The Minimal Polynomial of Some Matrices Via Quaternions

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Abstract

This work provides explicit characterizations and formulae for the minimal polynomials of a wide variety of structured 4×4 matrices. These include symmetric, Hamiltonian and orthogonal matrices. Applications such as the complete determination of the Jordan structure of skew-Hamiltonian matrices and the computation of the Cayley transform are given. Some new classes of matrices are uncovered, whose behaviour insofar as minimal polynomials are concerned, is remarkably similar to those of skew-Hamiltonian and Hamiltonian matrices. The main technique is the invocation of the associative algebra isomorphism between the tensor product of the quaternions with themselves and the algebra of real 4×4 matrices.

1 Introduction

The minimal polynomial of a matrix is the unique monic polynomial of minimal degree which annihilates the matrix. It has several theoretical and practical uses. It provides information about the Jordan structure of the matrix, and in some situations can nearly determine it. Its principal utility, arguably, is in computing functions of a matrix such as the matrix exponential and the Cayley transform. While any annihilating polynomial can be used for this purpose, the complexity of the resultant expression is naturally minimal when the minimal polynomial is used.

If one knows the Jordan structure of the matrix then its minimal polynomial is easily computed. However, since the former is difficult to arrive at, this is rarely advisable. Essentially any mechanism which *explicitly* detects linear dependence at the earliest stage in the matrices $I, A, A^2, \ldots, A^n$ (in that order) will yield the minimal polynomial, [7, 8]. In this work we use quaternions to achieve the same for specific structured 4×4 matrices. Whilst, 4×4 matrices are amenable to the techniques of [7, 8], the corresponding calculations can be quite difficult, and would not produce the closed form expressions for minimal polynomials presented herein. More discussion on this issue is presented in Section 5. In the method proposed here one replaces matrix calculations (specifically computing A^k) via quaternion calculations. Not only does this simplify such calculations, but it also yields elegant geometric interpretations of situations wherein the minimal polynomial is a particular polynomial. Of course, this methodology does not extend to higher dimensions immediately (see, however, the discussion in Section 5), but 4×4 matrices already cover several important applications to warrant the investigation of such a technique. In quantum computation, quantum optics, computer graphics, robotics etc., much of the analysis is reducible to the study of 4×4 matrices [see, for instance, [3, 4, 16]].

The isomorphism of $H \otimes H$ with $M(4, R)$ is a central point in the theory of Clifford algebras. So a natural question is whether Clifford algebra isomorphisms can be used for similar purposes. In fact, the interesting work of [1] uses the symbolic and numerical computation the (real) minimal polynomial of matrices via their Clifford algebra representatives, for exponentiation of matrices. The difference between the work of [1] and the results here, insofar as the problem of computation of minimal polynomials of matrices in $M(4, R)$ is concerned, is that the structural (i.e., "geometrical") conditions given here on the entries of a matrix for it to possess a given minimal polynomial are missing in [1]. There are of course other differences. Section 5 discusses this issue briefly.

It is appropriate at this point to record some history of the linear algebraic applications of the isomorphism between $H \otimes H$ and $M_4(R)$. This isomorphism is central to the theory of Clifford algebras, [10]. However, it is only relatively recently been put into use for linear algebraic purposes. To the best of our knowledge, the first instance seems to be the work of [9], where it was used in the study of linear maps preserving the Ky-Fan norm. Then in [6], this connection was used to obtain the Schur canonical form explicitly for real 4×4 skew-symmetric matrices. Next, is the work of [5, 11, 12], wherein this connection was put to innovative use for solving eigenproblems of several classes of structured 4×4 matrices. In [14, 15], this isomorphism was used to explicitly calculate the exponentials of a wide variety of 4×4 matrices. Finally, in [2] it was used to obtain, among other things, the polar decomposition of 4×4 symplectic matrices via the solution of 2×2 linear systems of equations.

The balance of this paper are organized as follows. In the next section basic notation and preliminary facts are reviewed. The next section contains all the main results on minimal polynomials obtained by our method. Since many of the proofs are similar we provide proofs for only a part of the announced results. The fourth section contains three applications. The first is to the complete determination of the Jordan structure of 4×4 skew-Hamiltonian matrices. The second illustrates the usage of the results on minimal polynomials to calculate the Cayley transform in closed form. The final application is to the determination of the singular values of 3×3 real matrices. The next section discusses extension of the results of section 3 via the use of Clifford Algebras. In particular, classes of matrices are uncovered which behave very similar to skew-Hamiltonian matrices insofar as minimal polynomial matrices are concerned. Their block structures do not suggest this similarity. This section also provides a brief comparison of our technique with that of [1, 8]. The final section offers some conclusions.

2 Notation and Preliminary Observations

The classes of real matrices discussed in this work are as follows:

- Skew-symmetric matrices, i.e., X satisfying $X^T = -X$.

- Hamiltonian matrices, i.e., matrices H satisfying $H^T J_{2n} = -J_{2n} H$, where $J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$.

- Perskewsymmetric matrices, i.e, matrices X satsifying $X^T R_n = -R_n X$, where R_n is the $n \times n$ matrix containing 1s on its main anti-diagonal and 0s elsewhere. R_n is sometimes denoted F and is called the flip matrix.

- For a matrix $X \in M(n, R)$, we denote by X_F, its adjoint with respect to the non-degenerate bilinear form defined by R_n, i.e., it is the matrix $R_n X^T R_n$.

- For a matrix $X \in M(2n, R)$, we denote by X_H, its adjoint with respect to the non-degenerate bilinear form defined by J_n, i.e., it is the matrix $-J_{2n} X^T J_{2n}$.

- Symmetric matrices, i.e., X with $X^T = X$.

- Skew-Hamiltonian matrices, i.e, matrices X satisfying $X^T J_{2n} = J_{2n} X$.

- Special orthogonal matrices, i.e., X with $X^T X = X X^T = I_n$ and $det(X) = 1$.

These classes were picked because i) they are ubiquitous in applications; and ii) in most cases, as will be seen subsequently, elegant geometric conditions on their quaternionic representations can be given which ensure their possessing a certain minimal polynomial. Matrices such as persymmetric and symplectic matrices do not seem that amenable by the latter consideration, and therefore are not considered here. We note, however, that in the final section we discuss how quaternion techniques can be used to *compute the minimal polynomial of a general matrix in* $M(4, R)$.

Definition 2.1 H stands for the real division algebra of the quaternions. $\mathcal{P}$ stands for the purely imaginary quaternions.

We will tacitly identify an element of $\mathcal{P}$ with the corresponding vector in R^3. With this understood, the following two identities will be frequently used.

- Let $p, q \in \mathcal{P}$. Then $pq = -(p.q)1 + p \times q$.

- Let $p, q, r \in R^3$. Then

$$p \times (q \times r) = (p.r)q - (p.q)r \tag{2.1}$$

$H \otimes H$ **and** $gl(4, R)$: The algebra isomorphism between $H \otimes H$ and $M_4(R)$ (also denoted by $gl(4, R)$), which is central to this work, may be summarized as follows:

- Associate to each product tensor $p \otimes q \in H \otimes H$, the matrix, $M_{p\otimes q}$, of the map which sends $x \in H$ to $px\bar{q}$, identifying R^4 with H via the basis $\{1, i, j, k\}$. Here, $\bar{q} = q_0 - q_1 i - q_2 j - q_3 k$

- Extend this to the full tensor product by linearity. This yields an associative algebra isomorphism between $H\otimes H$ and $M_4(R)$. Furthermore, a basis for $gl(4, R)$ is provided by the sixteen matrices $M_{e_x\otimes e_y}$ as e_x, e_y run through $1, i, j, k$. In particular, R_4, the matrix intervening in the definition of perskewsymmetric matrices, and J_4, the matrix used in the definition of Hamiltonian and skew-Hamiltonian matrices, represented respectively, by $M_{j\otimes i}$ and $M_{1\otimes j}$, belong to this basis.

Quaternion Representations of Special Classes of Matrices: Throughout this work, the following list of $H \otimes H$ representations of the above classes of matrices will be used:

- *Skew-Symmetric Matrices:* $s \otimes 1 + 1 \otimes t$ with $s, t \in \mathcal{P}$.

- *Hamiltonian Matrices:* $b(1 \otimes j) + p \otimes 1 + q \otimes i + r \otimes k$, with $b \in \mathcal{R}$ and $p, q, r \in \mathcal{P}$.

- *Perskewsymmetric Matrices:* $r \otimes i + j \otimes s + \alpha(1 \otimes i) + \beta(j \otimes 1)$, with $r, s \in \mathcal{P}$ and $\alpha, \beta \in \mathcal{R}$.

- *Symmetric Matrices:* $a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k$, with $a \in \mathcal{R}$ and $p, q, r \in \mathcal{P}$.

- *Skew-Hamiltonian Matrices:* $b(1 \otimes 1) + p \otimes j + 1 \otimes (ci + dk)$, with $b, c, d \in \mathcal{R}$ and $p \in \mathcal{P}$.

