CERN-2010 LPTHE-2010

Four twisted differential operators for the N=4 superconformal algebra

Laurent Baulieu *

Theoretical Division CERN†
LPTHE Université Pierre et Marie Curie ‡

Abstract

The $\mathcal{N} = 4, d = 4$ Yang–Mills conformal supersymmetry exhibits a very simple sub-sector described by four differential operators. The invariance under this subalgebra is big enough to determine the $\mathcal{N} = 4$ theory. Some attempts are done to interpret these differential operators.

*email address: baulieu@lpthe.jussieu.fr
† CH-1211 Genève, 23, Switzerland
‡ 4 place Jussieu, F-75252 Paris Cedex 05, France

1 Introduction

The N=4 super-Yang–Mills action

$$S = \frac{1}{g^2}\int d^4x tr\Big(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar\lambda^{\dot\alpha}_A \not{D}_{\dot\alpha\beta}\lambda^{\beta A} - i\lambda^A_\alpha \not{D}^{\alpha\dot\beta}\bar\lambda_{A\dot\beta} + \frac{1}{2}(D_\mu\bar\phi_{AB})(D^\mu\phi^{AB})$$
$$-\sqrt{2}\bar\phi_{AB}\{\lambda^{\alpha A},\lambda^B_\alpha\} - \sqrt{2}\phi^{AB}\{\bar\lambda^{\dot\alpha}_A,\bar\lambda_{\dot\alpha B}\} + \frac{1}{8}[\phi^{AB},\phi^{CD}][\bar\phi_{AB},\bar\phi_{CD}]\Big) \tag{1}$$

is an interacting theory between a Yang–Mills field A_μ, 6 scalar fields ϕ^{AB} and 4 Majorana spinors $\lambda^A = (\lambda^{A\alpha},\lambda^A_{\dot\alpha})$, where the indices A, B label the R-symmetry. It is invariant under the Poincaré supersymmetry transformations with 16 supersymmetry transformations, $\delta_\epsilon A^\mu = \bar\epsilon\gamma^\mu\lambda$, $\delta_\epsilon\phi^{AB} = \epsilon^A\lambda^B$ and $\delta\lambda = (\Gamma^{\mu\nu}F_{\mu\nu} + iD_\mu\Gamma^\mu\phi - [\phi,\phi])\epsilon$. The N=4 theory is obtained by dimensionally reducing the N=1 d=10 theory or, equivalently, the N=2 d=8 theory. The 16 supersymmetries as a whole close only up to field equations, as follows,

$$\{\delta_\epsilon,\delta_{\hat\epsilon}\} \sim -2i\bar\epsilon\gamma^\mu\hat\epsilon\partial_\mu - 2i\delta^{gauge}(\bar\epsilon\gamma^\mu A_\mu\hat\epsilon) \tag{2}$$

where the symbol $\sim$ means modulo (spinor) field equations of motion. In a light-cone approach, one can select eight supersymmetries that close off-shell. However, in this physical approach, Lorentz invariance is difficult to recover and, moreover, light-cone gauge field propagators are ill-defined, even in perturbative quantum field theory. In Euclidean space, the Lorentz symmetry is $SO(4)$ and the R-symmetry of the theory is $SO(5,1)$ (instead of $SO(6)$ in Minkowski space). It is in fact possible to reduce the global $SO(4)\times SO(5,1)$ invariance down to a $SO'(4)\times SL(2,R)$ invariance, a so-called twist operation [1][2]. In this way the fermion and scalar representations becomes reducible, and one can covariantly select nine supersymmetry generators [3] that close "off-shell", that is, constitute a reduced super-algebra where no field equations occur in the closure relations of transformations. Moreover, the gauge transformation in the right-hand side of the closure relations can be eliminated, provided one introduces new fields called shadow fields [4]. Eventually, one can do a gauge-fixing of the theory, such that one has Ward identities that allow one to control both gauge invariance and supersymmetry at the quantum level. This provides a solid framework for studying the supersymmetric properties of the N=4 theory within the framework of quantum field theory [4].

The twist [1][2] of the N=4 superPoincaré symmetry is done by taking the diagonal $SU'(2)$ of one of the $SU(2) \subset SO(5,1)$ subgroups of the R-symmetry (there are 3 possible choices [2]) and one of both SU(2) factors of the Lorentz group. What is left is the new Lorentz group $SO'(4) = SU(2)_L \otimes SU'(2)$ and a part of the R-symmetry that contains at least a $U(1)$ symmetry, for instance, a $SL(2,R)$ symmetry in the first twist. Under the new Lorentz group, the supersymmetry generators become scalars, vectors and (anti)self-dual tensors. The 16 super-charges are decomposed as follows,

$$(Q^A_\alpha,\bar Q_{A\dot\alpha}) \quad\to\quad (Q_0,\bar Q_0,Q_\mu,\bar Q_\mu,Q_{\mu\nu^-},\bar Q_{\mu\nu^-}) \tag{3}$$

The nine charges $Q_0, \bar{Q}_0, Q_\mu$ and $Q_{\mu\nu^-}$ turn out to build an off-shell closed algebra, and, moreover, they can be geometrically constructed from the point of view of topological field theory [3] as anticipated in [5]. Furthermore, the $N = 4$ action is uniquely defined by the invariance under the symmetry with the 6 generators $Q_0, \bar{Q}_0$ and Q_μ. The 10=16-6 other supersymmetry generators are overdetermining and appear as accidental, but welcome, symmetries of the N=4 action. For proving the finiteness of 1/2 BPS operators, one only need the 5-generator subalgebra made of $Q_0, \bar{Q}_0$ and $Q_{\mu\nu^-}$[4]. This may challenge us to find direct ways of constructing these maybe more fundamental smaller symmetries of the N=4 theory. The aim of this paper is to show that a non-trivial part of the superconformal symmetry can be also directly built by generalizing the framework of reference [3]. This will provide a much smaller number of generators than the 32 ones of the twisted superconformal algebra [7]. It determines the N=4 action, while keeping track of a non-trivial part of the superconformal algebra.

