\lambda} \begin{pmatrix} b & 0 \\ -p & -b \end{pmatrix} + \begin{pmatrix} q & p - 2q^2 - t \\ \frac{1}{2} & -q \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \psi, \quad (2.11a)$$

$$\frac{\partial \psi}{\partial t} = - \left\{ \begin{pmatrix} q & 0 \\ \frac{1}{2} & -q \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \psi. \quad (2.11b)$$

Indeed, the HTW-pair (2.11) is a Lax pair for Hamilton's equations (2.3) [12]. By applying Proposition 5.1, it can be shown that equations (2.11) imply equations (2.7).

□

3 Solving CP_{II} and \widetilde{CP}_{II}

The discussion in the introduction implies that starting with the well-known special solutions of P_{II} and \widetilde{P}_{II} , we can construct special solutions for CP_{II} and \widetilde{CP}_{II} . It also implies that we can solve, at least implicitly, the general initial problem.

3.1 A class of special solutions

First we recall the rational solutions of P_{II} and \widetilde{P}_{II} . There are two fundamental types of Bäcklund transformations for P_{II} and \widetilde{P}_{II} which were derived in [15], [5], [13]. Taking into consideration Remark 2.1, we express these transformations for Hamilton's equations (2.3):

- (i) Suppose that $(q(t; b), p(t; b))$ is a solution of equations (2.3) with constant b . Then

$$(\hat{q}(t), \hat{p}(t)) = \left(q(t; b) + \frac{b}{p(t; b)}, p(t; b) \right)$$

is a solution of equations (2.3) with constant $-b$.

- (ii) Suppose that $(q(t; b), p(t; b))$ is a solution for equations (2.3) with constant b . Then

$$(\hat{q}(t), \hat{p}(t)) = \left(-q(t; b), -p(t; b) + 2q^2(t; b) + t \right)$$

is a solution of equations (2.3) with constant $1 - b$.

⁴This pair is the so-called Harnad-Tracy-Widom pair (HTW-pair), which was first discovered by Harnad, Tracy and Widom in [9] and first written out explicitly by Joshi, Kitaev and Treharne in [12].

The transformations (i) and (ii) imply, respectively, $\hat{h}(t) = h(t)$ and $\hat{h}(t) = h(t) + q(t)$.

Starting from a particular solution $q = 0, p = t/2$ with $b = 1/2$, and applying the above Bäcklund transformations, we can obtain a class of rational solutions [15], [5], [13] for P_{II} and \widetilde{P}_{II} . For example:

$$q = \frac{2(t^3-2)}{t(t^3+4)}, p = \frac{t^3+4}{2t^2}, h = -\frac{t^2}{8} + \frac{1}{t} \text{ with } b = -\frac{3}{2};$$

$$q = \frac{1}{t}, p = \frac{t}{2}, h = -\frac{t^2}{8} \text{ with } b = -\frac{1}{2};$$

$$q = 0, p = \frac{t}{2}, h = -\frac{t^2}{8} \text{ with } b = \frac{1}{2};$$

$$q = -\frac{1}{t}, p = \frac{t^3+4}{2t^2}, h = -\frac{t^2}{8} + \frac{1}{t} \text{ with } b = \frac{3}{2};$$

By inverting the h -function and substituting the resulting t -function to the rational solutions for P_{II} and \widetilde{P}_{II} , we obtain the following solutions for CP_{II} and \widetilde{CP}_{II} :

•

$$q = \frac{2\left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^3 - 4}{\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3} \left[\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right]^3 + 4}, \quad (3.1a)$$

$$p = \frac{4 + \left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^3}{2\left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^2}, \quad (3.1b)$$

$$t = \left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3} \quad (3.1c)$$

with $b = -\frac{3}{2}$, where $D = 9 + (81 + 96h^3)^{1/2}$;

•

$$q = \frac{1}{2(-2h)^{1/2}}, p = (-2h)^{1/2}, t = 2(-2h)^{1/2} \quad (3.2)$$

with $b = -\frac{1}{2}$;

•

$$q = 0, p = (-2h)^{1/2}, t = 2(-2h)^{1/2} \quad (3.3)$$

with $b = \frac{1}{2}$;

•

$$q = -\left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^{-1}, \quad (3.4a)$$

$$p = \frac{4 + \left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^3}{2\left(\left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3}\right)^2}, \quad (3.4b)$$

$$t = \left(\frac{2}{3}\right)^{2/3} D^{1/3} - \left(\frac{2}{3}\right)^{1/3} 4hD^{-1/3} \quad (3.4c)$$

with $b = \frac{3}{2}$, where $D = 9 + (81 + 96h^3)^{1/2}$;

The four particular solutions computed above can be verified directly. For $|b| > 3/2$, in order to compute the corresponding solutions we need to solve polynomial equations of order higher than 4.