- *Special Orthogonal Matrices:* $u \otimes v$, with u, v unit quaternions, i.e., $\| u \| = \| v \| = 1$.

These can be easily obtained from the entries of the 4×4 matrix in question (see [11, 5, 12] for some instances). The key to this consists of the following two observations

- Conjugation in $H \otimes H$ corresponds to matrix transposition in $gl(4, R)$, i.e., $M_{\bar{p}\otimes\bar{q}} = (M_{p\otimes q})^T$. This is why, for instance, symmetric matrices correspond to $c(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$ with p, q, r purely imaginary, and skew-symmetric matrices correspond to $s \otimes 1 + 1 \otimes t$, $s, t \in P$.

- Hamiltonian (resp. skew-Hamiltonian) matrices are expressible as $J_{2n}S$, with S symmetric (resp. skew-symmetric). Similarly persymmetric (resp. perskew-symmetric) matrices are expressible as R_nS with S symmetric (resp. skew-symmetric). Thus, for instance, perskewsymmetric matrices are represented by $(j\otimes i)[s\otimes 1+1\otimes t]$, $s, t \in \mathcal{P}$. This simplifies to $p\otimes i+\alpha(j\otimes 1)+j\otimes q+\beta(1\otimes i)$ with $p \in span\ \{i, k\}, q \in span\ \{j, k\}, \alpha, \beta \in R$. If such a matrix is simultaneously symmetric, then $\alpha = \beta = 0$, etc.,

Combining these two observations with the explicit forms of the sixteen matrices, $M_{e_x\otimes e_y}$ leads to $H \otimes H$ representations, in terms of the entries of the matrices. For the first five classes, the expressions for the $H \otimes H$ representations are linear in the entries of the matrix. See [11, 5, 12] for these expressions. For special orthogonal matrices, the entries of the matrix are quadratic in u and v. See [4] for an algorithmic determination of the unit quaternions u and v from the entries of a special orthogonal matrix.

By way of illustration, the requisite expression for the quaternionic representation for a skew-Hamiltonian matrix is provided below.

Proposition 2.1 *[5]* Let X be a skew-Hamiltonian matrix. Its $H \otimes H$ representation is $b(1 \otimes 1) + p \otimes j + 1 \otimes (ci + dk)$, with $b, c, d \in R$ and p a purely imaginary quaternion. The formulae relating these to X's entries are as follows: i) $b = \frac{1}{2}(X_{11} + X_{22})$; ii) $p = \frac{1}{2}[(X_{32} - X_{14}), (X_{11} - X_{22}), (X_{12} + X_{21})]$; iii) $c = \frac{1}{2}(X_{12} - X_{21})$; iv) $d = \frac{1}{2}(X_{14} + X_{32})$.

We close this section with the notion of the reverse of a polynomial.

Definition 2.2 If $p(x) = \sum_{i=0}^{n} a_i x^i$ is a polynomial of degree n, then its reverse is the polynomial $p_{\mathrm{Rev}}(x) = \sum_{i=0}^{n} a_{n-i} x^i$.

3 Minimal Polynomials of Classes of 4×4 Matrices

We begin with a simple proposition, applicable in arbitrary dimensions, which reduces the list of possible minimal polynomials for some of the matrices to be considered here.

Proposition 3.1
- I) Let A^T be similar to $-A$. If the degree of the minimal polynomial of A is even, then its minimal polynomial is an even polynomial. If the degree of the minimal polyomial is odd, then it is an odd polynomial.
- II) Let A^{-1} be similar to A^T. Then the constant term in its minimal polynomial is either $+1$ or -1. If it is the former, then its minimal polynomial equals its reverse. If it is the latter it is minus its reverse.

Proof: I) Let $q_A(x) = x^k + \sum_{i=0}^{k-1} a_i x^i$ be the minimal polynomial of A (and thus of A^T). Then clearly the polynomial $p(x) = x^k + \sum_{i=0}^{k-1}(-1)^i a_i x^i$ annihilates the matrix $-A$. So, $p(x)$, which is monic, has to be the minimal polynomial of $-A$. Indeed, if $q_{(-A)}(x) = x^l + \sum_{i=0}^{l-1} b_i x^i$, was the minimal polynomial of $-A$ (with $l < k$) then $x^l + \sum_{i=0}^{l-1}(-1)^i b_i x^i$ annihilates A, contradicting the minimality of $q_A(x)$. Thus, since $-A$ and A^T are similar, $q_A(x) = p(x)$, and the result follows.

II) Let $q_A(x) = x^k + \sum_{i=0}^{k-1} a_i x^i$ be the minimal polynomial of A. Since A is invertible $a_0 \neq 0$. Then, we find that $p(x) = \frac{1}{a_0}(x^k + \sum_{i=0}^{k-1} a_{k-i} x^i)$ annihilates A^{-1}. Now, by an argument similar to I), $p(x)$ has to be the minimal polynomial of A^{-1}. The similarity of A^T and A^{-1} then implies the equality of $p(x)$ and $q_A(x)$. This forces a_0 to be either 1 or -1. This then means $p(x)$ equals its reverse when $a_0 = 1$ or it equals minus its reverse when $a_0 = -1$, and the result follows. $\diamondsuit$

Remark 3.1 Note that even if $det(A) = 1$, for matrices as in II) of Proposition (3.1), it is possible that $a_0 = -1$, and thus the minimal polynomial equals minus its reverse. This is in sharp contrast with the characteristic polynomial of such an A, which always equals its reverse.

Proposition (3.1) shortens the list of possible minimal polynomials for skew-symmetric, Hamiltonian, perskewsymmetric and special orthogonal matrices, since in each of these cases A^T is similar to either $-A$ or A^{-1}. When A^T is similar to A, there are no such general results.

Next follow our main results about minimal polynomials. As mentioned in Section 1, we detail only those cases where one has an "elegant" condition on the $H \otimes H$ representations of the matrix in question which is equivalent to the matrix having the said polynomial as its minimal polynomial. This already contains an extensive collection of useful matrices. Furthermore, since the proofs are similar, we present details only for some cases.

Theorem 3.1 *Minimal Polynomials of Antisymmetric Matrices:* Let S be antisymmetric, with representation $s \otimes 1 + 1 \otimes t$. Then,

- S has a quadratic minimal polynomial, which equals $x^2 + \lambda^2$, iff precisely one of s or t is equal to zero. Furthermore, in this case, λ^2 is either $s.s$ or $t.t$.

- S has a cubic minimal polynomial, which equals $p(x) = x^3 + (\lambda^2 + 2l)x$, iff

$$|| \, s \, ||^2 = || \, t \, ||^2 \neq 0$$

  Furthermore, in this case, $\lambda^2 = s.s + t.t$ and $l = || \, s \, ||^2$.

- If none of the above conditions hold, the minimal polynomial is the characteristic polynomial which equals $p(x) = x^4 + 2\lambda^2 x^2 - (4l^2 - \lambda^4)$, with λ and l as above.

Theorem 3.2 *Minimal Polynomials of Hamiltonian Matrices:* Let H be Hamiltonian with representation $b(1 \otimes j) + p \otimes 1 + q \otimes i + r \otimes k$. Then,

- H has a quadratic minimal polynomial, which equals $p(x) = x^2 - \omega$, with $\omega = -b^2 - p.p + q.q + r.r$, iff $p.r = p.q = 0$ and $r \times q = -bp$. Notice, if $b \neq 0$, then the first two conditions are subsumed by the last condition.

- H has a cubic minimal polynomial, which equals $p(x) = x^3 - (\omega + 2k)x$, with ω as in the quadratic minimal polynomial case and k as specified below, iff one of the following five mutually exclusive conditions hold [See Remark (3.2), below, for special cases of these conditions].

  1. $b \neq 0$ and the matrix $G = X^T X$, with $X = [p \mid q \mid r]$, has the the matrix

$$Y = \begin{pmatrix} b^2 + k & -p.q & -p.r \\ -p.q & r.r - k & -q.r \\ -p.r & -q.r & q.q - k \end{pmatrix}$$

with $k = \frac{1}{b}[p.(q \times r) - b(p.p)]$, as its "near inverse". Specifically $GY = b[p.(q \times r)]I_3$, In this case the coefficient k in the given cubic minimal polynomial is $\frac{1}{b}[p.(q \times r) - bp.p]$.

2. $b = 0$, $r \times q \neq 0$, $p = 0$ $r.q = 0$ and $q.q = r.r$. In this case $k = r.r + q.q$

3. $b = 0$, $r \times q \neq 0$, $p \neq 0$, $p.(q \times r) = 0$, $p.q = 0$, $r.q = 0$, $p.r \neq 0$ and $(r.r)^2 + (p.r)^2 = (q.q)(r.r)$. In this case $k = r.r$.

4. $b = 0$, $r \times q \neq 0$, $p \neq 0$, $p.r = 0$, $q.r = 0$, $p.q \neq 0$, and $(q.q)^2 + (p.q)^2 = (q.q)(r.r)$. In this case, $k = q.q$.

