Topological term in the non-linear σ model of the SO(5) spin chains

Dung-Hai Lee

Department of Physics, University of California at Berkeley, Berkeley, CA 94720, USA;

Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Guang-Ming Zhang

Department of Physics, Tsinghua University, Beijing 100084, China

Tao Xiang

Institute of Physics and Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China.

Abstract

We show that there is a topological (Berry phase) term in the non-linear σ model description of the SO(5) spin chain. It distinguishes the linear and projective representations of the SO(5) symmetry group, in exact analogy to the well-known θ-term of the SO(3) spin chain. The presence of the topological term is due to the fact that $\pi_2(\frac{SO(5)}{SO(3)\times SO(2)}) = \mathbb{Z}$. We discuss the implication of our results on the spectra of the SO(5) spin chain, and connect it with a recent solvable SO(5) spin model which exhibits valence bond solid ground state and edge degeneracy.

Key words: Non-linear sigma model, topological terms, spin chains

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1 Introduction

The effects of topological terms on the dynamics of Goldstone modes and the quantum number of solitons and instantons in non-linear σ (NLσ) models have a long history, and continue to attract strong interests from the physics community[1]. For example, in one spatial dimension (1D), quantum SO(3) spin chains have fundamentally different low energy properties, depending on whether the site representation is linear (spin integer) or projective (spin half-odd integer). For nearest neighbor isotropic exchange interaction, the former cases always have excitation gaps while the latter are gapless - the well known Haldane conjecture[2]. Aside from the usual stiffness terms, the (1+1)D NLσ models for these spin chains contain a topological (Berry phase) term (also known as the θ term)[2]. When the space-time configuration of the Neel order parameter wraps the target space (S^2) n times, the Berry phase factor is +1 for integer spin chains, while it is $(-1)^n$ for half-odd integer spin chains[2].

Using an algebraic approach, Chen, Gu and Wen[3] recently generalized an idea of Ref.[4] and argued that, in one spatial dimension, a gapped ground state which is invariant under translation and the global symmetry operation (we refer to this type of state as "totally symmetric" in the following) is obtainable when the site representation of the global symmetry group (which can be discrete) is linear. When the site representation is projective and nontrivial (i.e., not linear), a totally symmetric ground state must be gapless. In a projective representation D, the matrix product $D(g_1)D(g_2)$, where $g_{1,2}$ are group elements, can differ from $D(g_1.g_2)$ by a phase factor $e^{i\theta(g_1,g_2)}$. For the SO(3) group, an integer spin forms the linear representation while a half-odd integer spin forms the projective representation. Hence the spectral difference between the translational invariant integer and half-odd integer SO(3) spin chains constitutes a special example of the results of Ref.[3]. Thus there are two ways to view the difference between integer and half-odd integer SO(3) spin chain: one is geometrical (the Berry's phase)[2] and the other is algebraic[3].

SO(5) is a rank-2 classical Lie group. It also has linear and projective representations. For example, in the vector representation the generators of the Lie algebra are given by[5]

$$(L_{ab})_{jk} = -i\delta_{a,j}\delta_{b,k} + i\delta_{a,k}\delta_{b,j}, \tag{1}$$

where $a, b, = 1, ..5$, and $i, j = 1..5$. Two consecutive π rotations generated by, e.g., L_{12} give

$$U_{12}(\pi)U_{12}(\pi) = I_{5\times 5}. \tag{2}$$

The spinor representation, on the other hand, is given by[5]

$$L_{ab} = i[\Gamma_a, \Gamma_b]/4, \tag{3}$$

where $\Gamma_{a,b}$ are the 4×4 gamma matrices (e.g., $\Gamma_1 = -\sigma_y \otimes \sigma_x$, $\Gamma_2 = -\sigma_y \otimes \sigma_y$, $\Gamma_3 = -\sigma_y \otimes \sigma_z$, $\Gamma_4 = \sigma_x \otimes I_{2\times2}$, $\Gamma_5 = \sigma_z \otimes I_{2\times2}$). In this case it is simple to check that two consecutive π rotations generated by L_{12} yield

$$U_{12}(\pi)U_{12}(\pi) = -I_{4\times4}. \tag{4}$$

Thus the vector representation is linear, while the spinor representation is projective and non-trivial. According to Ref.[3], a spin chain of the former type can have a totally symmetric ground state with a gapped spectrum, while a spin chain of the latter type has to be gapless if it is totally symmetric. In the following we will seek for the geometric (Berry' phase) difference between the two cases.