Using the transformations (i) and (ii), we find the following result:

Proposition 3.1 (*Bäcklund transformations*)

(i) Suppose that $(q(h; b), p(h; b))$ is a solution of equations (2.10) with constant b . Then

$$(\hat{q}(h), \hat{p}(h)) = \left(q(h; b) + \frac{b}{p(h; b)}, p(h; b)\right)$$

is a solution of equations (2.10) with constant $-b$.

(ii) Suppose that $(q(h; b), p(h; b))$ is a solution of equations (2.3) with constant b . Then

$$(\hat{q}(\hat{h}), \hat{p}(\hat{h})) = \left(-q(h; b), \frac{-2h + 2bq(h; b)}{p(h; b)}\right)$$

is a solution of equations (2.3) with constant $1-b$ and independent variable \hat{h} , where $\hat{h} = h + q(h; b)$.

Remark 3.2 The solutions (3.1) – (3.4) can also be generated by employing Proposition 3.1. The main difficulty for the explicit computation of these solutions is the requirement of solving the equation $\hat{h} = h + q(h; b)$ for h in terms of \hat{h} , $h = h(\hat{h})$.

3.2 An implicit representation of the solution of the initial value problem

We study the following initial value problem (IVP) of \widetilde{CP}_{II} :

$$\frac{d^2 p}{dh^2} = 4 + \frac{8h}{p^2} - \frac{4b^2}{p^3}, \quad (3.5a)$$

$$p|_{h=h_0} = p_0, \quad p'|_{h=h_0} = p_1. \quad (3.5b)$$

\widetilde{CP}_{II} is equivalent to Hamilton's equations (2.10). Using these equations, we can find the initial values of q and q' at $h = h_0$:

$$q|_{h=h_0} = -\frac{b}{2p_0} - \frac{p_1}{4} := q_0, \quad q'|_{h=h_0} = -1 - \frac{2h_0 + 2bp_1}{p_0^2} - \frac{b^2}{p_0^3} := q_1.$$

Thus,

$$t_0 = T(p_0, q_0, h_0), \quad q|_{t=t_0} = q_0, \quad p|_{t=t_0} = p_0.$$

Next, from the Hamiltonian structure (2.3) of P_{II} , we obtain the following initial values:

$$\left. \frac{dq}{dt} \right|_{t=t_0} = 2p_0q_0 + b, \quad \left. \frac{dp}{dt} \right|_{t=t_0} = p_0 - q_0^2 - t_0/2.$$

The IVP of P_{II} with initial values $q|_{t=t_0}$ and $\left. \frac{dq}{dt} \right|_{t=t_0}$, can be solved via the isomonodromy method and yields

$$q = q(t).$$

Substituting this solution to equation (2.3), we obtain

$$p = p(t).$$

The h -function is obtained by

$$h(t) = H(p(t), q(t), t).$$

By the inverse function theorem, as least locally, we obtain

$$t = t(h).$$

Thus, the implicit solution of the IVP of \widetilde{CP}_{II} is given by

$$p = p(t(h)).$$

The IVP for CP_{II} can be solved in a similar way.

4 Conjugate equations of Painlevé I and IV

Let P_I denote the first Painlevé equation, namely

$$P_I : \frac{d^2q}{dt^2} = 6q^2 + t, \quad t, q \in \mathbb{C}. \quad (4.1)$$

P_I possesses the Hamiltonian H , where

$$H(p, q, t) = \frac{1}{2}p^2 - 2q^3 - tq. \quad (4.2)$$

Indeed, the associated Hamilton's equations are

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = 6q^2 + t, \quad (4.3a)$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p. \quad (4.3b)$$

Eliminating from equations (4.3) the variable p we find P_I :

$$\frac{d^2q}{dt^2} = 6q^2 + t.$$

Eliminating from equations (4.3) the variable q we find \widetilde{P}_I :

$$\widetilde{P}_I : \frac{d^2p}{dt^2} = 2p\left(6\frac{dp}{dt} - 6t\right)^{1/2} + 1, \quad t, p \in \mathbb{C}. \quad (4.4)$$

Indeed,

$$\frac{d^2p}{dt^2} = 12qp + 1.$$

Replacing in this equation q from equation (4.3a), we find equation (4.4).

