On dependence of the implied volatility on returns for stochastic volatility models

Mikhail Martynov, Olga Rozanova

Moscow State University

Abstract

We study the dependence of volatility on the stock price in the stochastic volatility framework on the example of the Heston model. To be more specific, we consider the conditional expectation of variance (square of volatility) under fixed stock price return as a function of the return and time. The behavior of this function depends on the initial stock price return distribution density. In particular, we obtain the "smile" effect near the mean value of the stock price return. For the Gaussian distribution this effect is strong, but it weakens and becomes negligible as the decay of distribution at infinity slows down.

Key words: stochastic volatility, the Heston model, conditional expectation of variance
1991 MSC: 91B24

1 Introduction

Stochastic volatility (SV) models are quite popular in recent decades due to a need for reliable quantitative analysis of market data. The most popular ones are the Heston (Heston 1993), Stein-Stein (Stein and Stein 1991) , Schöble-Zhu (Schöble and Zhu 1999), Hull-White (Hull and White 1987) and Scott (Scott 1987) models. We refer for reviews to (Miccichè, Bonanno, Lillo and Mantegna 2002; Mitra 2009; Fouque, Papanicolaou and Sircar 2000). The main reason for introducing the SV models is to find a realistic alternative approach to

Email address: rozanova@mech.math.msu.su (Olga Rozanova).
[1] Supported by DFG, project 436 RUS 113/823/0-1 (O.R.) and the special program of the Ministry of Education of the Russian Federation "The development of scientific potential of the Higher School", project 2.1.1/1399.

option pricing to capture the time varying nature of the volatility, assumed to be constant in the Black-Scholes approach.

Nevertheless, SV models can be used for investigation of another properties of financial markets. For example, in (Dragulescu and Yakovenko 2002) the time-dependent probability distribution of stock price returns was studied. While returns are readily known from a financial data, variance (square of the stock-price volatility) is not given directly, so it acts as a hidden stochastic variable. In (Dragulescu and Yakovenko 2002) the joint probability density function of returns and variance was found, then the integration over variance was performed and the probability distribution function of returns unconditional on variance was obtained. The latter PDF can be directly compared with the Dow-Jones data for the 20-years period of 1982–2001 and an excellent agreement was found. The tails of the PDF decay slower than the log-normal distribution predicts (the so-called "fat-tails" effect).

Technically our paper is connected with (Dragulescu and Yakovenko 2002). However, we study the dependence of the variance on fixed returns, thus, we estimate hidden stochastic variable through the variable that can be easily obtained from financial data. The result strongly depends on initial distribution of returns and variance. It is natural that the distributions change their shape with time. In particular, we show that for Gaussian initial distribution of returns the expectation of variance demonstrates the "smile" phenomenon near the mean value of returns.

To get a formula for the conditional expectation of variance at a fixed return we prove a lemma that simplifies appreciably the computations. This lemma can be useful for another problems, since it spares the calculation of the inverse Fourier transform.

2 General formulas for the conditional expectation and variance

Let us consider the stochastic differential equation system:

$$\begin{aligned} &dF_t = A dt + \sigma dW_1, \quad dV_t = B dt + \lambda dW_2, \\ &F_0 = f, \quad V_0 = v, \quad t \ge 0,\ f \in \mathbb{R}, v \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $W(t) = (W_1(t), W_2(t))$ is a two-dimensional standard Wiener process, $A = A(t, F_t, V_t)$, $B = B(t, F_t, V_t)$, $\sigma = \sigma(t, F_t, V_t)$, $\lambda = \lambda(t, F_t, V_t)$ are prescribed functions.

The joint probability density $P(t,f,v)$ of random values F_t and V_t obeys the Fokker–Plank equation (e.g.,(Risken 1989))

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial f}(AP) - \frac{\partial}{\partial v}(BP) + \frac{1}{2}\frac{\partial^2}{\partial f^2}\left(\sigma^2 P\right) + \frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\lambda^2 P\right) \tag{2.2}$$

with initial condition

$$P(0,f,v) = P_0(f,v), \tag{2.3}$$

determined by initial distributions of F_t and V_t.