5. $b = 0, r \times q \neq 0, p \neq 0$, $p.(q \times r) = 0$, $p.q \neq 0 \neq p.r$ and the following four quantities are all equal to $-\frac{q.p}{q.r}(p.r)$,

$$|| \, q \, ||^2 + \frac{p.r}{p.q}(r.q), || \, r \, ||^2 + \frac{p.q}{p.r}(r.q), || \, r \, ||^2 - \frac{[(q.p)^2 + (q.r)^2]}{|| \, q \, ||^2}, || \, q \, ||^2 - \frac{[(r.p)^2 + (q.r)^2]}{|| \, r \, ||^2}$$

In this case $k = -\frac{q.p}{q.r}(p.r)$.

- If none of the above conditions hold, the minimal polynomial is the characteristic polynomial, which equals

$$p(x) = x^4 - 2\omega x^2 - (4b^2 \, || \, p \, ||^2 + 8bp.(r \times q) + 4 \, || \, r \times q \, ||^2 - \omega^2 - 4(p.q)^2 - 4(p.r)^2)$$

Theorem 3.3 *Minimal Polynomials of Perskewsymmetric Matrices:* Let P be a perskewsymmetric matrix with representation $r \otimes i + j \otimes s + \alpha(1 \otimes i) + \beta(j \otimes 1)$. Then,

- P has a quadratic minimal polynomial iff one of the following three mutually exclusive sets of conditions hold. These are: i)$\alpha = 0, \beta \neq 0, s = 0$; ii)$\beta = 0, \alpha \neq 0, r = 0$; iii) $\alpha = \beta = 0$ and either $r \times j = 0$ or $s \times i = 0$. In each of these cases the minimal polynomial is $x^2 - \lambda^2$, with $\lambda^2 = r.r + s.s - \alpha^2 - \beta^2$.

- P has a cubic minimal polynomial iff $\alpha^2 - \beta^2 = s.s - r.r$ (without any of the conditions in the quadratic minimal polynomial case occurring). In this case the minimal polynomial is $x^3 - (\lambda^2 + 2\alpha^2 - 2(s.s))x$.

- If none of the above conditions hold the minimal polynomial is the characteristic polynomial which equals

$$p(x) = x^4 - 2\lambda^2 x^2 - [4\beta(s.s) - 4\alpha(r.r) + 4\alpha^2\beta^2 - 4 \, || \, r \times j \, ||^2 \, . \, || \, s \times i \, ||^2 - \lambda^4]$$

Sketch of the Proof: We will illustrate the calculations involved by proving the conditions for quadratic and cubic minimal polynomials for a Hamiltonian matrix H.

Quadratic Case: H^2's quaternionic representation is

$$H^2 = (-b^2 - p.p + q.q + r.r)(1 \otimes 1) + 2(r \times q + bp) \otimes j - (2p.q)1 \otimes j - (2p.r)1 \otimes k$$

According to Proposition (3.1) if at all H^2 is linearly dependent on a lower power of H, that power has to be $1 \otimes 1$. A necessary and sufficient condition for that to happen is evidently

$$r \times q = -bp, p.q = 0, p.r = 0$$

Clearly, if $b \neq 0$, this set of conditions is equivalent to $bp = q \times r$. If these conditions hold the minimal polynomial of H is $x^2 - \omega$, with

$$\omega = -b^2 - p.p + q.q + r.r$$

Cubic Case: By a direct calculation, which makes copious use of the vector triple identity [Equation (2.1)], one finds that

$$\begin{aligned} H^3 &= \omega H + 2[p.(q \times r) - bp.p]1 \otimes j + 2[-b^2 p + b(q \times r) + (p.q)q + (p.r)r] \otimes 1 \\ &+ 2[-(p.q)p - b(r \times p) + (r.r)q - (r.q)r] \otimes i \\ &+ 2[-(p.r)p - b(p \times q) + (q.q)r - (q.r)q)] \otimes k \end{aligned}$$

In view of Proposition (3.1), for H to have a cubic minimal polynomial, therefore there has to be a real k such that

$$[p.(q \times r) - bp.p] = kb \tag{3.2}$$

and further that

$$\begin{aligned} (b^2 + k)p - (p.q)q - (p.r)r &= b(q \times r) \\ -(p.q)p + (r.r - k)q - (r.q)r &= b(r \times p) \\ -(p.r)p - (q.r)q + (q.q - k)r &= b(p \times q) \end{aligned} \tag{3.3}$$

When this happens, in absence of the conditions for a quadratic minimal polynomial, the minimal polynomial of H is

$$p(x) = x^3 - (\omega + 2k)x$$

There are now two possibilities.

- $b \neq 0$, or
- $b = 0$.

In the former case, we find

$$k = \frac{1}{b}[p.(q \times r) - bp.p]$$

Next, noting that G is the Gram matrix of $X = [p \mid q \mid r]$, one finds that taking the inner product on both sides of Equation (3.3) successively with p, q, r yields

$$GY = b[p.(q \times r)]I_3$$

Conversely if $GY = b[p.(q \times r)]I_3$, one easily obtains Equation (3.3). For instance, if we denote by v the vector $(b^2 + k)p - (p.q)q - (p.r)r$, then, in view of the first column of Y, we find $v.r = 0 = v.q$, and hence v is proportional to $q \times r$. The remaining entry in this column of Y confirms that v is indeed $b(q \times r)$. Hence H has the stated minimal polynomial.

Now suppose, $b = 0$. Then we first need $r \times q \neq 0$, for otherwise we would have

$$2(bp + (r \times q)) = 0$$

which corresponds to the quadratic minimal polynomial case. Further, under the condition $b = 0$, Equations (3.2) and (3.3) reduce to

$$p.(q \times r) = 0 \tag{3.4}$$

and

$$\begin{aligned} kp - (p.q)q - (p.r)r &= 0 \\ -(p.q)p + (r.r - k)q - (r.q)r &= 0 \\ -(p.r)p - (q.r)q + (q.q - k)r &= 0 \end{aligned} \tag{3.5}$$

Now the analysis of the conditions equivalent to H having the stated minimal polynomial may be divided into two further cases:

- $p = 0$, or
- $p \neq 0$.

Suppose first that p is zero. Then Equation (3.4) and the first Equation in the system (3.5) are trivially satisfied, while the remaining two equations of Equation (3.5) yield

$$\begin{aligned}(r.r - k)q &= (r.q)r \\ (q.q - k)r &= (q.r)q\end{aligned}$$

These two equations contradict $r \times q \neq 0$, unless $r.r = q.q = k$ and $q.r = 0$. Conversely these two conditions are trivially sufficient to ensure that H has the said cubic minimal polynomial when $p = 0 = b$.

Next, suppose $p \neq 0$. Then certainly the linear independence of q and r, and the linear dependence of p on them is required. Now at least one of $p.q$ or $p.r$ is not zero, for otherwise p becomes zero, contradicting the starting assumption for this case.

Now the analysis may be divided into three cases:

- $p.q = 0$, but $p.r \neq 0$. Then taking the inner product of the first equation in (3.5) with q forces $r.q = 0$. Next, by taking the inner product of the same equation with r yields $k = r.r$. The second equation also forces $k = r.r$. The third equation is trivially satisfied upon taking inner product with q, while taking inner product with r forces $k = \frac{(q.q)(r.r)-(p.r)^2}{r.r}$. Hence, we necessarily require $(r.r)^2 + (p.r)^2 = (q.q)(r.r)$. Conversely, if these conditions hold, then the vectors formed by the left hand sides of Equation (3.5), which are in the span of q and r, are by construction orthogonal to q and r. Hence they must be zero. Thus, Equations (3.4) and (3.5) are satisfied and hence H has the stated cubic minimal polynomial.

- $p.r = 0$ and $p.q \neq 0$. Then, by an argument similar to the one above, a necessary and sufficient set of conditions are given by i) $q \times r \neq 0$; ii) $p.(q \times r) = 0$; iii)$r.q = 0$; iv) $(q.q)^2 + (p.q)^2 = (q.q)(r.r)$. In this case, $k = q.q$.

- Neither $p.q$ nor $p.r$ is zero. Then, first by taking inner product with q of the last equation of the system of (3.5), for instance, one sees that $r.q \neq 0$. Next taking the inner product with respect to q, first and then with respect to r of all equations in the system (3.5), one arrives at six possible expressions for k. Of these two are already equal to $-\frac{q.p}{q.r}(p.r)$.

The remaining four are

$$|| q ||^2 + \frac{p.r}{p.q}(r.q), || r ||^2 + \frac{p.q}{p.r}(r.q), || r ||^2 - \frac{[(q.p)^2 + (q.r)^2]}{|| q ||^2}, || q ||^2 - \frac{[(r.p)^2 + (q.r)^2]}{|| r ||^2}$$

Hence necessarily these four quantities are equal to each other and to $-\frac{q.p}{q.r}(p.r)$. Conversely, these conditions are sufficient to ensure that H has the said cubic minimal polynomial with $k = -\frac{q.p}{q.r}(p.r)$.