Another motivation for us to study the Berry's phase of the SO(5) spin chain is a recent exactly solvable 1D SO(5) spin model (in the vector representation) proposed by Tu et. al.[6]. The ground state is a translational invariant matrix product state, i.e., the valence bond solid state[7], and the excitation spectrum has a gap. Moreover, when the chain is subjected to the open boundary condition, there are edge states. These properties are reminiscent of the property of integer SO(3) spin chains[8].

Moreover, with the advance of cold atom physics, the SO(5) spin chain might not be a purely academic model. An SO(5) symmetric spin chain can in principle be realized experimentally when the hyperfine spin-3/2 cold fermions on an 1D optical lattice form the Mott-insulating state[9]. At quarter filling (one fermion per site), the effective spin chain is in the spinor representation, while for the half-filling (two fermions per site) the effective spin chain is in the SO(5) vector representation. Therefore the idea presented here might one day be tested experimentally.

2 Model formulation

Let us start by considering the following SO(5) invariant Hamiltonian

$$H = \sum_i \left[J_1 \left(\sum_{a<b} L^i_{ab} L^{i+1}_{ab} \right) + J_2 \left(\sum_{a<b} L^i_{ab} L^{i+1}_{ab} \right)^2 \right] \tag{5}$$

where L_{ab}'s are the SO(5) generators and $J_{1,2} > 0$. When L_{ab} are given by Eq.(1), the ground state is translational invariant and the spectrum has a gap in the parameter range $1/9 < J_2/J_1 < 1/3$[6]. Naively, one would not expect the NLσ model action of this model to contain a topological term. This is because in contrast to SO(3) spin chain where the space-time dimension $(1+1)$ matches the dimension of the target space[**?**] of the order parameter (S^2), for the SO(5) spin chain the dimension of target space is much larger than the $(1+1)$ space-time dimension.

To understand the structure of the target space for the SO(5) spin chain, we need to know how the presence of SO(5) "magnetic" moment breaks the global SO(5) symmetry. For that purpose it is sufficient to consider the following mean-field Hamiltonian of Eq. (5), where non-linear terms in L_{ab} are decoupled into linear ones with the order parameter $\langle L^i_{ab}\rangle = (-1)^i m_{ab}$

$$H_{\rm MF} = \left(-2J_1 + 2J_2\Delta^2\right)\sum_{i,a<b}(-1)^i m_{\rm ab}L^i_{\rm ab} + \sum_i\left(J_1\Delta^2 - 3J_2\Delta^4\right), \tag{6}$$

where $\Delta^2 = \sum_{a<b} m^2_{ab}$. The question at hand is for a fixed total magnitude of m_{ab} (i.e. fixed $\sum_{a<b} m^2_{ab}$) what is the most energetically favorable ratio between different components of m_{ab}. This can be answered simply by diagonalizing a single-site Hamiltonian

$$H_1 = -\sum_{a<b} m_{ab}L_{ab}. \tag{7}$$

and see what ratio gives the lowest ground state energy. (Of course we need to remember the sign of m_{ab} change from site to site.)

In the following we study the two irreducible representations given by Eqs.(1) and (3). First we consider the vector representation, Eq.(1). It is straightforward to show the energy spectrum of H_1 is $E = (-\Delta_1, -\Delta_2, 0, \Delta_2, \Delta_1)$ where $\Delta_{1,2} = \sqrt{A \pm \sqrt{A^2 + B - C}}$ and

$$\begin{aligned}
A &= \sum_{a<b} m^2_{ab}/2 = \Delta^2/2,\\
B &= 2\sum_{a<b<c<d}(m_{ac}m_{ad}m_{bc}m_{bd} - m_{ab}m_{ad}m_{bc}m_{cd} + m_{ab}m_{ac}m_{bd}m_{cd}),\\
C &= \sum_{a<b}\sum_{a<c<d} m^2_{ab}m^2_{cd}(1-\delta_{bc})(1-\delta_{bd}).
\end{aligned} \tag{8}$$