If $P(t,f,v)$ is known, one can find $E\left(V_t|F_t = f\right)$, which is the conditional expectation of value V_t at a fixed F_t at the moment t. This value can be found by the following formula (see, (Chorin and Hald 2006)):

$$E\left(V_t|F_t = f\right) = \lim_{L\to+\infty} \frac{\int_{(-L,L)} vP(t,f,v)dv}{\int_{(-L,L)} P(t,f,v)dv}, \tag{2.4}$$

when improper integrals from numerator and denominator converge.

Note that if we choose $P_0(f,v) = \delta(v - v_0(f))g(f)$, where $v_0(f)$ and $g(f)$ are arbitrary smooth functions, then $E\left(V_t|F_t = f\right) = v_0(f)$.

For some classes of systems (2.1) the conditional expectation $V(t,f)$ was found in (Risken 1989; Albeverio and Rozanova 2009; Albeverio and Rozanova 2010) within an absolutely different context.

Let us remark that sometimes it is easier to find the Fourier transform of $P(t,f,v)$ function over f,v variables, than the function itself. We will get formula allowing to express $E\left(V_t|F_t = f\right)$ in terms of Fourier transform of $P(t,f,v)$ and will apply it for finding an average variance of the stock price, which depends on known return rate. It is assumed that stochastic volatility here is modeled in compliance with the Heston law (Heston 1993).

Lemma 2.1 *Let $\hat{P}(t,\mu,\xi)$ be the Fourier transform of function $P(t,f,v)$ over (f,v) variables, which is the solution of problem* (2.2), (2.3), *and both integrals from* (2.4) *converge. Assume that $\hat{P}(t,\mu,0)$ and $\partial_\xi \hat{P}(t,\mu,0)$ are decreasing over μ at infinity faster than any power. Then $E\left(V_t|F_t = f\right)$ determined by* (2.4) *can be found as*

$$E\left(V_t|F_t = f\right) = \frac{i\mathbf{F}_\mu^{-1}\left[\partial_\xi \hat{P}(t,\mu,0)\right](t,f)}{\mathbf{F}_\mu^{-1}\left[\hat{P}(t,\mu,0)\right](t,f)}, \quad t \geq 0,\ f \in \mathbb{R}, \tag{2.5}$$

where $\mathbf{F}_\mu^{-1}$ and $\mathbf{F}_\xi^{-1}$ mean the inverse Fourier transforms over μ and ξ, respectively.

PROOF. Let us calculate denominator (2.4) formally (we mean the Fourier transform in the sense of distributions):

$$\int_{\mathbb{R}} P(t,f,v)\,dv = \int_{\mathbb{R}} \mathbf{F}_{\mu}^{-1}[\mathbf{F}_{\xi}^{-1}[\hat{P}(t,\mu,\xi)]]\,dv =$$

$$\mathbf{F}_{\mu}^{-1}[\Big(\mathbf{F}_{v}^{-1}[1](\xi),\hat{P}(t,\mu,\xi)\Big)_{\xi}] = \sqrt{2\pi}\,\mathbf{F}_{\mu}^{-1}[\Big(\delta(\xi),\hat{P}(t,\mu,\xi)\Big)_{\xi}] = \sqrt{2\pi}\mathbf{F}_{\mu}^{-1}[\hat{P}(t,\mu,0)].$$

Here $(.,.)_{\xi}$ means the action of a distribution on the test function of ξ.