$\diamondsuit$

Remark 3.2 There are some special cases of the above result for the stated cubic minimal polynomial for a Hamiltonian matrix H, which deserve mention.

- First if $b \neq 0, p \neq 0$, and p, q, r are collinear, then H has the given cubic minimal polynomial iff

  $$b^2 = p.p + q.q + r.r$$

  In this case $k = -p.p$. Indeed, in this case both the matrices G and Y are rank one matrices, and the condition $GY = 0$ then is equivalent to $b^2 = p.p + q.q + r.r$.

  Note this contains the special case that $q = r = 0$. In this case, H is also skew-symmetric, and we find that a necessary and sufficient condition for H to have the given polynomial as its minimal polynomial is $p.p = b^2$. This, as is easily seen, is in keeping with the conditions for a skew-symmetric matrix to have a cubic minimal polynomial.

- A second special case, diametrically opposed to the previous one, occurs when the vectors p, q, r are all non-zero, and satisfy $q \times r = \alpha p, r \times p = \beta q, p \times q = \gamma r$, for some non-zero real numbers α, β, γ. One then finds that $\alpha \neq b$ (for otherwise, we would have a quadratic minimal polynomial). In this case $k = (\alpha - b)b$ and G and Y are both diagonal. Then the condition $GY = bp.(q \times r)I_3$ is equivalent to $\beta = \gamma$ (equivalently $q.q = r.r$) and

  $$b\beta^2 + (\alpha - b)b - \alpha b^2 = 0$$

  These conditions are satisfied if, for instance, $b = \beta = \gamma$, $\alpha \neq b$ and $p.p = b^2$.

- Note when $b = 0 = p$, H is a symmetric, Hamiltonian matrix. The conditions stated above for a cubic minimal polynomial for H also follow from Theorem (3.5) below.

Remark 3.3 Note that there is an asymmetry in the role of p (vis a vis q, r) in the matrix Y intervening in the conditions for a cubic minimal polynomial for a Hamiltonian matrix H. This is not surprising since p stems from the anti-symmetric part of H, while q, r stem from the symmetric part of H.

Next we study minimal polynomials for skew-Hamiltonian and symmetric matrices. Now Proposition (3.1) does not apply. Nevertheless we will find that the former always have quadratic minimal polynomials, and this is an illustration of the utility of quaternions. For the latter, in order to minimize bookkeeping, we suppose they are traceless. Once the minimal polynomial of these are found, those of symmetric matrices with non-zero trace are easily found.

Theorem 3.4 *Minimal Polynomials of Skew-Hamiltonian Matrices:* Let W be skew-Hamiltonian with representation $b(1\otimes 1)+p\otimes j+1\otimes(ci+dk)$. Then W has a quadratic minimal polynomial, which equals $p(x) = x^2 - 2bx + \kappa$, with $\kappa = b^2 - \parallel p \parallel^2 + c^2 + d^2$. .

Theorem 3.5 *Minimal Polynomials of Symmetric Matrices:* Let S be a non-zero, traceless symmetric matrix with representation $p \otimes i + q \otimes j + r \otimes k$. Then

- i) S has the quadratic minimal polynomial $p(x) = x^2 - \lambda^2$ iff the rank of $X = [p, q, r]$ is one. In this case $\lambda^2 = (p.p + q.q + r.r)$.
- ii) S has the quadratic minimal polynomial $p(x) = x^2 - 2lx - \lambda^2$ iff $p \times q = lr, q \times r = lp, r \times p = lq$, for the same non-zero l and $\lambda \neq 0$, as in i) above.
- S has the cubic minimal polynomial $p(x) = x^3 - (\lambda^2 + 2\alpha)x$ iff the rank of X is two and one of the three following mutually exclusive conditions hold:
  - Upto cyclic permutations of p, q, r, $p.q = 0, r \times p = 0 = q \times r, p.p = q.q$. In this case $\alpha = p.p$.
  - Upto cyclic permutations of p, q and r, one has $p \times q = 0, r \times p \neq 0, q.r = 0, p.p + q.q = r.r$. In this case $\alpha = r.r$ (note the conditions $p \times q = 0$ and $q.r = 0$ imply $r.p = 0$ also).
  - None of $p \times q, q \times r$ or $r \times p$ is zero and the following set of equalities holds

$$\parallel r \parallel^2 - \frac{[(r.p)(r.q)]}{p.q} = \parallel q \parallel^2 - \frac{[(q.p)(r.q)]}{p.r} = \parallel p \parallel^2 - \frac{[(r.p)(p.q)]}{q.r}$$

In this case $\alpha = \| p \|^2 - \frac{[(r.p)(p.q)]}{q.r}$.

- When the degree of the minimal polynomial is four, the minimal (and characteristic) polynomial is $p(x) = x^4 - 2\lambda^2 x^2 - 8(p.(q\times r))x + [\lambda^4 - 4(\| q\times r \|^2 + \| r\times p \|^2 + \| p\times q \|^2]$.

Note: In the case of symmetric matrices, there are other cubic minimal polynomials. Expressions and conditions for them can be found, but they do not have elegant geometric interpretations, and so we omit them.

Sketch of the proof: Once again we illustrate the quadratic and cubic minimal polynomial case for traceless, symmetric matrices. One first finds that S^2 is given by

$$S^2 = (\| p \|^2 + \| q \|^2 + \| r \|^2)1\otimes 1 + 2(p\times q)\otimes k + 2(q\times r)\otimes i + 2(r\times p)\otimes j$$

Clearly then $p(x) = x^2 - \lambda^2$ can annihilate S iff $p\times q = q\times r = r\times p = 0$, i.e., iff $[p, q, r]$ has rank one. When this holds $\lambda^2 = p.p + q.q + r.r$.

Similarly, $p(x) = x^2 - 2lx - \lambda^2$ annihilates S iff $p\times q = lr, q\times r = lp, r\times p = lq$ for the same non-zero l. When this happens λ^2 has to be necessarily $p.p + q.q + r.r$.

Next a calculation shows that

$$\begin{aligned} S^3 &= \lambda^2 S + 6p.(q\times r)1\otimes 1 \\ &\quad +2[q\times(p\times q) - r\times(r\times p)\otimes i \\ &\quad +2[r\times(q\times r) - p\times(p\times q)]\otimes j + 2[p\times(r\times p) - q\times(q\times r)]\otimes k \end{aligned}$$

It follows that for S to have the desired minimal polynomial one needs $p.(q\times r) = 0$ and that the following condition,*and all cyclic permutations of it*, have to hold

$$(q\times p)\times q + (r\times p)\times r = \alpha p \tag{3.6}$$

for the *same* non-zero α.

The condition $p.(q\times r) = 0$ forces the rank of $X = [p, q, r]$ to be atmost two. It has to be two, since the rank one case corresponds to a quadratic minimal polynomial. Hence rank of $X^T X = 2$. Since $X^T X$ is positive semidefinite, at least one principal minor of order two has to be non-zero. Hence further analysis can be divided into three mutually exclusive cases:

- Precisely one 2×2 principal minor of $X^T X$ is non-zero - say the one corresponding to the pair (p, q). Thus $r \times p = 0 = q \times p$, but $p \times q \neq 0$. So the system (3.6) reduces to

$$q \times (p \times q) = \alpha p; (p \times q) \times p = \alpha q; \alpha r = 0$$

  Hence, the linear independence of p, q first forces $p.q = 0$ and $q.q = p.p = \alpha$. This implies $\alpha \neq 0$ and hence $r = 0$. Conversely these conditions are sufficient for S to have the stated minimum polynomial.

- Precisely two of the 2×2 principal minors of $X^T X$ are zero, say those corresponding to the pairs (r, p) and (q, r). In particular, $p \times q = 0$. Writing out the system (3.6) under these assumptions, we find that

$$r.p = 0, \alpha = r.r, r.q = 0, \alpha = p.p + q.q$$

  So the stated conditions are necessary and it is easy to see their sufficiency as well.

- None of the 2×2 principal minors of $X^T X$ are zero. Thus each of the pairs (p, q), (q, r) and (r, p) are linearly independent, but each of the three vectors is linearly dependent on the remaining two. Then the system (3.6) is equivalent to

$$\begin{aligned}
(q.q + r.r - \alpha)p &= (q.p)q + (r.p)r \qquad (3.7)\\
(r.r + p.p - \alpha)q &= (q.r)r + (q.p)p\\
(p.p + q.q - \alpha)r &= (r.p)p + (r.q)q
\end{aligned}$$

  Then the first equation in the last system says $q.p \neq 0$. Indeed, if $q.p = 0$, then the linear independence of the pair of vectors (r, p) would force $r.p = 0$ as well. But then, p being linearly dependent on the pair (q, r) would have to be zero. Similarly $r.p \neq 0$ and $q.r \neq 0$.