Proposition 4.1 *The conjugate equations of equations P_I and of \widetilde{P}_I , i.e., the conjugate equations of equations (4.1) and (4.4), are the following ODEs:*

$$CP_I : \frac{d^2q}{dh^2} = -\frac{1}{2q}\left(\frac{dq}{dh}\right)^2 + 4 - \frac{h}{q^3}, \quad h, q \in \mathbb{C}, \quad (4.5a)$$

$$\widetilde{CP}_I : \frac{d^2p}{dh^2} = \frac{2hp - p^3}{F(p', p, h)^4} + \frac{1 - pp'}{F(p', p, h)^2} + \frac{4p}{F(p', p, h)}, \quad h, p \in \mathbb{C}, \quad (4.5b)$$

where F is a solution of the following equation

$$4F^3 + p'F^2 + \frac{1}{2}p^2 - h = 0, \quad h, p, F \in \mathbb{C}.$$

Equations (4.5) possess the Hamiltonian T , where

$$T(p, q, h) = \frac{1}{2}\frac{p^2}{q} - 2q^2 - \frac{h}{q}. \quad (4.6)$$

Proof. The solution of the equation

$$h = \frac{1}{2}p^2 - 2q^3 - tq, \quad (4.7)$$

yields

$$t = T(p, q, h), \quad (4.8)$$

where the function T denotes the RHS of equation (4.6). The associated Hamilton's equations are:

$$\frac{dp}{dh} = \frac{\partial T}{\partial q} = -\frac{1}{2} \frac{p^2}{q^2} - 4q + \frac{h}{q^2}, \quad (4.9a)$$

$$\frac{dq}{dh} = -\frac{\partial T}{\partial p} = -\frac{p}{q}. \quad (4.9b)$$

Eliminating from equations (4.9) the function p we find,

$$\frac{d^2q}{dh^2} = -\frac{1}{2q} \left(\frac{dq}{dh} \right)^2 + 4 - \frac{h}{q^3},$$

which is equation (4.5a). Similarly, eliminating from equations (4.9) the function q we find equation (4.5b). □

Let P_{IV} denote the fourth Painlevé equation, namely

$$P_{IV} : \frac{d^2q}{dt^2} = \frac{1}{2q} \left(\frac{dq}{dt} \right)^2 + \frac{3}{2}q^3 + 2tq^2 + \left(\frac{t^2}{2} + a_1 + 2a_2 - 1 \right)q - \frac{a_1^2}{2q}, \quad t, q \in \mathbb{C}. \quad (4.10)$$

where a_1, a_2 are arbitrary complex constants. P_{IV} possesses the Hamiltonian H , where

$$H(p, q, t) = qp(p - q - t) - a_2q - a_1p. \quad (4.11)$$

The associated Hamilton's equations are

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -p^2 + 2pq + pt + a_2, \quad (4.12a)$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = -q^2 + 2pq - qt - a_1. \quad (4.12b)$$

Eliminating from equations (4.12) the variable p we find P_{IV} :

$$\frac{d^2q}{dt^2} = \frac{1}{2q} \left(\frac{dq}{dt} \right)^2 + \frac{3}{2}q^3 + 2tq^2 + \left(\frac{t^2}{2} + a_1 + 2a_2 - 1 \right)q - \frac{a_1^2}{2q}.$$

Eliminating from equations (4.12) the variable q we find \widetilde{P}_{IV} :

$$\widetilde{P}_{IV} : \frac{d^2p}{dt^2} = \frac{1}{2p} \left(\frac{dp}{dt} \right)^2 + \frac{3}{2}p^3 - 2tp^2 + \left(\frac{t^2}{2} - 2a_1 - a_2 + 1 \right)p - \frac{a_2^2}{2p}, \quad t, p \in \mathbb{C}. \quad (4.13)$$

Indeed,

$$\frac{d^2p}{dt^2} = -2p \frac{dp}{dt} + 2p(-q^2 + 2pq - qt - a_1) + 2 \frac{dp}{dt}q + \frac{dp}{dt}t + p.$$

Replacing in this equation q from equation (4.12a), we find equation (4.13).

Proposition 4.2 *The conjugate equations of equations P_{IV} and of \widetilde{P}_{IV} , i.e., the conjugate equations of equations (4.10) and (4.13), are the following ODEs:*

$$CP_{IV} : \frac{d^2q}{dh^2} = \frac{1+q'}{hq+a_2q^2} (q-2q^2(1+q')G_1+2(h+a_1G_1)+q'(h+2a_1G_1)),$$

$$h, q \in \mathbb{C},$$
(4.14a)

$$\widetilde{CP}_{IV} : \frac{d^2p}{dh^2} = \frac{1+p'}{hp+a_1p^2} (p+2p^2(1+p')G_2+2(h+a_2G_2)+p'(h+2a_2G_2)),$$

$$h, p \in \mathbb{C},$$
(4.14b)

where $G_1 = \left(-\frac{h+a_2q}{q+q'}\right)^{1/2}$, $G_2 = \left(\frac{h+a_1p}{p+pp'}\right)^{1/2}$.