Numerator can be found analogously:

$$\int_{\mathbb{R}} vP(t,f,v)\,dv = \int_{\mathbb{R}} \mathbf{F}_{\xi}^{-1}[\mathbf{F}_{\mu}^{-1}[\hat{P}(t,\mu,\xi)]]\,dv = \mathbf{F}_{\mu}^{-1}[\Big(\mathbf{F}_{v}^{-1}[v](\xi),\hat{P}(t,\mu,\xi)\Big)_{\xi}] =$$

$$-\sqrt{2\pi}i\,\mathbf{F}_{\mu}^{-1}[\Big(\delta'(\xi),\hat{P}(t,\mu,\xi)\Big)_{\xi}] = \sqrt{2\pi}i\,\mathbf{F}_{\mu}^{-1}[\Big(\delta(\xi),\partial_{\xi}\hat{P}(t,\mu,\xi)\Big)_{\xi}] =$$

$$i\sqrt{2\pi}\mathbf{F}_{\mu}^{-1}[\partial_{\xi}\hat{P}(t,\mu,0)].$$

Thus, the lemma is proved. □

Let us define the variance of V_t at a fixed F_t as

$$Var\,(V_t|F_t=f) = \lim_{L\to+\infty}\frac{\int_{(-L,L)} v^2 P(t,f,v)dv}{\int_{(-L,L)} P(t,f,v)dv} - E^2\,(V_t|F_t=f)\,, \tag{2.6}$$

provided that improper integrals exist.

Lemma 2.2 *With the same assumptions as in Lemma 2.1 the following formula holds:*

(2.7)

$$Var\,(V_t|F_t=f) = \frac{(\mathbf{F}_{\mu}^{-1}[\partial_{\xi}\hat{P}(t,\mu,0)])^2 - \mathbf{F}_{\mu}^{-1}[\partial_{\xi}^2\hat{P}(t,\mu,0)]\mathbf{F}_{\mu}^{-1}[\hat{P}(t,\mu,0)]}{(\mathbf{F}_{\mu}^{-1}[\hat{P}(t,\mu,0)])^2}(t,f).$$

The proof is absolutely similar to the proof of Lemma 2.1.

3 Example: the Heston model

Of course, there is no explicit formula for the joint probability density function $P(t,f,v)$ for arbitrary system (2.1). We will consider a particular, but important case

$$df_t = \left(\alpha - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_1, \tag{3.1}$$

$$dv_t = -\gamma(v_t - \theta)dt + k\sqrt{v_t} dW_2. \tag{3.2}$$

Here $\gamma,\ k,\ \theta$ are arbitrary positive constants.

Equation (3.2) describes the process that in financial literature is called Cox-Ingersoll-Ross (CIR) process, and in mathematical statistics — the Feller process (Fouque, Papanicolaou and Sircar 2000; Feller 1951). In (Feller 1951) it is shown that this equation has a nonnegative solution for $t \in [0, +\infty)$ when $2\gamma\theta > k^2$.

The first equation describes a return f_t on the stock price, in assumption that the stock price itself obeys a geometric Brownian motion with stochastic volatility. The second equation describes the square of volatility $\sigma_t^2 = v_t$ according to the Heston model.

The Fokker–Planck equation (2.2) for the joint density function $P(t, f, v)$ of return f_t and variance v_t takes here the following form:

$$\frac{\partial P(t,f,v)}{\partial t} = \gamma P(t,f,v) + (\gamma(v-\theta) + k^2)\frac{\partial P(t,f,v)}{\partial v} + \left(\frac{v}{2} - \alpha\right)\frac{\partial P(t,f,v)}{\partial f} +$$

$$\frac{k^2 v}{2}\frac{\partial^2 P(t,f,v)}{\partial v^2} + \frac{v}{2}\frac{\partial^2 P(t,f,v)}{\partial f^2}. \tag{3.3}$$

Now we can choose different initial distributions for return and variance in interesting cases. Note that it is natural to assume that initially the variance does not depend on return, below we consider constant initial variance.

Below we denote $E\left(v_t | f_t = f\right)$ as $V(t, f)$ for short.