  Now successively taking the inner product of the first equation in the above system with q, r; of the second equation with r, p; and the third equation with p, q, yields six possible expressions for α. Of these three are identical. Thus, we have three distinct expressions for α, which have therefore got to coincide, i.e., it is necessary that

$$r.r - \frac{[(r.p)(r.q)]}{p.q} = q.q - \frac{[(q.p)(r.q)]}{p.r} = p.p - \frac{[(r.p)(q.p)]}{q.r}$$

Conversely if the above equalities hold, then the vectors represented by the left hand sides of the system (3.7) are equal to the corresponding right hand sides. Hence these conditions are necessary and sufficient for S to have the stated cubic minimal polynomial.

Remark 3.4 It is not enough for $X = [p, q, r]$ to have rank 2 for a symmetric S to have the stated cubic minimal polynomial. The remaining conditions are needed. In fact, it turns out that all other cases when X has rank two correspond to fourth degree minimal polynomials.

We next consider matrices in $SO(4, R)$. II) of Proposition (3.1) applies to such matrices.

Theorem 3.6 *Minimal Polynomials of Special Orthogonal Matrices:* Let $G \neq I, -I$ be represented by $u \otimes v$. Let $u_0 = \mathrm{Re}(u), v_0 = \mathrm{Re}(v)$. Then,

- G has minimal polynomial $x^2 - 1$ iff $u_0 = v_0 = 0$.
- G has minimal polynomial $x^2 + ax + 1$ iff either (but not both) $\mathrm{I}m(u) = 0$ or $\mathrm{I}m(v) = 0$. In this case, $a = -2v_0$ (resp. $-2u_0$).
- G has minimal polynomial $x^3 - ax^2 + ax - 1$ iff $u_0 = v_0 \neq 0$. In this case $a = 4u_0v_0 - 1$.
- G has minimal polynomial $x^3 + ax^2 + ax + 1$ iff $u_0 = -v_0 \neq 0$. In this case $a = -(1 + 4u_0v_0)$.
- If none of the above conditions hold, G's minimal polynomial is its characteristic polynomial which equals $x^4 + ax^2 + bx^2 + ax + 1$, with $a = -4u_0v_0, b = 4u_0^2 + 4v_0^2 - 2$.

Sketch of the proof: First, since G is neither I nor $-I$, $\mathrm{I}m(u)$ and $\mathrm{I}m(v)$ cannot be simultaneously zero.

Next, using $u^2 = (2u_0^2 - 1) + 2u_0\mathrm{I}m(u)$ (and a similar expression for v^2), we see G^2 is represented by

$$(2u_0^2-1)(2v_0^2-1)(1\otimes 1)+2u_0(2v_0^2-1)(\mathrm{I}m(u)\otimes 1)+2v_0(2u_0^2-1)(1\otimes \mathrm{I}m(v))+4u_0v_0(\mathrm{I}m(u)\otimes \mathrm{I}m(v))$$

In view of Proposition (3.1) the only possible candidates for a quadratic minimal polynomial are $p(x) = x^2 - 1$ and $p(x) = x^2 + ax + 1$.

For $G^2 = I$, it is necessary and sufficient that all of the following to hold:

- $(2u_0^2-1)(2v_0^2-1)=1$.
- $2v_0(2u_0^2-1)=0$ or $\mathrm{I}m(v)=0$.
- $2u_0(2v_0^2-1)=0$ or $\mathrm{I}m(u)=0$
- $4u_0v_0=0$ or $\mathrm{I}m(u)=0$ or $\mathrm{I}m(v)=0$.

Suppose, $\mathrm{I}m(u)=0$. Then $u_0^2=1$ and $\mathrm{I}m(v)\neq 0$. So the second condition above forces $v_0=0$. But then the first condition above is not satisfied. Similarly the condition $\mathrm{I}m(v)=0$ is untenable. Hence, $\mathrm{I}m(u)\neq 0\neq \mathrm{I}m(v)$. Now the fourth and the first conditions together force $u_0=0=v_0$. Conversely, when $u_0=0=v_0$ we see, from conditions above, that $G^2=I$.

Next, consider $G^2+aG+I=0$. Suppose neither $\mathrm{I}m(u)$ nor $\mathrm{I}m(v)$ is zero. Then, by comparing the $\mathrm{I}m(u)\otimes\mathrm{I}m(v)$ coefficient, it is necessary that $a=-4u_0v_0$. But if $a=-4u_0v_0$, then (by comparing the $1\otimes 1$ coefficient) we see $u_0^2+v_0^2=1$, while by comparing the coefficients of $\mathrm{I}m(u)\otimes 1$ and $1\otimes\mathrm{I}m(v)$, we find $u_0=0=v_0$. Hence, necessarily precisely one of $\mathrm{I}m(u)$ or $\mathrm{I}m(v)$ is zero.

Suppose $\mathrm{I}m(u)=0$. Then u_0 is $+1$ or -1. By absorbing the negative coefficient in v, we may suppose $u_0=1$. In this case a further necessary condition is $a=-2v_0$. Similarly, if $\mathrm{I}m(v)=0$, $a=-2u_0$ is required. Conversely, these conditions are easily seen to be sufficient for $p(x)=x^2+ax+1$ to be the minimal polynomial of G.

Next we study the necessary and sufficient conditions for G to have cubic minimal polynomials. First, we find that

$$\begin{aligned}
G^3 &= u_0v_0(16u_0^2v_0^2-12u_0^2-12v_0^2+9)1\otimes 1\\
&\quad v_0[20u_0^2v_0^2-12u_0^2-6v_0^2+3]\mathrm{I}m(u)\otimes 1\\
&\quad u_0[20u_0^2v_0^2-12v_0^2-6u_0^2+3]1\otimes\mathrm{I}m(v)\\
&\quad [16u_0^2v_0^2-4u_0^2-4v_0^2+1]\mathrm{I}m(u)\otimes\mathrm{I}m(v)
\end{aligned}$$

By Proposition (3.1) it suffices to consider when $G^3+aG^2+aG+I=0$ or $G^3-aG^2+aG-I=0$ for suitable constants a.

Writing out the former we get

$$f_1(a,u_0,v_0)1\otimes 1+f_2(a,u_0v_0)\mathrm{I}m(u)\otimes 1+f_3(a,u_0v_0)1\otimes\mathrm{I}m(v)+f_4(a,u_0v_0)\mathrm{I}m(u)\otimes\mathrm{I}m(v)=0$$

for some polynomials $f_i, i = 1, \ldots, 4$, whose explicit form we omit for brevity. Necessarily $\mathrm{Im}(u) \neq 0 \neq \mathrm{Im}(v)$ (for otherwise we would be in the case of lower degree minimal polynomials). Hence a necessary and sufficient condition for $G^3 + aG^2 + aG + I = 0$ is $f_i(a, u_0, v_0) = 0, i =, \ldots, 4$. These four equalities are equivalent to $u_0 = -v_0$. Clearly we need $u_0 = -v_0 \neq 0$ to preclude $G^2 = I$. In this case, we also find $a = -(1 + 4u_0v_0)$. Similarly $x^3 - ax^2 + ax - 1$ is the minimal polynomial iff $u_0 = v_0 \neq 0$ and $a = 4u_0v_0 - 1$.

Remark 3.5 While (II) of Proposition (3.1) applies to other groups of matrices such as symplectic matrices, finding quaternioninc representations for them is quite arduous, and the formulae for such representations are not nearly as succinct as those for matrices in $SO(4, R)$. In [2], a quaternionic representation for $Sp(4, R)$ was obtained. In particular, this was used to find a closed form formula for the characteristic polynomial of such matrices. Extending this to find expressions for the minimal polynomial remains to be investigated.

4 Illustrative Applications

In this section we work out a few sample applications of the foregoing results. The first application shows that Jordan structure of skew-Hamiltonian matrices is determined completely by its minimal polynomial plus a single rank calculation (which can be performed in closed form). The second application works out the Cayley transform of skew-Hamiltonian matrices. Finally, we show how the minimal polynomial calculation of symmetric matrices can be used to determine the singular values of 3×3 real matrices.

Jordan Structure of Skew-Hamiltonian Matrices:

Proposition 4.1 Let W be a non-scalar, skew-Hamiltonian with quaternionic representation $b(1 \otimes 1) + p \otimes j + 1 \otimes (ci + dk)$. Then W is diagonalizable iff $\| p \|^2 \neq c^2 + d^2$. The Jordan normal form of W is either $\mathrm{d}iag(b + \mu, b + \mu, b - \mu, b - \mu)$ or $\mathrm{d}iag(J_2(b), J_2(b))$. Here $\mu = \sqrt{\| p \|^2 - c^2 - d^2}$, and $J_2(b)$ stands for the standard 2×2 Jordan block with b as the corresponding eigenvalue. Finally the characteristic polynomial of W is $p(x) = x^4 - 4bx^3 + (6b^2 - 2\mu^2)x^2 + (4b\mu^2 - 4b^3)x + b^4 + \mu^4 - 2\mu^2 b^2$.