The single site ground state energy reaches the minimum when $B = C$, where the energy spectrum of H_1 is $E = \{-\Delta, 0, 0, 0, \Delta\}$. Under such a condition using H_1 as one of the two Cartan generators[5], and choose the other Cartan generators H_2 to satisfy $\mathrm{Tr}(H_1H_2) = 0$, the root and weight diagrams are shown in

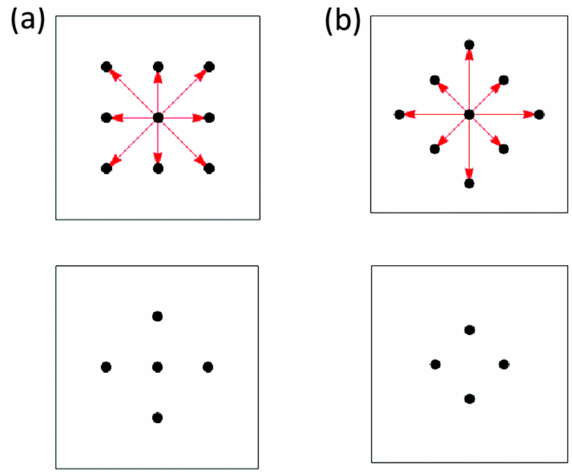

Fig. 1. (color on-line) The root (upper row) and weight (lower row) patterns of the vector (a) and spinor (b) representation of SO(5). The x and y coordinates of each dot corresponds to the eigenvalue of H_1 and H_2. The red arrows indicate how the rasing/lowering operators in the SO(5) Lie algebra change the eigenvalues of H_1 and H_2.The central dots in the root patterns are doubly degenerate.

Fig. (1a). Here the x and y coordinates of the dots are the eigenvalues of H_1 and H_2 respectively. The arrows in the root diagrams indicate how the rasing/lowering operators[5] in the SO(5) Lie algebra change the eigenvalues of the Cartan generators. Together with H_2 the rasing and lowering operators generate an SO(3) subgroup which commutes with H_1. As the result, the SO(5) symmetry is broken down to SO(3)×SO(2) with the SO(2) being generated by H_1 itself.

Second we consider the spinor representation, Eq.(3). It is straightforward to show that the energy spectrum of H_1 is $E = (-\Delta_1, -\Delta_2, \Delta_2, \Delta_1)$ with $\Delta_{1,2} = \sqrt{A \pm \sqrt{C-B}}/\sqrt{2}$. In this case the single site ground state energy reaches the minimum when $A = \sqrt{C-B}$ where the energy spectrum of H_1 is $E = \{-\Delta, 0, 0, \Delta\}$. The root and weight patterns are shown in Fig. (1)(b). Again the SO(5) symmetry is broken down to SO(3)×SO(2).

After fixing the ratio of m_{ab}, we assume that the low energy fluctuations correspond to smooth space-time dependent SO(5) rotations of such an order parameter pattern. The NLσ model precisely describes such smooth fluctuations. The second homotopy group of $\frac{SO(5)}{SO(3)\times SO(2)}$ is $\mathbb{Z}$ (Ref.[10]). Thus the corresponding NLσ model may contain a topological term, which can lead to a spectral difference between the vector and spinor representations.

3 The single-site Berry's phase

To study the possible topological term, we begin by analyzing the Berry's phase of a single SO(5) spin described by the following time-dependent Hamiltonian

$$H_1(t) = -\sum_{a<b} m_{ab}(t) L_{ab}, \tag{9}$$

where $m_{ab}(t)$ satisfy the constraints: (a) $\sum_{a<b} m_{ab}^2 = 1$, and (b) $H_1(t)$ possesses SO(3)×SO(2) symmetry. Both constraints can be satisfied by starting with a reference Hamiltonian $H_{1,0}$ satisfying (a) and (b) and perform time-dependent SO(5) conjugation, i.e.,

$$H_1(t) = U^\dagger(t) H_{1,0} U(t). \tag{10}$$

As usual, the Berry's phase is given by the loop integral of the Berry connection[1]. We can use Stoke's theorem to convert this loop integral to an areal integral over a disk with the loop as the boundary. The advantage of doing so is the Berry curvature rather than the Berry connection appears in the latter integral. As the result the integral has a local form and is gauge invariant:

$$S_B = \frac{i}{2}\int_0^1 du \int dt\ \epsilon^{\mu\nu}\mathrm{Tr}F_{\mu\nu}. \tag{11}$$