3.1 The uniform initial distribution of returns.

We begin with the simplest and almost trivial case. Let us assume that initially the rate of return is distributed uniformly in the interval $(-L, L)$, $(L = \text{const} > 0)$, and volatility is equal to some constant $a \geq 0$. Then the initial joint density distribution of f_t and v_t is

$$P(0, f, v) = \frac{1}{2L}\delta(v - a). \tag{3.4}$$

To simplify further calculations we will exclude randomness for $t = 0$, i.e. we will assume $a = 0$.

The function $\hat{P}(t, \mu, \xi)$, the Fourier transform of P over (f, v) and it satisfies the equation

$$\frac{\partial \hat{P}(t, \mu, \xi)}{\partial t} + \frac{1}{2}\left(\mu + i\mu^2 + 2\gamma\xi + ik^2\xi^2\right)\frac{\partial \hat{P}(t, \mu, \xi)}{\partial \xi} + i(\gamma\theta + \xi\mu\alpha)\hat{P}(t, \mu, \xi) = 0 \tag{3.5}$$

with the initial condition

$$P(0, \mu, \xi) = \frac{\pi}{L}\delta(\mu). \tag{3.6}$$

The first-order equation (3.5) can be integrated. The solution of problem (3.5), (3.6) takes the form

$$\hat{P}(t, \mu, \xi) = \frac{\pi}{L}\delta(\mu)\left(\frac{4\gamma^2 e^{2\gamma t}}{\left(2\gamma e^{\gamma t} + ik^2\xi\left(e^{\gamma t} - 1\right)\right)^2}\right)^{\frac{\gamma\theta t}{k^2}} \tag{3.7}$$

It is easy to calculate that

$$\hat{P}(t, \mu, 0) = \frac{\pi}{L}\delta(\mu), \quad \partial_\xi \hat{P}(t, \mu, 0) = \frac{\pi}{L}\delta(\mu) i\theta\left(e^{-\gamma t} - 1\right) \tag{3.8}$$

And finally from (3.8), (2.5) and (2.7) we get

$$E(v_t | f_t = f) = V(t, f) = \frac{i\mathbf{F}_\mu^{-1}[\partial_\xi \hat{P}(t, \mu, 0)](t, f)}{\mathbf{F}_\mu^{-1}[\hat{P}(t, \mu, 0)](t, f)} = \theta\left(1 - e^{-\gamma t}\right), \tag{3.9}$$

$$Var(v_t | f_t = f) = \frac{\theta k^2}{2\gamma}\left(1 - e^{-\gamma t}\right)^2. \tag{3.10}$$

It is evident that here there is no dependence on f and the result is the same as we could obtain from calculation of mathematical expectation and variance of v_t from equation (3.5).

3.2 The Gaussian initial distribution of returns.

Let us assume that initially rate of return is distributed according to the Gaussian law. Then we have the following initial condition:

$$P(0,f,v)=\frac{m}{\sqrt{\pi}}e^{-m^2f^2}\delta(v),\quad m>0. \tag{3.11}$$

When $a=0$, the Fourier transform of initial data over (f,v) is $\hat{P}(0,\mu,\xi)=e^{-\frac{\mu^2}{4m^2}}$.

Solution of the problem (3.3), (3.11) takes the form:
(3.12)

$$\hat{P}(t,\mu,\xi)=\frac{\sqrt{\pi}}{m}\left(-\frac{\mu(\mu-i)+\frac{\gamma^2}{k^2}}{\mu^2+k^2\gamma^2-i(2\gamma\xi+\mu)}\right)^{\frac{\gamma\theta}{k^2}}\exp\left(-\frac{\mu^2}{4m^2}-(\alpha\mu i-\frac{\gamma^2\theta}{k^2})t\right)*$$

$$\left(-\cosh\left(\frac{t}{2}\sqrt{k^2\mu(\mu-i)+\gamma^2}-i\arctan\left(\frac{-k^2\xi+i\gamma}{\sqrt{k^2\mu(\mu-i)+\gamma^2}}\right)\right)\right)^{-\frac{2\gamma\theta}{k^2}}.$$