Proof: First since W is non-scalar, the quantity $\theta^2 = \| p \|^2 + c^2 + d^2$ is non-zero. Per Theorem (3.4), W has minimal polynomial $x^2 - 2bx + \kappa$, where $\kappa = b^2 - \| p \|^2 + c^2 + d^2$. This polynomial has roots $(b+\mu, b-\mu)$, which are distinct iff $\mu \neq 0$, whence the first conclusion. The algebraic multiplicity of both the roots, $b+\mu$ and $b-\mu$, as roots of the characteristic polynomial has to be two each, for any other configuration of algebraic multiplicities would not yield $\mathrm{T}r(W) = 4b$. This yields the stated characteristic polynomial. Note that when $\mu = 0$, the sole eigenvalue is b with algebraic multiplicity four.

Next, when $\mu \neq 0$, W is diagonalizable and, in view of the algebraic multiplicities mentioned above, the corresponding Jordan form is $\mathrm{d}iag(b+\mu, b+\mu, b-\mu, b-\mu)$.

When $\mu = 0$, b is a two-fold root of the minimal polynomial. Hence the size of the largest Jordan block corresponding to the sole eigenvalue, b, has to be 2. Thus, the remaining Jordan blocks are either a single Jordan block of size 2 or two Jordan blocks each of size 1. To determine which possibility occurs, recall that for an $n \times n$ matrix W [7]

$$n_i = r_{i-1} - 2r_i + r_{i+1}, i = 0, 1, \ldots, n-1$$

where n_i stands for the number of Jordan blocks of size i corresponding to a given eigenvalue λ of W, and $r_k = \mathrm{r}ank(W - \lambda I)^k$, with the convention that $r_{n+1} = r_n = n - \nu$, with ν being the algebraic multiplicity of λ as a root of the characteristic polynomial.

Let $Y = W - bI$. In view of the only possibilities for the Jordan form of W (when $\mu = 0$), it is obvious that the rank of Y is either 1 or 2. From this we see that W has 2 Jordan blocks of size 2 each iff $\mathrm{r}ank(Y)$ is 2, while it has one Jordan block of size 2 and two Jordan blocks of size 1 each iff $\mathrm{r}ank(Y)$ is 1. We will now show that only the former possibility can occur.

This can be seen in a variety of ways. For instance, $\mathrm{r}k(Y) = \mathrm{r}k(Y^TY)$ and the latter has rank 2 precisely when at least one of its 2×2 principal minors is non-zero. We will now show that at least one 2×2 principal minor M_{ij} of Y^TY has to be non-zero.

Since $Y = p \otimes j + 1 \otimes (ci + dk)$, a simple calculation yields

$$Y^TY = \begin{pmatrix} \theta^2 - 2cp_3 + 2dp_1 & 2cp_2 & -2dp_3 & 2dp_2 \\ 2cp_2 & \theta^2 + 2cp_3 + 2dp_1 & 2dp_2 & 2dp_3 - 2cp_1 \\ -2cp_1 - 2dp_3 & 2dp_2 & \theta^2 - 2dp_1 + 2cp_3 & -2cp_2 \\ 2dp_2 & 2dp_3 - 2cp_1 & -2cp_2 & \theta^2 - 2dp_1 - 2cp_3 \end{pmatrix} \tag{4.8}$$

Here, as before, $\theta^2 = || p ||^2 + c^2 + d^2$. We will now show that even 5 of the principal minors being zero leads to the contradiction $\theta^2 = 0$.

Specifically suppose

- $(\theta^2 + 2dp_1)^2 - 4c^2p_3^2 - 4c^2p_2^2 = 0$ ((1, 2) minor).
- $\theta^4 - (2cp_3 - 2dp_1)^2 - (2cp_1 + 2dp_3)^2 = 0$ ((1, 3) minor).
- $(\theta^2 - 2cp_3)^2 - 4d^2p_1^2 - 4d^2p_2^2 = 0$ ((1, 4) minor).
- $(\theta^2 + 2cp_3)^2 - 4d^2p_1^2 - 4d^2p_2^2 = 0$ ((2, 3) minor).
- $(\theta^2 - 2dp_1)^2 - 4c^2p_3^2 - 4c^2p_2^2 = 0$ ((3, 4) minor).

Now the above facts regarding the $(1, 2)$ and $(3, 4)$ minors are equivalent to $dp_1 = 0$ (since $\theta^2 \neq 0$). Similarly the facts about the $(1, 4)$ and $(2, 3)$ minors are equivalent to $cp_3 = 0$. Hence these last two facts used in the $(1, 4)$ and $(1, 2)$ minor say $\theta^4 = 4d^2p_2^2 = 4c^2p_2^2$. This means d and c are non-zero. Hence, necessarily $p_1 = p_3 = 0$. Using this last piece of information in the fact concerning the $(1, 3)$ minor gives $\theta^4 = 0$ - a contradiction. Hence $\mathrm{r}k(Y^TY) = \mathrm{r}k(Y) = 2$.

Cayley transform of skew-Hamiltonian matrices: The Cayley transform of matrices provides a relationship between matrix Lie groups and their Lie algebras. It is interesting to compute it even for matrices not belonging to a Lie algebra. We do this below for skew-Hamiltonian matrices.

Let $\psi_C(A) = (I - A)(I + A)^{-1}$ be the Cayley transform of A which is assumed to be 4×4 skew-Hamiltonian. Since $\psi_C(A)$ is not defined if -1 is an eigenvalue of A, we suppose that b equals neither $-(1 + \mu)$ nor $-1 + \mu$. We know from the results above that this ensures that -1 is not in the spectrum of A.

Since A's minimal polynomial is quadratic, we know $\psi_C(A) = c_0I + c_1A$, with c_0, c_1 some constants. So we get

$$c_1A^2 + (c_1 + c_0 + 1)A + (c_0 - 1)I = 0$$

and hence, in view of the minimal polynomial of A,

$$(2bc_1 + c_1 + c_0 + 1)A + (c_0 - \kappa c_1 - 1)I = 0$$

This leads to the following system of equations for c_0, c_1:

$$\begin{aligned} c_0 - \kappa c_1 &= 1 \qquad (4.9) \\ c_0 + (2b + 1)c_1 &= -1 \end{aligned}$$

This yields $c_0 = \frac{2b+1-\kappa}{2b+1+\kappa}$ and $c_1 = \frac{-2}{2b+1+\kappa}$.

Hence,

$$\psi_C(A) = \frac{2b+1-\kappa}{2b+1+\kappa} I + \frac{-2}{2b+1+\kappa} A$$

Singular values of 3×3 real matrices One can use the geometric characterizations of minimal polynomials of traceless 4×4 real symmetric matrices to infer information about the singular values of real 3×3 matrices. This follows from the results in [11], wherein the eigenvalues of the symmetric matrix $X = p \otimes i + q \otimes j + r \otimes k$ are related to the singular values of the real matrix $Y = [p \mid q \mid r]$. Thus, if $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the singular values of Y and $\tau = sgn\ detY$ ($\tau = 0$ if Y is singular), then the eigenvalues of X are $\lambda_1 = \sigma_1 + \sigma_2 + \tau\sigma_3, \lambda_2 = \sigma_1 - \sigma_2 - \tau\sigma_3, \lambda_3 = -\sigma_1 + \sigma_2 - \tau\sigma_3, \lambda_4 = -\sigma_1 - \sigma_2 + \tau\sigma_3$.

From this expression and the fact that X's minimal polynomial has to have distinct roots, one can infer the following relation between X's minimal polynomial, and therefore the corresponding geometric conditions on p, q, r stated in Theorem (3.5), and Y's singular values:

- $\sigma_2 = 0 = \sigma_3, \sigma_1 \neq 0$iff X has minimal polynomial $x^2 - c^2$.
- $\sigma_1 = \sigma_2 \neq 0$ and $\sigma_3 = 0$ iff X has minimal polynomial $x^3 + cx$.
- $\sigma_1 = \sigma_2 = \sigma_3 \neq 0$ iff X has minimal polynomial $x^2 - 2lx - \lambda^2$.

Remark 4.1 The above list only contains those statements regarding the singular values of $Y = [p, q, r]$ corresponding to the list of minimal polynomials in Theorem (3.5). One can also infer the following statements regarding the singular values of Y by invoking the diagonalizability of X. Alternatively, the statements below about the singular values of Y can be used to augment the list of minimal polynomials of X. The corresponding conditions on p, q, r are too cumbersome to state. Partly because of this, and partly since that would have been contrary to the spirit of the paper, these minimal polynomials were not presented in Theorem (3.5) (cf., the note, immediately following the statement of Theorem (3.5) and Remark (3.4).

- If Y has rank 2 and $\sigma_1 \neq \sigma_2$ then X has a quartic minimal polynomial.
- Y has a cubic minimal polynomial other than $x^3 + cx$ iff $\tau \neq 0$ and either i) $\sigma_1 = \sigma_2 \neq \sigma_3$. In this case no eigenvalue of X is zero; or ii) $\sigma_2 = \sigma_3 \neq \sigma_1$. In this case X has a zero

eigenvalue iff $\sigma_1 = 2\sigma_2$;

5 Extensions

There are a few potential extensions of this work which we will discuss in this section.