In the above $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ and $A_\mu = -i\langle\Omega|\partial_\mu|\Omega\rangle$. Here $|\Omega(t,u)\rangle$ is the ground state of

$$H_1(t,u) = U^\dagger(t,u) H_{1,0} U(t,u). \tag{12}$$

In Eq. (12), $U(t,u)$ is the extension of the $U(t)$ in Eq. (10) to the disk. Because π_1(SO(5)/SO(3)×SO(2))=0, we can always construct the extension so that $U(u=1,t) = U(t)$ and $U(u=0,t) = U_0$, where U_0 is a certain reference SO(5) element.

Using the first order perturbation theory for wave functions, it is simple to show that

$$\mathrm{Tr}F_{\mu\nu} = -i\sum_k \frac{\langle\Omega\,|\partial_\mu H|\,k\rangle\,\langle k\,|\partial_\nu H|\,\Omega\rangle - (\mu\leftrightarrow\nu)}{(E_0-E_k)^2}. \tag{13}$$

Here k labels the excited states. Since all Hamiltonians described by Eq. (12) are unitary conjugate of one another, they have the same eigenspectrum. Under that condition we have

$$\partial_\mu H = \sum_k (E_k - E_0)(|\partial_\mu k\rangle\langle k| + |k\rangle\langle\partial_\mu k|). \tag{14}$$

In writing down the above equation we have made a shift of the zero of energy so that $E_0 \to 0$. Substitute Eq. (14) into Eq. (13) and use the fact that $\langle\Omega|k\rangle = \langle k|\Omega\rangle = 0$ we find

$$\mathrm{Tr}F_{\mu\nu} = -i\mathrm{Tr}(Q[\partial_\mu Q, \partial_\nu Q]), \tag{15}$$

where Q is the ground state projection operator

$$Q(t,u) = |\Omega\rangle\langle\Omega| = U(t,u)PU^\dagger(t,u), \tag{16}$$

where $P = |0\rangle\langle 0|$ is the ground state projector operator of $H_{1,0}$. Substituting Eq. (15) into Eq. (11) we obtain

$$S_B = \int_0^1 du \int dt \mathrm{Tr}(Q[\partial_u Q, \partial_t Q]). \tag{17}$$

Eq. (17) actually applies for any target space. For example in the case of SO(3)/SO(2) we have

$$U(t,u) = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \text{ and } P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $z_{1,2}(t,u)$ satisfy

$$(\bar{z}_1, \bar{z}_2) \cdot \vec{\sigma} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \hat{n}(t,u).$$

Substitute the above two expressions into Eq. (16) and Eq. (17) we obtain

$$S_B = \frac{i}{2}\int_0^1 du \int dt \left(\hat{n} \cdot \partial_u \hat{n} \times \partial_t \hat{n}\right), \tag{18}$$

which is the well known expression for the Berry's phase of a spin-1/2[2]. Eq.(17) has also been used in Ref.[11] in the study of the SU(N) spin chain.

Because the dimension of our target space is eight, there are more than one disks having the loop in question as the boundary. Therefore we need to ask whether Eq. (17) yields the same answer for the Berry phase when different extensions of $U(t) \to U(t,u)$ are used. The difference in the Berry phase using two different disks as extension can be calculated by integrating the Berry curvature over a closed two dimensional surface formed by joining the two disks at their common boundary. The resulting closed surface has the topology of a 2-sphere. Because the second homotopy group of our target space is $\mathbb{Z}$, all 2-spheres in the target space are topologically the multiple of a basic sphere. Hence all we need to check is whether the Berry curvature integral is integer multiple of 2π when t and u in

Eq. (17) parameterize the basic 2-sphere[12]. In the following we perform such a calculation.