We see that $\hat{P}(t,\mu,\xi)$ exponentially decreases over μ. That is why we can use formula (2.5) and obtain (after cumbersome transformations) the following integral expression:

(3.13)

$$V(t,f)=2\gamma\theta\,\frac{\int\limits_{\mathbb{R}}\Psi(t,\mu,f)\frac{\sinh\left(\frac{t}{4}\sqrt{k^2(4\mu^2+1)+4\gamma^2}\right)}{\left(\cosh\left(\frac{t}{4}\sqrt{k^2(4\mu^2+1)+4\gamma^2}\right)\sqrt{k^2(4\mu^2+1)+4\gamma^2}+2\gamma\sinh\left(\frac{t}{4}\sqrt{k^2(4\mu^2+1)+\gamma^2}\right)\right)}d\mu}{\int\limits_{\mathbb{R}}\psi(t,\mu,f)d\mu},$$

where

$$\Psi(t,\mu,f)=e^{\frac{-\mu^2+i\mu(4f-4t\alpha-1)}{4m^2}}*$$

$$\left(\frac{k^2(4\mu^2+1)+\gamma^2}{\left(\cosh\left(\frac{t}{4}\sqrt{k^2(4\mu^2+1)+4\gamma^2}\right)\sqrt{k^2(4\mu^2+1)+4\gamma^2}+2\gamma\sinh\left(\frac{t}{4}\sqrt{k^2(4\mu^2+1)+\gamma^2}\right)\right)^2}\right)^{\frac{\gamma\theta}{k^2}}.$$

Let us remark that if $a\neq 0$, we can also get a similar formula, but it will be more cumbersome.

3.3 "Fat-tails" initial distribution of return

Integral formula, analogous to (3.13) can be obtained for initial distributions intermediate between uniform and Gaussian ones. For example, as initial distribution we can take

$$P(0,f,v) = K(1+m^2f^2)^q\,\delta(v), \quad m>0,\, q<0,$$

with an appropriate constant K. Exact formula for the Fourier transform $\hat{P}(t,\mu,\xi)$ can be found for $q=-\frac{1}{2}$, $-n$, $n\in\mathbb{N}$. For all these cases $\hat{P}(t,\mu,\xi)$ decays as $|\mu|\to\infty$ sufficiently fast and Lemma 1 can be applied for calculation of $V(t,f)$.

For example, for $q=-1$ the difference with (3.12) is only in the multiplier $e^{-\frac{\mu^2}{4m^2}}$: it should be changed to

$$\left(e^{-\frac{\mu}{m}} - e^{\frac{\mu}{m}}\right) H\left(\mu\right) + e^{\frac{\mu}{m}},$$

with the Heaviside function H.

3.4 "Smile" volatility effect and asymptotic behavior for small time parameters

It turns out that if in the Heston model the average volatility is considered as a function of the rate on return, we will observe a "smile" volatility effect. This "smile" is different from one which can be seen on plot of volatility against strike price in the case of a standard option model (e.g. (Derman and Kani 1998)). The effect appears in numerical calculation of both integrals in (3.13) with the use of standard algorithms. Thus, we plotted the function $V(t,f)$ for three consequent moments of time (Fig. 1) and for the following values of parameters:

$$\gamma=1,\ k=1,\ \theta=1,\ \alpha=1,\ m=1.$$

The "smile" volatility effect can be found by analytical methods as well. Indeed, let us fix rate of return f. Then from (3.13) by expansion of integrand functions into formal series as $t\to 0$ up to the forth component and by further term-wise integration (series converge at least for small f and m) we will get that

$$V(t,f) = \gamma\theta t - \frac{1}{2}\gamma^2\theta t^2 + \frac{1}{6}\gamma\theta\left(\gamma^2 + 2f^2m^4k^2 - fm^2k^2 - m^2k^2\right)t^3 -$$

$$\frac{1}{6}\gamma\theta\left(8\gamma\, k^2f^2m^4 - 4\left(\gamma\, + 4\,m^2\alpha\right)k^2m^2f - 4\left(\gamma+\alpha\right)m^2k^2 + \gamma^3\right)\,t^4 + O(t^5).$$