One trivial way to extend the above results is to consider block diagonal matrices, with each block 4×4. The minimal polynomial of such a matrix is the least common multiple of the minimal polynomials of the individual blocks. Thus, when each of these blocks belongs to any of the classes of matrices considered here, one can find in closed form their minimal polynomials.

A second extension is to apply the theory of Clifford Algebras to calculate minimal polynomials, since each Clifford algebra arises as a suitable matrix algebra. In this regard we mention the interesting work of [1], where a symbolic calculation of the so-called real minimal polynomial is used to calculate exponentials of matrices. This, however, does not take into account the involutions of Clifford algebras, and thus the structure of the matrix is not used in finding minimal polynomials. In particular, there are no analogues of the geometric conditions on quaternions in the previous sections.

To understand the crux of the differences between our work and that in [1], it is useful to note the three features of $H \otimes H$ which enable our approach :

- i) $H \otimes H$ has a basis in which every element squares to plus or minus 1. Furthermore, any two elements in this basis commute or anti-commute.
- ii) The matrix analogue of the natural conjugation on $H \otimes H$ is matrix transposition.
- iii) The multiplication in $H \otimes H$ is intimately related to the geometry of vectors in R^3.

For Clifford algebras the first feature goes through verbatim. The second feature's effect is somewhat diluted, inasmuch as the natural involutions of the theory of Clifford Algebras (Clifford conjugation and reversion), [10, 13], have easy matrix theoretic interpretations only in certain cases. Finally, the third feature is completely lost. In the work of [1], only the first feature is used. Hence the structural (i.e., geometric) conditions in this work on a matrix's $H \otimes H$ representation, for it to have a specific minimal polynomial, have no analogues in [1].

As mentioned in the previous paragraph, the three enabling features for the $H \otimes H$ isomoprhism of $M(4, R)$ are diluted for Clifford algebra isomorphisms of matrix algebras. Nevertheless, there are two ways in which the theory of Clifford algebras can be used for the purpose at hand. First, one can uncover more classes for 4×4 matrices whose minimal polynomials can be calculated, and whose Jordan structure is akin to those of skew-Hamiltonian matrices. This is achieved by first considering matrices in $M(4, R)$ as elements of suitable Clifford algebras and inspecting their behaviour under Clifford conjugation and/or reversion, and then representing such matrices via quaternions. In some cases the $H \otimes H$ representations of these matrices enables a complete characterization of their minimal polynomial. Arguably, one would have not been lead to consider these classes otherwise. Secondly, one can use Clifford algebra representations of matrices of larger size to give a partial characterization of their minimal polynomials. Whilst a complete characterization of possible minimal polynomials for such matrices is the subject of future work, one can already say more than what would be possible without using Clifford algebras.

Let us now explore the first extension. To that end, note that there are two standard involutions in the theory of Clifford algebras - **reversion and Clifford conjugation.**, [10, 13] These are both anti-automorphisms. The matrix versions of these two involutions are easy for two classes of Clifford algebras. For $Cl(n, 0)$, reversion is Hermitian conjugation, while for $Cl(0, n)$ Clifford conjugation is Hermitian conjugation. Representing $Cl(p + 1, q + 1)$ as $M(2, Cl(p, q))$ (the algebra of 2×2 matrices with entries in $Cl(p, q)$), it is known that Clifford conjugation is represented as follows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{CC} = \begin{pmatrix} D^{rev} & -B^{rev} \\ -C^{rev} & A^{rev} \end{pmatrix}$$

while reversion is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{rev} = \begin{pmatrix} D^{cc} & B^{cc} \\ C^{cc} & A^{cc} \end{pmatrix}$$

Here, and in the balance of this section, Z^{cc} (respectively, Z^{rev}) stands for the Clifford conjugation (respectively, reversion) of a matrix (or its Clifford representation) Z.

Let us illustrate how this can be used to find minimal polynomials for matrices stemming $Cl(2, 2)$. Since $Cl(1, 1)$ is $M(2, R)$, it follows that $Cl(2, 2)$ is $M(4, R)$. On $Cl(1, 1)$ reversion sends X to $R_2X^TR_2$ (which, in the notation introduced in Section 2, is X_F), while Clifford conjugation sends X to $-J_2X^TJ_2$, which is X_H. Equivalently, since X is 2×2, X^{CC} is a$dj(X)$, where, as usual, a$dj(X)$ is the classical adjugate of X. Thus, on $Cl(2,2)$ we get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{rev} = \begin{pmatrix} D_H & B_H \\ C_H & A_H \end{pmatrix}$$

Thus, if $X \in Cl(2,2)$ equals its own reversion, then A and D are each other's adjugates, while B and C are 2×2 skew-Hamiltonian. A basis of 1-vectors for $Cl(2,2)$ consists of the following four matrices (written in $H\otimes H$ form):

$$f_1 = -j\otimes k, f_2 = i\otimes k, f_3 = -1\otimes i, f_4 = -1\otimes j \tag{5.10}$$

This yields expressions for 2-vectors etc., which we omit. Now, since X equals its own reverse, it must be a linear combination of the identity, 1-vectors and 4-vectors. The last equation, for a basis of one-vectors for $Cl(2,2)$, thus yields the following $H\otimes H$ representation of of the most general $X \in Cl(2,2)$ satisfying $X^{rev} = X$.

$$a(1\otimes 1) + p\otimes k + 1\otimes s \tag{5.11}$$

with p, s pure-quaternions, *with the latter having no k-component.* Thus such an X is remarkably similar to skew-Hamiltonian matrices, with the difference that the *roles of j and k have been interchanged.* Thus, we find, for instance that $X^2 = (p.p - s.s - a^2)1\otimes 1 + 2aX$ and hence such an X's minimal polynomial is quadratic. We omit the similar statements about the Jordan structure of such matrices, that this minimal polynomial yields. If $X \in Cl(2,2)$ satisfies $X^{rev} = -X$, then it has an $H\otimes H$ representation akin to that for a Hamiltonian matrix, and therefore an analogue of Theorem (3.2) applies to it.

Similarly, if $X \in Cl(2,2)$ is minus its own reversion, then i) $A = D^F$ and B and C are both perskewsymmetric; and ii) the $H\otimes H$ representation of X is given by

$$X = a(1\otimes 1) + k\otimes p + q\otimes 1 \tag{5.12}$$

with $a \in \mathcal{R}, p, q \in \mathcal{P}$ and $q.k = 0$. Once again, this yields a quadratic minimal polynomial for X.

Remark 5.1
- A calculation shows that matrices represented by Equation (5.11) [resp. Equation (5.12)] are precisely those self-adjoint with respect to the non-degenerate bilin-

ear form on R^4 whose defining matrix is $M_{1\otimes k}$ (resp. $M_{k\otimes 1}$). Thus, the considerations of the previous paragraphs yield a natural Clifford theoretic interpretations for these bilinear forms.

- By passing to $Cl(3,1)$ and performing an analysis akin to the one above for $Cl(2,2)$ one can show that those $X \in Cl(3,1)$ satisfying $X^{rev} = X$ are again given by Equation (5.11), while those satisfying $X^{CC} = X$ are skew-Hamiltonian matrices.

- It is worth emphasizing that the block structures of the matrices considered in the previous paragraphs do not themselves reveal the simplicity of their minimal polynomials. It is only by passing to their $H \otimes H$ representations that we are lead to these results.

For higher dimensional matrix algebras arising from Clifford Algebras, we do not (yet) have an exhaustive set of results. Nevertheless, some conclusions can be drawn, which would have been difficult to arrive at without passing to Clifford Algebras. Let us illustrate this via $Cl(0,6)$. This is $M(8,R)$. Furthermore, Clifford conjugation is precisely matrix transposition in this case and thus a matrix is anti-symmetric iff it is minus its Clifford conjugation. Since Clifford conjugation of a p-vector in $Cl(0,6)$ is minus itself iff $p = 1,2,5,6$, an 8×8 matrix is anti-symmetric iff it is a linear combination of of these p-vectors. We use the following basis of 1-vectors:

$$\begin{aligned} e_1 &= \sigma_z \otimes \sigma_x\sigma_z \otimes I_2 \\ e_2 &= \sigma_z\sigma_x \otimes I_4 \\ e_3 &= \sigma_x \otimes \sigma_z\sigma_x \otimes \sigma_x \\ e_4 &= \sigma_x \otimes \sigma_z\sigma_x \otimes \sigma_x \\ e_5 &= \sigma_x \otimes I_2 \otimes \sigma_x\sigma_z \\ e_6 &= \sigma_z \otimes \sigma_x \otimes \sigma_z\sigma_x \end{aligned} \tag{5.13}$$

Here the σ's are the usual Pauli matrices. Using this one can write down a basis of p-vectors for $p = 2,5,6$ which we omit for brevity. The typical 8×8 anti-symmetric matrix is thus a real linear combination

$$X = \sum_{i=1}^{6} p_i e_i + \sum_{i<j} p_{ij} e_{ij} + \sum_{i<j<k<l<m} p_{ijklm} e_{ijklm} + p_{123456} e_{123456} \tag{5.14}$$

One can now list a set of mutually exclusive conditions on these coefficients which are necessary and sufficient for X to have a quadratic minimal polynomial. From Proposition (3.1) we know that the minimal polynomial has to have the form $p(x) = x^2 - \lambda^2$. Due to the more complicated structure of Clifford multiplication on $Cl(0,6)$ this list of conditions, even for the quadratic case, are far too long to enlist. Therefore, we will just give sample instances of these conditions.