2 The superPoincaré twist of the N=4 Yang–Mills theory

To proceed, we need to give more details for the superPoincaré structure. In the first twisted formalism, the gluino decomposes analogously as the supersymmetry generators[1],

$$(\lambda^a_\alpha, \lambda_{a\dot{\alpha}}) \quad \to \quad (\eta, \bar{\eta}, \psi_\mu, \bar{\psi}_\mu, \chi_{\mu\nu^-}, \bar{\chi}_{\mu\nu^-}) \equiv (\eta^\alpha, \psi^\alpha, \chi^{I\alpha}) \tag{4}$$

where the $SL(2,R)$ indices $1 \leq \alpha \leq 2$ label the barred and unbarred fields. So, after the twist, the $16 = 4 \times 4$ spinorial degrees of freedom of the conventional theory are expressed as $16 = (2 + 2 \times 4 + 2 \times 3)$ $SO'(4)$-tensor degrees of freedom.

The 6 components of the $SO(5,1)$-valued scalar are twisted as follows (the indices i label the SL(2,R) adjoint representation),

$$\phi \ \to (\ \phi^i, h_{\mu\nu^-}) \equiv (\phi^i, h^I) \tag{5}$$

The symmetry with the nine generators $Q_0, \bar{Q}_0, Q_\mu, Q_{\mu\nu^-}$ closes off-shell, by including among the fields an auxiliary fields with 7 components, organized as a vector T_μ and a selfdual 2-form $H_{\mu\nu^-}$. The balanced system of fields, denoted as (9,16,7) multiplet, is

$$(A_\mu, \phi^i,\ h_{\mu\nu^-}, \Psi_\mu, \bar{\Psi}_\mu, \chi_{\mu\nu^-}, \bar{\chi}_{\mu\nu^-}, \eta,\ \bar{\eta},\ T_\mu,\ H_{\mu\nu^-}) \tag{6}$$

A brute force change of variables of the known on-shell $N = 4$ transformation can compute the action on the fields of the nine generators, with on-shell closure. Standard physicist methods can provide their modifications to get off-shell closure by introducing the auxiliary fields $T_\mu, H_{\mu\nu^-}$. There is in fact a direct, and maybe more profound, construction that we will explain,

[1] It is convenient to identify antiselfdual 2-forms as $SU(2) \subset SO'(4)$ valued scalars, using the three independent Kahler 2-forms $J^I_{\mu\nu^-}$, $(I = 1, 2, 3)$, according to the invertible identity $X^I \equiv J^I_{\mu\nu^-} X^{\mu\nu^-}$, so that $h_{\mu\nu^-} \sim h^I$.

since it suggests to us the way to incorporate superconformal transformations and determine straightforwardly an interesting off-shell closed sub-sector of the superconformal algebra.

To find the relevant supersymmetries, it is best to start from eight dimensions, using the TQFT methods, and to compactify the results in four dimensions. Indeed, in eight dimensions, triality indicates immediately the possibility of mapping the 16 supersymmetry spinorial generators on twisted tensor generators, as follows,

$$(Q^{\alpha}, Q_{\dot{\alpha}}) \to (Q_0, Q_M, Q_{MN^-}) \tag{7}$$

where $1 \le M, N \le 8$ are $SO(8)$ indices and the self-duality index MN^- is defined by using the $Spin(7) \subset SO(8)$-invariant selfdual tensor t_{MNPQ}. One has $t_{8abc} = c_{abc}$ where the c_{abc}'s are the octonion structure coefficient. Using this 4-tensor, any given $SO(8)$ 2-form can be decomposed as $28 = 7 \oplus 21$, in a $Spin(7) \subset SO(8)$-invariant way. Thus Q_{MN^-} stands for 7 generators.

The 9=1+8 generators Q_0 and Q_M can be determined explicitly from the methods of TQFT [5][6]. They satisfy

$$Q_0^2 = Q_M Q_N + Q_N Q_M = 0, \quad Q_0 Q_M + Q_N Q_0 = \partial_M \tag{8}$$

This equation implies off-shell closure, modulo gauge transformations and is obtained thank's to the introduction of an auxiliary field that is a self-dual 2-form T_{MN^-}. In seven dimensions, it becomes a 7-vector auxiliary field T_a, $1 \le a \le 7$. The non-closure relations are cornered in the sector of the selfdual generator Q_{MN^-}. Getting rid of Q_{MN^-}, one has 9=1+8 off-shell closed generators, which is a property that survives after dimensional reduction, e.g., for the N=4, d=4 theory.

The off-shell closed representation of the $N = 2, d = 8$ theory is thus given by the balanced (9,16,7) multiplet

$$(A_M, \Phi, \bar{\Phi}, \Psi_M, \chi_{MN^-}, \eta, T_{MN^-}) \tag{9}$$

This 8-dimensional formulation exists in curved space, provided the manifold has $Spin(7)$ holonomy, that is, one has a constant spinor, which allows one to map all spinors on forms [2]. This is the triality property. In flat space, it can be understood as a mere $Spin(7) \subset Spin(8)$ invariant changes of variables, using the invariant tensor t_{MNPQ}. So, the 8-dimensional twist, we are concerned with, only preserves the $Spin(7) \subset Spin(8)$ invariance.