Equations (4.14) possess the Hamiltonian T , where

$$T(p, q, h) = p - q - \frac{a_2}{p} - \frac{a_1}{q} - \frac{h}{pq}. \quad (4.15)$$

Proof. The solution of the equation

$$h = qp(p - q - t) - a_2q - a_1p, \quad (4.16)$$

yields

$$t = T(p, q, h), \quad (4.17)$$

where the function T denotes the RHS of equation (4.15). The associated Hamilton's equations are:

$$p' = \frac{\partial T}{\partial q} = \frac{h}{q^2p} + \frac{a_1}{q^2} - 1, \quad (4.18a)$$

$$q' = -\frac{\partial T}{\partial p} = -\frac{h}{p^2q} - \frac{a_2}{p^2} - 1. \quad (4.18b)$$

Eliminating from equations (4.18) the function p we find (4.14a). Similarly, eliminating from equations (4.18) the function p we find equation (4.14b). □

Remark 4.3 *Every hamiltonian has the gauge freedom $\tilde{H} = H + f(t)$, where $f(t)$ is an arbitrary function of t . This implies that we can associate infinitely many ODEs with each Painlevé equation. Among these ODEs, the ODEs presented here are expected to have the simplest form.*

Remark 4.4 *We note that conjugate Painlevé equations are of the form $y'' = F(y, y', t)$, where F is algebraic in y, y' . The corresponding conjugate Hamiltonian systems are of the form*

$$p' = F_1(p, q, h), \quad (4.19a)$$

$$q' = F_2(p, q, h), \quad (4.19b)$$

where F_1 and F_2 are rational in p, q .

5 Lax pairs for conjugate equations

The following proposition provides a method for constructing Lax pairs for conjugate Painlevé equations.

Proposition 5.1 *An explicit Lax pair for the Hamiltonian form of any Painlevé equation, leads an explicit Lax pair for the Hamiltonian form of the corresponding conjugate Painlevé equation. The relevant construction involves the following steps:*

- (i) *Substitute the t -function into the Lax pair of a given Painlevé equation, so that the new independent variables become λ and h (instead of λ and t).*
- (ii) *Replace in the resulting Lax pair the unknown functions by the associated explicit functions of (p, q, h) .*

Proof. Let $H(p, q, t)$ be a Hamiltonian of a given Painlevé equation, i.e., the given Painlevé equation is equivalent to Hamilton's equations

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}. \quad (5.1)$$

Let $T(p, q, h)$ be the associated conjugate Hamiltonian, i.e., the associated conjugate Painlevé equation is equivalent to Hamilton's equations

$$\frac{dp}{dh} = \frac{\partial T}{\partial q}, \quad \frac{dq}{dh} = -\frac{\partial T}{\partial p}. \quad (5.2)$$

Suppose equations (5.1) admit the following Lax pair:

$$\frac{\partial \psi}{\partial \lambda}(\lambda, t) = A(p(t), q(t), t, \lambda)\psi(\lambda, t), \quad (5.3a)$$

$$\frac{\partial \psi}{\partial t}(\lambda, t) = B(p(t), q(t), t, \lambda)\psi(\lambda, t), \quad \lambda \in \mathbb{C}, \quad (5.3b)$$

where A and B are two known $k \times k$ matrix-valued functions of (p, q, t) , and the function ψ is a $k \times k$ matrix-valued function of λ and t (for some $k > 1$). Equations (5.3) imply Lax's equation

$$\partial_t A - \partial_\lambda B + [A, B] = 0,^5 \quad (5.4)$$

where $[\cdot, \cdot]$ denotes the usual matrix commutator.