Hence for small t we find that

$$V(t,f) = \frac{1}{3}\gamma\,\theta\,t^3(1-\gamma\,t)m^4k^2f^2 - \left(\frac{2}{3}\,\alpha\,m^2\,t + \frac{1}{6}\,(1\,-\gamma\,t)\right)t^3\gamma\,\theta m^2k^2f +$$

$$\frac{1}{6}\,(\gamma\,t-(1-\alpha\,t))\,\theta\,\gamma t^3m^2k^2 + \gamma\,\theta\,t - \frac{1}{2}\,\gamma^2\theta\,t^2 + \frac{1}{6}\,\gamma^3\theta\,t^3 - \frac{1}{24}\,\gamma^4\theta\,t^4$$

is a quadratic trinomial over f with a minimum in point $f = \frac{4\,m^2\alpha\,t-\gamma\,t+1}{4\,m^2(1-\gamma\,t)}$, for $t > 0$.

The "smile" effect holds for initial "fat-tails" power initial distributions as well. Nevertheless, this effect weakens as the decay of the distribution at infinity becomes slower.

Fig.2 presents the function $V(t,f)$ for three consequent moments of time for the initial distribution of return $p(f)$ given by formula

$$p(f) = \frac{1}{\pi}\frac{1}{1+f^2}.$$

The values of parameters are

$$\gamma = 10,\ k = 1,\ \theta = 0.1,\ \alpha = 10.$$

It seems that the curves are strait lines, but the analysis of numerical values shows that the "smile" still persists near the mean value of return. One can still find formal asymptotics of $V(t,f)$ as $t \to 0$,

$$V(t,f) = \gamma\theta t - \frac{1}{2}\gamma^2\theta t^2 - \gamma\theta\frac{R_4(f,\gamma,k)}{R_6(f)}t^3 + \gamma^2\theta\frac{R_8(f,\gamma,k,\alpha)}{R_4(f)}t^4 + O(t^5),$$

where we denote by R_k a polynomial of order k with respect to f. We do not write down these polynomial, let us only note that $\frac{R_4(f,\gamma,k)}{R_6(f)} \sim \frac{1}{2f^2}\left(\frac{k^2}{8}-\gamma^2\right)$ and $\frac{R_8(f,\gamma,k,\alpha)}{R_4(f)} \sim \frac{16f^4}{3}\left(2\gamma^2-k^2\right)$ as $|f| \to \infty$.

It is very interesting to study the limiting behavior of $V(t,f)$ as $|f| \to \infty$ and $t \to \infty$. We do not dwell here on this quite delicate question at all and reserve it for future research. Some hints can be found in (Dragulescu and Yakovenko 2002; Gulisashvili and Stein 2006; Gulisashvili and Stein 2010).

3.5 Modifications of the Heston model

Let us analyze the situation when the coefficient γ from equation (3.2) depends on time. For some interesting cases of this dependence one can find the Fourier

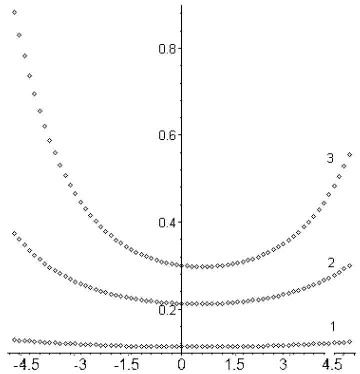

Figure 1.