One can then dimensionally reduce all formula in seven dimensions, with $A_M \to (A_a, L)$, $\Psi_M \to (\Psi_a, \bar{\eta})$, $\chi_{MN^-} \to \bar{\Psi}_a$, $T_{MN^-} \to \bar{T}_a$, where $a = 1, \ldots, 7$ is a $Spin(7)$ vector index. In seven dimensions, the balanced 8-dimensional multiplet (9) becomes the following one

$$(A_a, \Phi^i, \Psi_a^{\alpha}, \eta^{\alpha}, T_a) \tag{10}$$

[2] In fact the existence of such a constant spinor ζ warrantees the existence of the $Spin(7) \subset Spin(8)$-invariant tensor $t_{MNPQ} = {}^t\zeta \Gamma_{MNPQ} \zeta$, which allows one to split any given 2-form in a selfdual and antiselfdual 2-form.

The $SL(2,R)$ indices $\alpha = 1,2$ and $i = 1,2,3$ arise naturally in the dimensional reduction, giving a $Spin(7) \times SL(2,R)$ covariance [3]. The 8 generators Q_M become a 7-vector Q_a and a scalar $\bar{Q}_0$. The 7 generators Q_{MN^-} become another 7-vector $\bar{Q}_a$, which enforces a global $SL(2,R)$ covariance of the algebra, by pairing together Q_a and $\bar{Q}_a$. The non-closure relations are now cornered in the anti-commutation relations between the 7-vector generators $\bar{Q}_a$ and Q_a, by equations of motion that appear in $\{\bar{Q}_a, Q_b\}$, proportionally to the antisymmetric octonionic tensor c_{abc}. By introducing a seven-dimensional vector parameter k^a, which is shared by both vector generators $\bar{Q}_a$ and Q_a, and two independent scalar parameters k^0 and $\bar{k}^0$ for both $\bar{Q}_0$ and Q_0, one finds an off-shell closed algebraic structure for the four differential operators $k^0 Q_0, \bar{k}^0 \bar{Q}_0, k^a Q_a$ and $k^a \bar{Q}_a$,. Their action on the fields expresses an off-shell closed supersymmetry in seven dimensions, with 9=1+1+7 independent parameters.

One can do a further dimensional reduction in four dimensions. The 7 auxiliary field T_a decompose into a vector and a self-dual tensor in four dimensions $(T_\mu, H_{\mu\nu^-})$, and the 7 generators Q_a decompose into a vector and a self-dual tensor. The nine off-shell closed generators are $Q_0, \bar{Q}_0, Q_\mu, Q_{\mu\nu^-}$.

One can in fact do a more subtle selection of generators, to reestablish the $SL(2,R)$ covariance in four dimensions. One decomposes the seven generators $\bar{Q}_a$ into a vector and a self-dual tensor in four dimensions, and one retains the $SL(2,R)$ covariant set of 10 generators, $Q_0, \bar{Q}_0, Q_\mu$ and $\bar{Q}_\mu$. The 6 generators $Q_0, \bar{Q}_0$ and Q_μ build an off-shell closed subalgebra, but the off-shell closure is broken between Q_μ and $\bar{Q}_\mu$, by equations of motion proportional to the antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$. However, analogously as in seven dimensions, one can introduce a constant four-dimensional vector parameter k^μ and two scalar ones k^0 and $\bar{k}^0$ and define the 4 differential operators $s^\alpha = (k^0 Q_0, \bar{k}^0 \bar{Q}_0)$ and $\delta^\alpha = (k^\mu Q_\mu, k^\mu \bar{Q}_\mu)$, $\alpha = 1,2$, for a total of 6=1+1+4 independent parameters. One can then set $k^0 = \bar{k}^0 = 1$ and one finds the following $SL(2,R)$ and Lorentz covariant graded differential algebra in four dimensions[3],

$$\{s^\alpha, s^\beta\} = \sigma^{i\alpha\beta} \delta_{gauge}(\Phi_i) \ , \ \{\delta^\alpha, \delta^\beta\} = \sigma^{i\alpha\beta} \delta_{gauge}(|k|^2 \Phi_i) \ , \ \{s^\alpha, \delta^\beta\} = \epsilon^{\alpha\beta}(\mathcal{L}_k + \delta_{gauge}(i_k A)) \tag{11}$$

This expresses in a very compact way the off-shell closure of maximal supersymmetry in four dimensions, with equivariant closure relations.

A possible direct and geometrical construction of these relations in the TQFT language [3] follows from identities such as

$$\begin{aligned} &(s + \delta + \bar{s} + \bar{\delta})(A + c) + (A + c)^2 = \\ &F + \psi + \bar{\psi} + g(k)(\eta + \bar{\eta}) + g(J^I k)(\chi^I + \bar{\chi}^I) + (1 + |k|^2)(\bar{\Phi} + L + \Phi) \end{aligned} \tag{12}$$

[3]The notation $\delta_{gauge}(i_k A)$ stands for a gauge transformation with the field dependent parameter $i_k A \equiv k^\mu A_\mu$, and the coefficients $\sigma_i^{\alpha b}$ are for a basis of three 2×2 $SL(2,R)$ matrices. One also defines the graded Lie derivative $\mathcal{L}_\kappa = i_\kappa d + d i_\xi$.