For the vector representation, we choose $H_{1,0} = L_{12}$, and pick $U(t,u)$ so that

$$U^\dagger(t,u) H_{1,0} U(t,u) = \hat{w}(t,u) \cdot \vec{L}, \tag{19}$$

where $\vec{L} = (L_{13}, L_{23}, L_{12})$,

$$\hat{w}_i(t,u) = (\sin(\pi u) \cos\frac{2\pi t}{\beta}, \sin(\pi u) \sin\frac{2\pi t}{\beta}, \cos(\pi u)),$$

and β is the period of the imaginary time. For the spinor representation, we take $H_{1,0} = (L_{12} - L_{34})/\sqrt{2}$, and $\vec{L} = (L_{13} + L_{24}, L_{14} - L_{23}, L_{12} - L_{34})/\sqrt{2}$.

Using Eq. (17), or equivalently Eqs.(11) and (13), we find that

$$S_{B,\text{basic sphere}}^{\text{vector}} = 4\pi, \quad S_{B,\text{basic sphere}}^{\text{spinor}} = 2\pi. \tag{20}$$

Thus the condition for the uniqueness of the Berry phase is satisfied. As we shall see, the difference between the vector and spinor Berry phase in Eq. (20) serves to distinguish the vector and spinor SO(5) chains.

4 The lattice Berry's phase, gapful versus gapless, and the edge states

We now extend the above single-site Berry phase analysis to the one dimensional lattice. If the order parameter is perfectly "anti-ferromagnetically" correlated at all time, the ground state of $H_1(t)$ are $|\Omega\rangle(t)$ on the even sublattice and its conjugate $|\bar{\Omega}(t)\rangle = R|\Omega(t)\rangle^*$ on the odd sublattice, where R is the operator that satisfies $RL_{ab}^* R^{-1} = -L_{ab}$. For the vector representation, R is given by $R_{15} = -R_{24} = R_{33} = -R_{42} = R_{51} = -1$ and $R_{ij} = 0$ otherwise. For the spinor representation, $R = -iI_{2\times 2} \otimes \sigma_y$. Using the above result, it is straightforward to show that $\langle\Omega|L_{ab}|\Omega\rangle = -\langle\bar{\Omega}|L_{ab}|\bar{\Omega}\rangle$, and $RU^*R^{-1} = U$ for all $U \in$ SO(5). This, plus the invariance of the trace under matrix transposition, allows one to show that $\epsilon^{\mu\nu}\text{Tr}\bar{Q}\partial_\mu\bar{Q}\partial_\nu\bar{Q} = -\epsilon^{\mu\nu}\text{Tr}Q\partial_\mu Q\partial_\nu Q$. As the result, the Berry's phases associated with neighboring sites tend to cancel each other. Let r label the center of mass position of site i and $i+1$ for $i =$ odd, the total lattice Berry's phase is equal to

$$S_B^{\text{tot}} = \sum_r \sum_{\epsilon=\pm 1} (-1)^{\epsilon-1} \int_0^1 du \int dt \text{Tr}(Q_{r+\epsilon/2}[\partial_u Q_{r+\epsilon/2}, \partial_t Q_{r+\epsilon/2}]), \tag{21}$$

where Q is a smooth function of spacial coordinates. Under such a condition, Eq. (21) has a continuum limit

$$S_B^{\rm tot} = \frac{1}{2}\int dx \int dt {\rm Tr} Q[\partial_t Q, \partial_x Q], \tag{22}$$

the factor of 1/2 arises from the density of odd lattice sites. A similar expression for the SU(N) spin chains has been presented in Ref.[11]. Under open boundary condition, Eq. (22) becomes

$$\begin{aligned} S_B^{\rm tot} =& \frac{1}{2}\int dx dt {\rm Tr} Q[\partial_t, Q\partial_x Q] \\ &+\frac{1}{2}\int_0^1 du \int dt \left\{ {\rm Tr}[Q[\partial_u, Q\partial_t Q]]_R - {\rm Tr}[Q[\partial_u Q, \partial_t Q]]_L \right\}, \end{aligned} \tag{23}$$

where the subscript "R" and "L" labels the right and the left ends. This topological term together with the stiffness term from the energetics, constitute the NLσ model for the SO(5) spin chain.