To that end, it is first noted that this set contains conditions of two types. The first consists of conditions which merely equate some of the coefficients, $p_J, J \subseteq \{1,2,3,4,5,6\}$ in Equation (5.14) to zero. The latter consist of more complicated algebraic relations between the p_J. To understand the difference between the two, it is first noted that a p-vector and a q-vector either commute or anti-commute. Conditions of the first type arise precisely when all the summands in Equation (5.14) anti-commute. Under these circumstances the minimal polynomial of X is clearly quadratic. The latter set of condition arises when there are some commuting summands in Equation (5.14). In this case the corresponding coefficients have to satisfy certain relations to ensure that $p(x) = x^2 - \lambda^2$ is the minimal polynomial of X. By carefully considering the commutation relations between the $1, 2, 5$ and 6-vectors in $Cl(0,6)$ one can arrive at the aforementioned conditions.

Enlisted below are instances, first of the first type of conditions and then of the second type of conditions.

- i) $X = p_i e_i + \sum_{k<i} p_{ki} e_{ki} \sum_{j>i} p_{ij} e_{ij} + p_{\alpha\beta\gamma\delta\epsilon} e_{\alpha\beta\gamma\delta\epsilon}$, with $i \notin \{\alpha, \beta, \gamma, \delta, \epsilon\}$.
- ii) $X = p_i e_i + p_{\alpha\beta\gamma\delta\epsilon} e_{\alpha\beta\gamma\delta\epsilon} + p_{123456} e_{123456}$, with $i \notin \{\alpha, \beta, \gamma, \delta, \epsilon\}$.

Examples of the second type of conditions are

- $X = p_1 e_1 + p_2 e_2 + p_{13} e_{13} + p_{23} e_{23}$ with $p_1 p_{23} = p_2 p_{13}$.
- $X = p_1 e_1 + p_{23} e_{23} + p_{45} e_{45} + p_{12345} e_{12345}$ with $p_1 = p_{45}, p_{23} = p_{12345}$ and p_1 equal to p_{23} upto sign.

Finally, in all the cases above λ^2 is the Euclidean length squared of the vector of coefficients describing X.

Octonions and Quadratic Minimal Polynomials: One special class of 8×8 matrices which always have quadratic minimal polynomials can be obtained via octonions. Whilst the octonions are not associative, one can attach two 8×8 matrices, $\omega(a), \theta(a)$ to an octonion a, [17]. The former describes the effect on an octonion upon left multiplication by a, while the latter does the same for right multiplication by a. To describe them express the octonion a by a pair of quaternions, $a = (a_1, a_2), a_i \in H, i = 1, 2$, via the Cayley doubling procedure, [10]. Then

$$\omega(a) = \begin{pmatrix} M_{a_1 \otimes 1} & -M_{1 \otimes \bar{a_2}} I_{1,3} \\ M_{a_2 \otimes 1} I_{1,3} & M_{1 \otimes \bar{a_1}} \end{pmatrix}$$

and

$$\theta(a) = \begin{pmatrix} M_{1 \otimes \bar{a_1}} & -M_{\bar{a_2} \otimes 1} \\ M_{a_2 \otimes 1} & M_{1 \otimes a_1} \end{pmatrix}$$

Here $I_{1,3} = \mathrm{d}iag(1, -1, -1, -1)$.

Then, as shown in [17], the alternating identities yield

$$\begin{aligned} \omega(a^2) &= (\omega(a))^2 \\ \theta(a^2) &= (\theta(a))^2 \end{aligned} \tag{5.15}$$

Now since any octonion a satisfies $a^2 - 2\mathrm{R}e(a)a + \mid a \mid^2 = 0$, we see that the 8×8 matrices $\omega(a)$ and $\theta(a)$ have the quadratic polynomial $p(x) = x^2 - 2\mathrm{R}e(a)x + \mid a \mid^2$ (as long as $a \neq 0$. Next, since the octonions are not associative one cannot expect $\omega(ab)$ (resp. $\theta(ab)$) to equal the product $\omega(a)\omega(b)$ (resp. $\theta(a)\theta(b)$). Nevertheless, if $ab \neq 0$, $\omega(ab)$ (resp. $\theta(ba)$) is similar to $\omega(a)\omega(b)$ (resp. $\theta(a)\theta(b)$), [17]. Thus, the matrix $\omega(a)\omega(b)$ (resp. $\theta(a)\theta(b)$) has a quadratic minimal polynomial, $p(x) = x^2 - 2\mathrm{R}e(ab)x + \mid ab \mid^2 = x^2 - 2 < a, \bar{b} > x + \mid a \mid^2 \mid b \mid^2$ (resp. $q(x) = x^2 - 2\mathrm{R}e(ba)x + \mid ba \mid^2$).

This is significant since the structure of the matrices $\omega(a)\omega(b)$ (resp. $\theta(a)\theta(b)$) *is more complicated* than that of $\omega(c)$ (resp. $\theta(c)$), for an octonion c.

We end this section with a discussion of how the method of [8] can be combined with those of this work to compute minimal polynomials of 4×4 matrices not covered above. The same discussion will also reveal why the method of [8] requires more computation than that proposed here.

We first briefly recall the method of [8] for computing the minimal polynomial of a matrix X of size $n \times n$. One first associates to the sequence $\{I, X, X^2, \ldots\}$ the matrices $G_i, i =$

$1, \ldots, n$, where G_i is the Gram matrix of the set of matrices $\{I, X, X^2, \ldots, X^i\}$ with respect to the inner product $< Y, Z >= \mathrm{T}r(Y^T Z)$ (here, for brevity, all matrices are assumed to be real). Thus, for instance

$$G_2 = \begin{pmatrix} \mathrm{T}r(I) & \mathrm{T}r(X) & \mathrm{T}r(X^2) \\ \mathrm{T}r(X^T) & \mathrm{T}r(X^T X) & \mathrm{T}r(X^T X^2) \\ \mathrm{T}r((X^T)^2) & \mathrm{T}r((X^T)^2 X) & \mathrm{T}r(X^T)^2 X^2) \end{pmatrix}$$

The method then, in essence, consists of two steps:

- One computes the ranks of the G_i's. Then the degree of the minimal polynomial of X is r iff the first i for which the rank of G_i is lower than $i+1$ is r.

- In this case it is also known that the kernel of G_r is of dimension one. Furthermore, it is guaranteed that there is a vector in the kernel of G_r whose last coefficient is non-zero. Normalizing this coefficient to one yields a vector $(a_0, a_1, \ldots, a_{r-1}, 1)$ in the kernel of G_r. This vector yields the minimal polynomial of X to be $p(x) = x^r + \sum_{i=0}^{r-1} a_i x^i$.

Thus, this method requires two steps i) Calculating the G_i and their ranks successively till one detects a drop in rank. Thus, this step requires a requisite number of trace calculations plus one's favourite method to compute ranks; ii) Computing a non-zero vector in the kernel of G_r.

The first step is amenable to the methods used in this work, since to find the trace of a matrix being represented in quaternion (or Clifford Algebra) form, one has to only find the coefficient of the $1 \otimes 1$ term in the matrix. This rarely requires the full quaternionic expansion of the matrix. However, even for the classes of structured matrices considered here, these calculations involve more than those required by our methods. We illustrate this issue via the case of 4×4 real symmetric matrices. To detect a quadratic minimal polynomial, our method requires finding only X^2. However, to find G_3 and check if its rank is two, one needs terms such as $\mathrm{T}r(X^3)$. While, this does not require the full calculation of X^3, it requires more than a calculation of X^2, because one has to find the $1 \otimes 1$ term in X^3.

Even when the ranks of the G_i have been computed and the degree of the minimal polynomial found, one has to still find a non-zero element of the kernel of G_r. This is typically difficult to do in closed form, whereas the methods used here do produce the minimal polynomials (for the classes of matrices considered here) in closed form.

6 Conclusions

In this work a complete characterization of the minimal polynomials of several important classes of 4×4 real matrices, including those of interest in applications, was provided. These were illustrated by relevant applications such as the determination of the Jordan structure of 4×4 skew-Hamiltonian matrices. Extensions of these results via the usage of Clifford algebras was indicated. In particular, classes of matrices were found whose block structures bely their close similarity, vis a vis minimal polynomials, to skew-Hamiltonian and Hamiltonian matrices. Extensions of the preliminary results announced here for $M_8(R)$ will be the subject of future investigations.

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