In view of these simplifications, one can consider a reverse construction. On can start from these anticommutation relations, solve them and determine the transformation laws under s^α and δ^α for the fields $A \equiv A_\mu dx^\mu,\ \Psi^\alpha \equiv \Psi^\alpha_\mu dx^\mu,\ \chi^I_\alpha,\ \eta_\alpha,\ \Phi_i,\ h^I,\ T \equiv T_\mu dx^\mu,\ H^I$.

For this, one uses power counting and grading conservation, as well as the covariance under the $SL(2,R) \times SO'(4)$ symmetry. By denoting $g(k) \equiv g_{\mu\nu} k^\mu dx^\nu$ and $g(J^I k) \equiv J^I_{\mu\nu} k^\mu dx^\nu$, the solution of the algebra acting on the fields of the balanced (9,16,7) 4-dimensional multiplet is

$$
\begin{aligned}
s^\alpha A &= \Psi^\alpha & & \\
s^\alpha \Psi_\beta &= \delta^\alpha_\beta T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i & s^\alpha h^I &= \chi^{\alpha\, I} \\
s^\alpha \Phi_i &= \frac{1}{2} \sigma_i{}^{\alpha\beta} \eta_\beta & s^\alpha \chi^I_\beta &= \delta^\alpha_\beta H^I + \sigma^i{}_\beta{}^\alpha [\Phi_i, h^I] \\
s^\alpha \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha [\Phi_i, \Phi_j] & s^\alpha H^I &= \frac{1}{2}[\eta^\alpha, h^I] + \sigma^{i\,\alpha\beta}[\Phi_i, \chi^I_\beta] \\
s^\alpha T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i, \Psi_\beta] & &
\end{aligned}
$$

$$
\begin{aligned}
\delta^\alpha A &= g(\kappa)\eta^\alpha + g(J_I\kappa)\chi^{\alpha\, I} \\
\delta^\alpha \Psi_\beta &= \delta^\alpha_\beta \big(i_\kappa F - g(J_I\kappa) H^I\big) + \sigma^i{}_\beta{}^\alpha g(J_I\kappa)[\Phi_i, h^I] - 2\sigma^{ij}{}_\beta{}^\alpha g(\kappa)[\Phi_i, \Phi_j] \\
\delta^\alpha \Phi_i &= -\frac{1}{2}\sigma_i{}^{\alpha\beta} i_\kappa \Psi_\beta \\
\delta^\alpha \eta_\beta &= -\delta^\alpha_\beta i_\kappa T + \sigma^i{}_\beta{}^\alpha i_\kappa d_A \Phi_i \\
\delta^\alpha T &= \frac{1}{2} d_A i_\kappa \Psi^\alpha - g(J_I\kappa)\big([\eta^\alpha, h^I] + \sigma^{i\,\alpha\beta}[\Phi_i, \chi^I_\beta]\big) + g(\kappa)\sigma^{i\,\alpha\beta}[\Phi_i, \eta_\beta] - \mathscr{L}_\kappa \Psi^\alpha \\
\delta^\alpha h^I &= -i_{J^I\kappa}\Psi^\alpha \\
\delta^\alpha \chi^I_\beta &= \delta^\alpha_\beta\big(i_\kappa d_A h^I + i_{J^I\kappa} T\big) + \sigma^i{}_\beta{}^\alpha i_{J^I\kappa} d_A \Phi_i \\
\delta^\alpha H^I &= \frac{1}{2}[i_\kappa \Psi^\alpha, h^I] + i_{J^I\kappa} d_A \eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i, i_{J^I\kappa}\Psi_\beta] - i_\kappa d_A \chi^{\alpha\, I}
\end{aligned} \tag{13}
$$

One can then show that the N=4 action (1) is uniquely determined in twisted form by the $s, \bar{s}$, δ (or $\bar{\delta}$) invariances, that is from a symmetry with 6 parameters $(k_0, \bar{k}_0, k^\mu)$. Moreover, it can be written as a $s\delta$-exact term [3],

$$
I_{N=4} = \int \frac{1}{|k|} s\delta \left[g(k)(AdA + \frac{2}{3}A^3) + g(J^I k)^* \epsilon_{IJK} h^J dh^K + \bar{s}\bar{\delta}(\frac{1}{2}h^I h_I - \frac{2}{3}\Phi_i \Phi^i) \right] \tag{14}
$$

This action is independent on choice of the constant vector κ. An even more symmetrical expression of the action is

$$
S_{N=4} = -\frac{1}{2}\int_M \mathrm{Tr}\ F_\wedge F + s^\alpha \delta_\alpha \mathscr{G} \tag{15}
$$

where

$$
\mathscr{G} = \int_M \mathrm{Tr}\ \big(-\frac{1}{2} g(\kappa)_\wedge \big(AF - \frac{1}{3}A^3\big) - \frac{1}{2} \star \varepsilon_{IJK} h^I i_{J_J\kappa} d_A h^K + \star s^\alpha \delta_\alpha\ \big(\frac{1}{2}\, h_I h^I\ -\ \frac{2}{3}\, \Phi^i \Phi_i\big)\big) \tag{16}
$$

3 Third twist and supersymmetric observables

The passage by twist from the superPoincaré representation to the first twisted representation is a linear mapping between fields, using Pauli matrices, giving equations that are invariant under a subgroup $SO'(4) \times SL(2,R) \subset SO(4) \times SO(5,1)$ [2]. The third twisted representation can be in fact obtained from the first one by the following invertible κ-dependent field redefinitions

$$V_\mu \equiv \kappa^\nu (h_{\mu\nu^-} + g_{\mu\nu} L) \quad \tilde{\Psi}_\mu \equiv \kappa^\nu (\bar{\chi}_{\mu\nu^-} + g_{\mu\nu}\bar{\eta}) \quad \bar{\Psi}_\mu \equiv \kappa^\nu (\chi_{\mu\nu^+} + g_{\mu\nu}\tilde{\eta}) \tag{17}$$