Eqs. (20) and (22) have important implications. The fact that the mapping from the space-time to the target space is classified by integer homotopy classes implies that the space-time order parameter configurations can be grouped into topological classes distinguished by an integer topological invariant. This is similar to the SO(3) NLσ model where the topological invariant, the Pontryagin index[13], is the number of times which the order parameter configurations cover the target space S^2. In our case there is an analogous integer topological index, which we will refer to as the Pontryagin index as well. Eq. (20) implies that for the vector SO(5) spin chain the Berry phase associated with the order parameter configuration having different Pontryagin indices are all the same because $\exp(i4\pi/2 \times \text{integer}) = +1$. Given the facts that (i) the topological term has no effect (hence the NLσ model has only the stiffness terms), and (ii) the target space dimension is high, it is easy to believe that the vector SO(5) spin chain should have a quantum disordered, i.e., translational invariant gapped, phase. For the spinor SO(5) chain, however, the order parameter configurations with even Pontryagin index have the Berry phase $\exp(i2\pi/2 \times \text{even integer}) = +1$, while those with odd Pontryagin index have Berry phase $\exp(i2\pi/2 \times \text{odd integer}) = -1$. This result is exactly analogous to the Berry's phase in the spin-1/2 representation of the SO(3) antiferromagnetic Heisenberg chain. In view of the result of Ref.[3], we conclude that the above non-trivial Berry's phase also implies the lack of an energy gap as long as the translation symmetry is unbroken.

Now we comment on the edge state of the SO(5) "valence bond solid state" in

Ref.[6]. By tuning the ratio of J_1 and J_2 in Eq.(5), Tu et al were able to show that a short-range entangled, translational invariant matrix product state is the exact ground state. In addition, under the open boundary condition the ground state wavefunction becomes $4 \times 4 = 16$ fold denervate. According to Eq. (23), the boundary of a vector spin chain should exhibit the following Berry phase

$$\frac{1}{2}\int_0^1 du \int dt \mathrm{Tr} Q[\partial_u, Q\partial_t Q]. \tag{24}$$

When $Q(t, u)$ is a unit Pontryagin index order parameter configuration, the value of Eq.(24) is $\frac{1}{2}4\pi = 2\pi$. This is consistent with the spinor Berry phase in Eq. (20). Therefore the edge state of the vector SO(5) spin chain carries the spinor representation. Because the latter is 4-dimensional, each end of the chain independently yields a 4-fold degeneracy of the ground state, resulting in a total $4 \times 4 = 16$ fold degenerate ground state for the open chain.

Before closing a technical remark is in order. The readers might wonder what if the ratio between different components of m_{ab} fluctuates away from the the optimal value. When that happens the single site spectrum will become $\{-\Delta_1, -\Delta_2, 0, \Delta_2, \Delta_1\}$ and $\{-\Delta_1, -\Delta_2, \Delta_2, \Delta_1\}$ for the vector and spinor representation respectively.In this case the SO(5) symmetry is broken down to SO(2)×SO(2). The second homotopy group of $\frac{SO(5)}{SO(2)\times SO(2)}$ is $Z \oplus Z$ rather than Z. In other words, the image of the space-time in the target space is topologically the multiple of two basic spheres. As $\Delta_2 \to 0$, one of these spheres shrinks to a point. We have checked that so long as Δ_2 is small, i.e., when the ratio between m_{ab} does not deviate from the optimal value too drastically, the Berry phase is only sensitive to the Pontryagin index of the dominant (large) sphere. Hence all the results discussed earlier remain unchanged.

5 Conclusion

We have studied the Berry's phase of the antiferromagnetic SO(5) spin chain, and shown the existence of a topological term in the non-linear σ model description of the system. The quantum phase factor associated with this topological term differentiates the vector (linear) and spinor (projective) representations. We argue that this leads to the spectral difference as long as the translation symmetry is unbroken. More specifically, the vector spin chain can have a totally symmetric ground state while having an energy gap. The spinor chain, on the other hand, must be gapless if there is no symmetry breaking. Under the open boundary condition, we find the boundary Berry's phase of the vector spin chain is consistent with the derived edge degeneracy of an exactly solvable model. The present result can be straightforwardly generalized to other irreducible representations, leading to two classes of SO(5) spin chain: in one class the site representation is linear, and in the other the site representation is projective. While the first class can have a totally symmetric ground state while maintaining a gapped spectrum, the second class must have gapless spectrum if there is no symmetry breaking. This result generalizes Haldane's seminal works[2] to a higher rank Lie group.

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