⁵We mention that $\partial_t A = \frac{\partial A}{\partial p} \frac{dp}{dt} + \frac{\partial A}{\partial q} \frac{dq}{dt} + \frac{\partial A}{\partial t}$,

Let $h = h(t)$ denote the h -function and let $t = t(h)$ denote the t -function, which is the inverse of the h -function. Let $\phi(\lambda, h) = \psi(\lambda, t(h))$. Replacing in equations (5.3) ψ by ϕ , we find

$$\frac{\partial \phi}{\partial \lambda}(\lambda, h) = A(p(t(h)), q(t(h)), t(h), \lambda)\phi(\lambda, h), \quad (5.5a)$$

$$\frac{\partial \phi}{\partial h}(\lambda, h) = \frac{dt}{dh}B(p(t(h)), q(t(h)), t(h), \lambda)\phi(\lambda, h). \quad (5.5b)$$

Both the h -function and the t -function are unknown functions (the knowledge of these functions requires solving Hamilton's equations). However, by the definition of the conjugate Hamiltonian, we have

$$t(h) = T(p(t(h)), q(t(h)), h). \quad (5.6)$$

Moreover, the conjugate Hamiltonian structure (5.2) implies

$$\frac{dt}{dh} = \frac{\partial T}{\partial h}. \quad (5.7)$$

Using in equations (5.5) equations (5.6) and (5.7) to replace the unknown functions in terms of explicit functions, we obtain the following Lax pair for equations (5.2):

$$\frac{\partial \phi}{\partial \lambda} = A(p, q, T(p, q, h), \lambda)\phi, \quad (5.8a)$$

$$\frac{\partial \phi}{\partial h} = \frac{\partial T}{\partial h}B(p, q, T(p, q, h), \lambda)\phi. \quad (5.8b)$$

Indeed, Lax's equation reads:

$$\partial_h A - \frac{\partial T}{\partial h} \partial_\lambda B + \frac{\partial T}{\partial h} [A, B] = 0. \quad (5.9)$$

Noting that

$$\partial_h = \frac{dt}{dh} \partial_t = \frac{\partial T}{\partial h} \partial_t,$$

we find that equation (5.9) is just equation (5.4) using h as the independent variable.

□

Example 5.2 (A Lax pair for CP_I and \widetilde{CP}_I) Recall the Jimbo-Miwa pair [11][12] for Hamilton's equations (4.3) of Painlevé I:

$$\frac{\partial \psi}{\partial \lambda} = \left\{ \begin{pmatrix} -p & q^2 + t/2 \\ -4q & p \end{pmatrix} + \lambda \begin{pmatrix} 0 & q \\ 4 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \psi, \quad (5.10a)$$

$$\frac{\partial \psi}{\partial t} = \left\{ \begin{pmatrix} 0 & q \\ 2 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \right\} \psi. \quad (5.10b)$$

Replacing in equations (5.10) t by

$$\frac{1}{2} \frac{p^2}{q} - 2q^2 - \frac{h}{q}$$

and using

$$\frac{\partial T}{\partial h} = -\frac{1}{q},$$

we obtain the following Lax pair for CP_I and \widetilde{CP}_I :

$$\frac{\partial \psi}{\partial \lambda} = \left\{ \begin{pmatrix} -p & \frac{p^2}{4q^2} - \frac{h}{2q} \\ -4q & p \end{pmatrix} + \lambda \begin{pmatrix} 0 & q \\ 4 & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \psi, \quad (5.11a)$$

$$\frac{\partial \psi}{\partial h} = -\frac{1}{q} \left\{ \begin{pmatrix} 0 & q \\ 2 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} \right\} \psi. \quad (5.11b)$$

This Lax pair can be verified directly.

6 Conclusions

We have introduced a novel class of integrable ODEs, which are related to P_I , P_{II} , P_{IV} and \widetilde{P}_I , \widetilde{P}_{II} and \widetilde{P}_{IV} . The relation between the new ODEs and the Painlevé equations is *implicit*. We recall, that there exist analogous implicit relations among integrable PDEs, namely the relations derived via the so-called hodograph transformations. For example, the celebrated Korteweg-de Vries and Harry-Dym equations are related by precisely such a transformation [3].

Hodograph type transformation do *not* preserve the Painlevé property (for example, solutions for the Harry-Dym equation do *not* possess this property [3]). Similarly, we do not expect that conjugate Painlevé equations to possess the Painlevé property. Nevertheless, these equations *are* integrable. Indeed, it is possible to construct a large class of solutions of the conjugate equations. Furthermore, in principle, it is possible to express the solution of the general initial value problem in terms of the solutions of the initial value problem of the associated Painlevé equation. However, the most efficient way to solve the initial value problem of a given conjugate ODE, is to use its associated Lax pair. For the conjugate equations of P_{II} and P_I , relevant Lax pairs are given by equations (2.7) and (5.11). For other conjugate ODEs, similar Lax pairs can be constructed using Proposition (5.1).

Taking into consideration the relation between the implicit transformations discussed here and hodograph type transformations, it is natural to expect that the ODEs introduced here might appear as ODE reductions of integrable PDEs (such as the Harry-Dym equation), which are related to well known integrable PDEs (such as the Korteweg-de Vries equation) via hodograph transformations.

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