The vector parameter κ, which is necessary for doing all changes of variables, disappears, modulo a boundary term, when one changes variables from the first twisted Lagrangian to the third twisted one. The third twisted variables are the most most appropriate to show the existence of supersymmetric variables, such as the supersymmetric Wilson-loop, which have interesting finiteness properties [8][10][9]. The third twist formulation has analogy with a complexified expression of the twisted N=2,d=4 TQFT, with $A \to A + iV$, as sketched below. It is useful to give details on this in view of the forthcoming analysis of the conformal supersymmetry. The set of fields in the third twist is

$$(A_\mu,\ \Phi_i,\ h^I\ \ \Psi^\alpha_\mu,\ \eta^\alpha,\ \chi^{I\alpha},\ T_\mu,\ G^I) \to (A_\mu,\ V_\mu,\ \Phi,\ \bar{\Phi}\quad \Psi_\mu,\ \tilde{\Psi}_\mu,\ \chi_{\mu\nu^\pm},\ \eta,\ \tilde{\eta}\quad, H_{\mu\nu^\pm},\ H) \tag{18}$$

It has only an internal $U(1) \times SO(4)$ covariance. One has a Q-symmetry that can be recognized as a combination of two of the symmetries s and δ, governed by two parameters u and v. It can be shown to satisfy the complex equation [13]

$$(Q + d)(A + iV + c) + (A + iV + c) = F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)\Phi \tag{19}$$

By defining

$$Q\chi_\pm = H_\pm - [c, \chi_\pm] \tag{20}$$

one finds that the $N = 4$ action can be recomputed as a Q-exact term

$$I = \int d^4x \frac{1}{u^2+v^2} QRe[\ \chi_- + i\chi_-)(u + iv)(F_{A+iV} + H_- + iH_+) + \dots\ ] \tag{21}$$

The action is independent on u and v. On can restrict to a particular Q-symmetry, by setting $u = iv$ [10]. For this restriction of the parameters, the Q-transformations for A and V are given by

$$Q(A + iV) = -D_{A+iV}\ c = \delta_{gauge}(c)(A + iV) \tag{22}$$

Therefore the Wilson loop with argument $A + iV$ is automatically Q-invariant

$$Qexp \int dx^\mu (A_\mu + iV_\mu) = 0 \tag{23}$$

as well as any given gauge-invariant functional of $A+iV$. This defines Q-supersymmetric observables for $u=iv$, which have been extensively studied in [10]. These supersymmetric Wilson loops can be expressed in the first twist formulation, since the latter is related to the third twist by a mere κ-dependent change of variables, which leaves invariant the action. It follows that the mean value

$$< exp \int_\Gamma dx^\mu (A_\mu + i\kappa^\nu (J^J_{\mu\nu} h^I + g_{\mu\nu} L)) > \tag{24}$$

is independent on all possible local deformations, in particular on those of the contour Γ.

4 Stochastic quantization and relation to Chern Simons action

This section is devoted to a possible interpretation of the scalar generators of the N=4 theory as the scalar supersymmetry of the stochastic quantization of a three-dimensional theory. The expression of the $s\delta$-antecedent of the N=4 action (15) strongly suggests the influence of a three-dimensional Chern simons action for the four dimensional theory. In fact, the general ideas of stochastic quantization [11] formally indicate that, if the contour is three-dimensional, one can either use the three-dimensional action

$$\int d^3x (AdA + 2/3A^3 +^* \epsilon_{IJK} h^I D^J h^K +^* LD_I h^I) \tag{25}$$

or the complex one

$$\int d^3x ((A+iV)d(A+iV) + 2/3(A+iV)^3) \tag{26}$$

to compute certain 3-dimensional observables for the N=4 theory. The action (25) has only real gauge invariance while the action (26) has complex gauge invariance [12], that is a double gauge symmetry. In fact, the former action differs from the later one by a covariant gauge-fixing of the vector V_μ. We will formally show that stochastic quantization of both actions leads one to supersymmetric theories, whose actions are identical either to the first or to the third twisted N=4 actions, modulo Q-exact terms. The latter terms are irrelevant for the computation of Q-invariant observables, such as the above mentioned Wilson loops

In flat or curved Euclidian three-dimensional space one can define quantization by introducing a fourth (stochastic) time. The time evolution is then governed by a Langevin equation. Its drift force is the sum of a force along gauge-invariant directions, which is the equation of motion of the gauge-invariant action, and of a force along gauge orbits, which is equal to a gauge transformation where the parameter (real or complex) can be any given arbitrary function, possibly field dependent. The later parameter can be promoted to an independent field over which one can functionally integrate. The reasoning is that the expectation values of gauge-invariant three-dimensional observables donnot depend on the choice of the parameters of the drift forces along gauge orbits, so that one can consider a summation over these fields, since it yields no modification of the value of gauge-invariant observables. This is how gauge covariance can be enforced in the fourth dimension. The additional fields become the fourth component A_0 of the gauge field for the action (25), or the fourth components A_0 and V_0 of both fields A_i and V_i for the action (26). The actions (25) and (26) are thus expected to generate the N=4 action in their first and third twist formulations, with a segregation of the fourth component x^0 as a somehow irrelevant variable. However one must consider observables at equal time x^0, and takes the limit $x^0 \to \infty$. Let us see how this can happen.