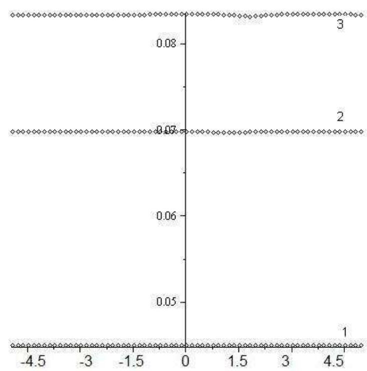

Figure 2.

transform of $P(t, f, v)$ and formula for $V(t, f)$. For example, if we set $\gamma = \frac{1}{T-t}$, then we get a Brownian bridge-like equation (see, (Øksendal 2002)) describing square of volatility behavior with start at $v_0 = a > 0$ and end at $v_T = b \geq 0$. Here the solution will be represented in terms of integrals of Bessel functions and the solution is cumbersome.

It may seem that the described approach, which helps to find the conditional expectation of volatility under fixed returns in the Heston model, can be successfully applied in other variations of this model. This is true when initial rate on return has a uniform distribution. However, this situation is trivial, because the answer does not contain f and is equal to the expectation of return obtained from the second equation of model. In the case of non-uniform initial distribution of return (for instance, Gaussian) formula (2.5) may be non-applicable, even when explicit expression for $\hat{P}(t, \mu, \xi)$ can be found. The cause is that $\hat{P}(t, \mu, \xi)$ increases as $|\mu| \to \infty$. For example, if we replace equation (3.2) with

$$dv_t = -\gamma(v_t - \theta)dt + kdW_2, \quad \gamma, \theta, k > 0, \tag{3.14}$$

under initial data (3.11), $a = 0$, we will get

$$\hat{P}(t, \mu, \xi) = \frac{\sqrt{\pi}}{m} \exp\left(\frac{k^2 t}{8\gamma^2}\mu^4 - i\frac{k^2 t}{4\gamma^2}\mu^3 - \left(\frac{\theta\, t}{2} + \frac{k^2 t}{8\gamma^2} + \frac{1}{4m^2}\right)\mu^2 + i\left(\frac{\theta}{2} - \alpha\right) t\mu\right),$$

whence it follows that the coefficient of μ^4 in exponent power is positive when t is positive. This means that integrals from (2.5) are divergent.

4 Possible application

Basing on our results one can introduce a rule for estimation of the company's rating based on stock prices. The natural presumption is that company's rating increases when return on assets increases and volatility decreases. Hence for estimation of the company's rating one can use (very rough) index $R(t, f) = f/V(t, f)$, where $V(t, f)$ is calculated by formula (3.13). Figs. 3 and 4 shows the plot function $R(t, f)$ for three consequent time points for Gaussian and power distributions, respectively. Parameters as in Figs. 1 and 2. We can see that in the Gaussian case the index does not rise monotonically with return.

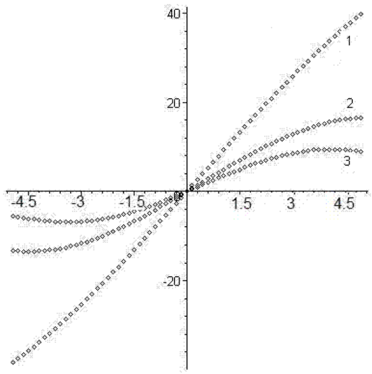

Figure 3.

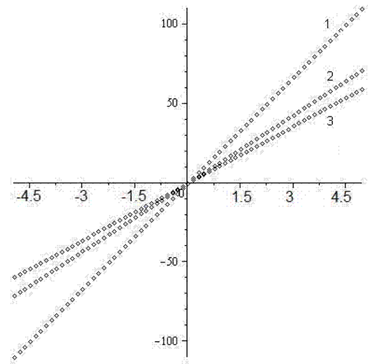

Figure 4.