Taking the action (26), the covariant Langevin equations that govern its quantization are

$$F_{0,i} - \epsilon_{ijk}F^{jk} - [V_0, V_i] + \epsilon_{ijk}[V_j, V_K] = b_i \qquad D_{[0}V_{i]} = \bar{b}_i \tag{27}$$

One can express the Langevin process as path integral with a Gaussian dependance in the noises b and $\bar{b}$, doing a change of variable between $b, \bar{b}$ and A and V. This necessitates the insertion of Jacobians. The latter can be expressed as a path integral over fermions, which will be interpreted as the fermions of the twisted theory. Since one has zero modes in the gauge covariant Langevin equations, their gauge-fixing introduces fermionic auxiliary fields, with the occurrence a super-Jacobian, which yields a functional representation with the commuting scalar fields Φ and $\bar{\Phi}$. It goes as follows. One uses 4-dimensional notations, so that the Langevin equations can be rewritten as

$$F_{\mu\nu^+} - V_{[\mu}V_{\nu]^+} = b_{\mu\nu^+} \qquad D_{[\mu}V_{\nu]^+} = \bar{b}_{\mu\nu^+} \tag{28}$$

One defines a new covariant equation for the stochastic evolution of V_0 as

$$D^\mu V_\mu = \bar{b} \tag{29}$$

By doing standard steps of inserting delta functions and determinants in a Gaussian path integral representation as in [11], one ends up with the following action for describing the Langevin process

$$\begin{aligned} I_{GF} = \int dtdxTrs_{top}(\chi_{\mu\nu^+}(F_{\mu\nu^+} - V_{[\mu}V_{\nu]^+} + D_{[\mu}V_{\nu]^+} - \frac{1}{2}\bar{b}_{\mu\nu^+}) \\ +\bar{\chi}_{\mu\nu^+}(F_{\mu\nu^+} - V_{[\mu}V_{\nu]^+} - D_{[\mu}V_{\nu]^+} - \frac{1}{2}b_{\mu\nu^+}) + \chi(D_\mu V_\mu - \frac{1}{2}\bar{b}) + \bar{\Phi}D_\mu\Psi_\mu + \bar{c}(\partial_\mu A_\mu - \frac{1}{2}b)) \end{aligned} \tag{30}$$

where

$$\begin{aligned} s_{top}A_\mu &= \Psi_\mu + D_\mu c \qquad & s_{top}c &= \Phi - cc \\ s_{top}\Psi_\mu &= D_\mu\Phi - [c, \Psi_\mu] \qquad & s_{top}\Phi &= -[c, \Phi] \end{aligned}$$

$$
\begin{aligned}
s_{top}\chi_{\mu\nu^{\pm}} &= b_{\mu\nu^{+}} & s_{top}b_{\mu\nu^{\pm}} &= 0 \\
s_{top}\bar{\Phi} &= \eta & s_{top}\eta &= 0 \\
s_{top}\bar{c} &= b & s_{top}b &= 0
\end{aligned}
\tag{31}
$$

One can identify Q and s_{top} and the action (30) is identical to the N=4 theory in the third twist, modulo Q-exact terms. The latter terms contain the quartic scalar field interactions. Their omission does not change the expectation values for Q-invariant observables. One obtains a similar result for the first twist, starting from the three-dimensional action (25) for the fields A_i, L, V_i, which yields the action of the N=4 theory in the first twist, modulo Q-exact terms.

There is no obstruction to do this formal construction in curved three-dimensional space. Moreover, one can introduce non-flat metrics components g_{oi}, which could maybe ease certain practical computations. We leave open the problem of directly computing Q-invariant Wilson loops of the four-dimensional theory with three-dimensional contours, directly in the Chern–Simons three-dimensional theory.

We see that the N=4 theory relies on building blocks that are much more elementary than expected. In what follows, we will see that one can extend these idea, and incorporate elements of special supersymmetry from the beginning. It is indeed interesting to introduce the superconformal algebra in a constructive way, with no redundancy.

5 Inclusion of part of the conformal symmetry in the Q symmetry

The conformal Yang–Mills supersymmetry is governed by 32 generators, with spinor parameters ϵ and η,

$$
\begin{aligned}
\delta A_\mu &= \lambda\Gamma_\mu(\epsilon + x^\mu\gamma_\mu\eta) \\
\delta\vec{\varphi} &= (\epsilon + x^\mu\gamma_\mu\eta)\vec{\tau}\lambda \\
\delta\lambda &= (\Gamma^{\mu\nu}F_{\mu\nu} + iD_\mu\Gamma^\mu\varphi - [\varphi,\varphi])(\epsilon + x^\mu\gamma_\mu\eta) + 2i\varphi\eta
\end{aligned}
$$

In [7], this superconformal symmetry has been twisted by reducing the product of its R-symmetry and conformal symmetry $SO(5,1)\times SO(5,1)$, in a way that generalises the mixing between the Lorentz symmetry and the R-symmetry for the superPoincaré case.

We will follow a different root. We generalize the algebra (11) for the four scalar generators, by replacing the constant vector k^μ into a local one that is proportional to the coordinate x^μ. We retain the same field representations as in the first twist. Some compensating transformations must be done for absorbing non-homogeneous terms in x^μ, using the existing global symmetries. After some thoughts, one concludes that one must consider the following distorted algebra

$$
\{s^\alpha, s^\beta\} = 2\sigma^{i\,\alpha\beta}\delta_{\text{gauge}}(\Phi_i) \qquad \{\delta^\alpha_x, \delta^\beta_x\} = 2\sigma^{i\,\alpha\beta}\delta_{\text{gauge}}(|x|^2\Phi_i) \tag{32}
$$

$$\{s^\alpha, \delta_x^\beta\} = \varepsilon^{\alpha\beta}\big(\mathcal{L}_x + \delta_{\rm gauge}(i_x A) + w\big) + \Delta^{\alpha\beta}_{SL(2,R)} \tag{33}$$

$\Delta_{SL(2,R)}$ is a $SL(2,R)$ transformation and w is a $U(1)$ symmetry, which counts the conformal weight of fields. The requirement of $SO(4)\times SL(2,R)$ covariance and the respect of the various gradings determine the structure of this algebra. One must look for possible representations, in terms of fields, and one is led to check whether some constants A, Z, W, W' exist, such that the following transformation laws fulfill the above anti-commutation relations.

$$\begin{aligned}
s^\alpha A &= \Psi^\alpha \\
s^\alpha \Psi_\beta &= \delta^\alpha_\beta T - \sigma^i{}_\beta{}^\alpha d_A \Phi_i & \qquad s^\alpha h^I &= \chi^{\alpha\, I} \\
s^\alpha \Phi_i &= \frac{1}{2}\sigma_i{}^{\alpha\beta}\eta_\beta & s^\alpha \chi^I_\beta &= \delta^\alpha_\beta H^I + \sigma^i{}_\beta{}^\alpha[\Phi_i, h^I] \\
s^\alpha \eta_\beta &= -2\sigma^{ij}{}_\beta{}^\alpha[\Phi_i,\Phi_j] & s^\alpha H^I &= \frac{1}{2}[\eta^\alpha, h^I] + \sigma^{i\,\alpha\beta}[\Phi_i, \chi^I_\beta] \\
s^\alpha T &= \frac{1}{2} d_A \eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i, \Psi_\beta]
\end{aligned} \tag{34}$$

$$\begin{aligned}
\delta^\alpha_x A &= g(x)\eta^\alpha + g(J_I x)\chi^{\alpha\, I} \\
\delta^\alpha_x \Psi_\beta &= \delta^\alpha_\beta\big(i_x F - g(J_I x) H^I\big) + \sigma^i{}_\beta{}^\alpha g(J_I x)[\Phi_i, h^I] - 2\sigma^{ij}{}_\beta{}^\alpha g(x)[\Phi_i,\Phi_j] \\
\delta^\alpha_x \Phi_i &= -\frac{1}{2}\sigma_i{}^{\alpha\beta} i_x \Psi_\beta \\
\delta^\alpha_x \eta_\beta &= -\delta^\alpha_\beta i_x T + \sigma^i{}_\beta{}^\alpha i_x d_A \Phi_i + A\sigma_i{}^{\alpha\beta}\Phi^i \\
\delta^\alpha_x T &= \frac{1}{2} d_A i_x \Psi^\alpha - g(J_I x)\big([\eta^\alpha, h^I] + \sigma^{i\,\alpha\beta}[\Phi_i, \chi^I_\beta]\big) + g(x)\sigma^{i\,\alpha\beta}[\Phi_i, \eta_\beta] - \mathscr{L}_x \Psi^\alpha + Z\delta^\alpha_\beta \Psi^\beta \\
\delta^\alpha_x h^I &= -i_{J^I x}\Psi^\alpha \\
\delta^\alpha_x \chi^I_\beta &= \delta^\alpha_\beta\big(i_x d_A h^I + W h^I + i_{J^I x} T\big) + \sigma^i{}_\beta{}^\alpha i_{J^I x} d_A \Phi_i \\
\delta^\alpha_x H^I &= \frac{1}{2}[i_x \Psi^\alpha, h^I] + i_{J^I x} d_A \eta^\alpha + \sigma^{i\,\alpha\beta}[\Phi_i, i_{J^I x}\Psi_\beta] - i_x d_A \chi^{\alpha\, I} + W' \chi^{I\alpha}
\end{aligned} \tag{35}$$

One gets after a lengthy computation that there is indeed a unique solution, given by $Z = -1, A = 2, W = 1, W' = 2$.

One thus obtains the intriguing result that, "special", i.e., x-dependent δ_x transformations, exist that are very simply related to the twisted vector super Poincaré supersymmetry transformation, as follows

$$\delta_x = x^\mu(\delta_\mu + g_{\mu\nu}\frac{x^\nu}{x^2} C) \tag{36}$$

Here the operator δ_μ is identical to that of the vector supersymmetry of the superPoincaré algebra, and the operator C is the further modification brought by the special supersymmetry, as it is implied by the graded commutation relations. The action of C is only non-zero for η^α, $\chi^{I\alpha}$, T and H^I and can be read from Eqs. (35).

One can verify that the above 4 symmetries can be identified as combinations of the twisted ones that are obtained by computing the first-twist of the 32 generators of the superconformal transformation [7]. Here, they have arised in a somehow very elementary geometrical construction, and they capture an interesting part of the maximal conformal supersymmetry with its 32 generators. Indeed, one can verify that the N=4 action, in first twisted form is completely determined by its invariance under both graded Poincaré operators $s, \bar{s}$ and both special supersymmetry operators $\delta_x, \bar{\delta}_x$.

Does it help to discuss special supersymmetric observables as in [9], and determine some of them? One can generalize the trick that we used by combining the scalar and vector symmetries for the ordinary supersymmetric observables, and define

$$Q = us + v\delta_x \tag{37}$$

where the scalar parameters u, v are commuting ones. One then finds that Q satisfies an equation as in Eq. (19), by a simple comparison between the δ- and δ_x- transformations.