5 Conclusion and further work

In this article the Heston model has been analyzed not in terms of finding an option fair price based on suggestion about stochastic behavior of volatility. Our goal is to determine volatility itself given rate on return data. This problem has been solved by calculation of the average volatility under a fixed rate on return and under the supplementary condition on initial distribution of return and volatility. Namely, different cases of initial distribution of returns have been studied: uniform, Gaussian and "fat-tails" distribution, intermediate between them. We found the effect of the "smile" volatility near the mean value of the stock price return for the Gaussian initial distribution and for certain distributions decreasing at infinity slower then the Gaussian one (for which we succeed to find the Fourier transform of the joint probability density of return and variance explicitly). For the Gaussian distribution this effect is strong, but it weakens and becomes negligible as the decay of distribution at infinity slows down. This effect is different from the traditional volatility "smile" in the sense that the plot has been considered as a function of return, but not a strike price function.

Formulas for the conditional variance at fixed return $V(t, f)$ are obtained in the present work in the integral form, we compute the integrals numerically using standard algorithms and did only simple formal asymptotic analysis of the formulas for small time. Of course, analysis of the formulas for larger t and f and the asymptotics of $V(t, f)$ as $|f| \to \infty$ and $t \to \infty$ is very important open question. Moreover, the dependence of the characteristics of the "smile" on the properties of the initial distribution of returns has to be studied in general case, not only for separate examples, as it was done here.

References

Albeverio, S., and O. Rozanova (2009): The Non-Viscous Burgers Equation Associated with Random Positions in Coordinate Space: a Threshold for Blow up Behavior. *Mathematical Models and Methods in Applied Sciences*, 19(5), 1–19.

Albeverio, S., and O. Rozanova (2010): Suppression of Unbounded Gradients in a SDE Associated with the Burgers Equation. *Proc. Amer. Math. Soc.*, 138, 241–251.

Chorin, A. J., and O. H. Hald (2006): *Stochastic Tools in Mathematics and Science*, New York: Springer.

Derman, E., and I. Kani (1998): Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *Int. J. Theor. Appl. Finance*, 1, 61–110.

Dragulescu, A. A., and V. M. Yakovenko (2002): Probability Distribution of Returns in the Heston Model with Stochastic Volatility. *Quantitative Finance*,2, 443–453.

Feller, W. (1951): Two Singular Diffusion Problems. *Annals of Mathematics*, 54, 173–182.

Fouque, J. P., G. Papanicolaou, and K. R. Sircar (2000): *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge: Cambridge University Press.

Gulisashvili, A., and E. M. Stein (2010): Asymptotic Behavior of the Stock Price Distribution Density and Implied Volatility in Stochastic Volatility Models. *Mathematical Finance*, 30, 447–477.

Gulisashvili, A., and E. M. Stein (2006): Asymptotic Behavior of the Distribution of the Stock Price in Models with Stochastic Volatility: the Hull-White Model. *C. R. Acad Sci. Paris, Ser.I*, 343, 519–523.

Heston, S. L. (1993): A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6, 327–343.

Hull, J., and A. White (1987): The Pricing of Options on Asset with Stochastic Volatilities. *J. Finance*, 42, 281–300.

Miccichè, S., G. Bonanno, F. Lillo and R. N. Mantegna (2002): Volatility in Financial Markets: Stochastic Models and Empirical Results. *Physica A*, 314, 756-761.

Mitra, S.(2009): A Review of Volatility and Option Pricing. E-print:0904.1392.

Oksendal, B. (2002): *Stochastic Differential Equations: An Introduction with Applications*. 5th ed., Heidelberg: Springer.

Risken, H. (1989): *The Fokker-Planck Equation. Methods of solution and applications*, 2ed, New York: Springer.

Schöobel, R., and J. Zhu (1999): Stochastic Volatility with an Ornstein - Uhlenbeck Process: An Extension. *Europ. Finance Rev.*, 4, 23–46.

Scott, L. (1987): Option Pricing when the Variance Changes Randomly: Theory, Estimaton and an Applications. *J. Finan. Quant. Anal.*, 22, 419–438.

Stein, E.M., and J.C. Stein (1991): Stock Price Distributions with Stochastic Volatility: An Analytic Approach. *Rev. Finan. Stud.*, 4, 727–752.