It follows that the following 1-form :

$$A + \frac{1}{x^2}\big(i_{J^J x}h^I + L\big) = dx^\mu\big(A_\mu + \frac{x^\nu}{x^2}\big(J^J_{\mu\nu}h^I + g_{\mu\nu}L\big)\big) \tag{38}$$

transforms under $s + \bar{\delta}$ simply by a gauge transformation when $u^\alpha = -iv^\alpha$, provided that

$$x^2 = 1 \tag{39}$$

The verification is as in section 3, except that the constant κ^μ has been replaced by x^μ. Therefore the following special Wilson loop is $s + \bar{\delta}$ invariant :

$$\big(s + \bar{\delta}\big)\exp i\int_{\Gamma_{x^2=1}} \big(A + \frac{1}{x^2}\big(i_{J^J x}h^I + L\big)\big) = 0 \tag{40}$$

Notice that because $x^2 = 1$, the term $dx^\mu g_{\mu\nu}x^\nu L$ disappears on the contour. The Wilson loop invariance equation reduces therefore to the following one

$$\big(s + \bar{\delta}\big)\exp i\int_{\Gamma_{x^2=1}} dx^\mu\big(A_\mu + \frac{1}{x^2}h_{\mu\nu^-}x^\nu\big) = 0 \tag{41}$$

So, the special supersymmetric Wilson loop must be defined on a circle, and only depends on 3 of the scalar fields. Its origin is analogous to that of the ordinary Wilson loop $\exp i\int_\Gamma\big(A + iV\big)$ in the third twist. The reason is because the special supersymmetry has a lot in common with the twisted vector symmetry and because the manipulations of using the third twist mapping are similar. One may question about the finiteness of observables, the topological properties, etc.. of such special observables that are invariant under $s + \bar{\delta}$ and computed by mean of the $N = 4$ action, which we have shown is s and $\bar{\delta}$ invariant, and thus $s + \bar{\delta}$-invariant. However, the use of Ward identities is complicated by the lack of translational invariance, due to the dependance on x of the transformations, so the topological properties are presumably lost, and the observables probably depend on the detail of the metrics. Interestingly, specific examples have shown that such observables are not automatically topological [9]. Perhaps, computing these special Wilson loops within a 3-dimensional framework will be useful, as suggested by the results of stochastic quantization.

Acknowledgments

I thank G. Bossard and K. Zarembo for interesting and constructive discussions on the subject.

References

[1] C. Vafa and E. Witten, A strong coupling test of S-duality, Nucl. Phys. B 431 (1994) 3, [arXiv:hep-th/9408074].

[2] N. Marcus, The other topological twisting of $\mathcal{N} = 4$ Yang–Mills, Nucl. Phys. B 452 (1995) 331, [arXiv:hep-th/9506002]; M. Blau and G. Thompson, Aspects of $\mathcal{N}_T \geqslant 2$ topological gauge theories and D-branes, Nucl. Phys. B 492 (1997) 545, [arXiv:hep-th/9612143].

[3] L. Baulieu, G. Bossard, New results on N=4 superYang-Mills theory, Phys. Lett. B632 (2006) 131-137, [arXiv:hep-th/0507003]; Supersymmetric renormalization prescription in N = 4 superYang–Mills theory, Phys. Lett. B643 (2006) 294-302, [arXiv:hep-th/0507003].

[4] L. Baulieu, G. Bossard and S. P. Sorella, Shadow fields and local supersymmetric gauges, Nucl. Phys. B 753, 273 (2006), [arXiv:hep-th/0603248]; Finiteness properties of the N=4 superYang–Mills theory in supersymmetric Gauge, Nucl. Phys. B753 (2006) 252-272, hep-th/0605164.

[5] L. Baulieu, H. Kanno and I. M. Singer, Special quantum field theories in eight and other dimensions, Commun. Math. Phys. 194 (1998), 149, hep-th/9704167.

[6] L. Baulieu, G. Bossard and A. Tanzini, Topological vector symmetry of BRSTQFT and construction of maximal supersymmetry, [arXiv:hep-th/0504224].

[7] P. Medeiros, J. Figueroa-O'Farrill, C. Hull, B. Spence, Conformal topological Yang-Mills theory and de Sitter holography", JHEP 0208 (2002) 055, [arXiv:hep-th//0111190].

[8] K. Zarembo, Supersymmetric Wilson loops, Nucl. Phys. B643 (2002) 157-171, [arXiv:hep-th//0205160].

[9] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, [arXiv:hep-th/0712.2824]; A. Dymarsky, V. Pestun, Supersymmetric Wilson loops in N=4 SYM and pure spinors, JHEP 1004:115, 2010, [arXiv:hep-th//0911.1841]; S. Giombi, V. Pestun, The 1/2 BPS 't Hooft loops in N=4 SYM as instantons in 2d Yang-Mills, [arXiv:hep-th/0909.4272].

[10] A. Kapustin, E. Witten, Electric-magnetic duality and the geometric Langlands program, [arXiv:hep-th/0604151].

[11] L. Baulieu, B. Grossman, A topological interpretation of stochastic Quantization, Phys. Lett. 212B, 351, 1988; L. Baulieu, Stochastic and topological gauge theories, Phys. Lett. B232, 479, 1989.

[12] E. Witten, Quantization of Chern-Simons gauge theory with complex gauge group, Commun. Math. Phys. 137:29-66,1991.

[13] L. Baulieu, N=4 Yang–Mills theory as a complexification of the N=2 theory, Nucl.Phys. Proc. Suppl.192-193:27-39, 2009, [arXiv:hep-th/0906.1289].