

A PEDAGOGICAL INTRODUCTION TO THE SLAVNOV FORMULATION OF QUANTUM YANG–MILLS THEORY

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Abstract

Over the last few years, Slavnov has proposed a formulation of quantum Yang–Mills theory in the Coulomb gauge which preserves simultaneously manifest Lorentz invariance and gauge invariance of the ghost field Lagrangian. This paper presents in detail some of the necessary calculations, i.e. those dealing with the functional integral for the S-matrix and its invariance under shifted gauge transformations. The extension of this formalism to quantum gravity in the Prentki gauge deserves consideration.

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I. INTRODUCTION

Within the framework of a global approach to quantum theory [1], the path integrals of ordinary quantum mechanics [2] and the functional integrals of quantum field theory remain a tool of fundamental importance and well suited for a Lagrangian, and hence fully relativistic, quantization. In the sixties, the work of Refs. [3–5] led to a unified description of quantum Yang–Mills and quantum general relativity, and deeper foundations were developed along the years until recent times [1]. More recently, new formal developments were obtained in Ref. [6], i.e.

(i) A formulation of quantum Yang–Mills theory which is manifestly Lorentz invariant and leads to gauge invariance of the ghost-field Lagrangian.

(ii) The problem of Gribov copies [7] is avoided.

(iii) Perturbative renormalization still holds [8], despite the occurrence of a propagator that does not decrease at infinity sufficiently fast [6].

The starting point of the analysis in Ref. [6] is the functional-integral representation of the S-matrix in the Coulomb gauge for an $SU(2)$ gauge model, i.e.

$$S = \int d\mu \exp \left\{ i \int [L_{YM} + \lambda^a \partial_j A_j^a] dx \right\} \quad (1.1)$$

where the measure $d\mu$ includes the Faddeev–Popov determinant [5], and L_{YM} is the standard Yang–Mills Lagrangian

$$L_{YM} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a, \quad (1.2)$$

built from the field strength

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc} A_\mu^b A_\nu^c. \quad (1.3)$$

Unlike the Abelian case, gauge invariance of the action in the integral (1.1) is broken not only by the gauge-fixing term but also by the Faddeev–Popov [5] ghost Lagrangian. To avoid this, the author of Ref. [6] has proposed consideration of the functional integral

$$S = \int \exp \left\{ i \int [L_{YM} + (D_\mu \varphi^*)(D_\mu \varphi) - (D_\mu \chi^*)(D_\mu \chi) + (D_\mu b^*)(D_\mu e) + (D_\mu e^*)(D_\mu b)] dx \right\} \delta(\partial_j A_j) d\gamma, \quad (1.4)$$

where the (formal) measure $d\gamma$ differs from $d\mu$ in (1.1) by the product of differentials of the scalar fields

$$(\varphi, \varphi^*, \chi, \chi^*, b, b^*, e, e^*),$$

where the fields φ, χ are commuting while b, e are anticommuting.

On making in the integral (1.4) the shift of integration variables

$$\varphi \rightarrow \varphi + \frac{\hat{a}}{g}, \quad (1.5)$$

$$\chi \rightarrow \chi - \frac{\hat{a}}{g}, \quad (1.6)$$

where

$$\hat{a} \equiv \left(0, \frac{a}{\sqrt{2}}\right), \quad (1.7)$$

a being a constant parameter, the formula (1.4) yields

$$\begin{aligned} \tilde{S} = & \int \exp \left\{ i \int \left[L_{YM} + (D_\mu \varphi^*)(D_\mu \varphi) + \frac{1}{g}(D_\mu \varphi^*)(D_\mu \hat{a}) \right. \right. \\ & + \frac{1}{g}(D_\mu \hat{a}^*)(D_\mu \varphi) + \frac{1}{g}(D_\mu \chi^*)(D_\mu \hat{a}) + \frac{1}{g}(D_\mu \hat{a}^*)(D_\mu \chi) \\ & \left. \left. - (D_\mu \chi^*)(D_\mu \chi) + (D_\mu b^*)(D_\mu e) + (D_\mu e^*)(D_\mu b) \right] dx \right\} \delta(\partial_j A_j) d\gamma. \end{aligned} \quad (1.8)$$

Interestingly, using different types of scalars for the kinetic terms, (φ, χ) being commuting and (b, e) being anticommuting, ensures that the modified theory is equivalent to the original Yang–Mills model. Moreover, the action in the exponent (1.8) turns out to be invariant under the shifted gauge transformations [6]

$$\delta A_\mu^a = \partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c, \quad (1.9)$$

$$\delta \varphi^0 = \frac{g}{2} \varphi^a \eta^a, \quad (1.10)$$

$$\delta \varphi^a = -\frac{a \eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \varphi^b \eta^c - \frac{g}{2} \varphi^0 \eta^a, \quad (1.11)$$

$$\delta \chi^a = \frac{a \eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \chi^b \eta^c - \frac{g}{2} \chi^0 \eta^a, \quad (1.12)$$

$$\delta \chi^0 = \frac{g}{2} \chi^a \eta^a, \quad (1.13)$$

$$\delta b^a = -\frac{g}{2} \varepsilon^{adc} b^d \eta^c - \frac{g}{2} b^0 \eta^a, \quad (1.14)$$

$$\delta b^0 = \frac{g}{2} b^a \eta^a, \quad (1.15)$$

$$\delta c^a = -\frac{g}{2}\varepsilon^{adc}e^d\eta^c, \quad (1.16)$$

$$\delta e^0 = \frac{g}{2}e^a\eta^a, \quad (1.17)$$

where the scalar field φ has been represented in terms of Hermitian components in the form [6]

$$\varphi = \left(\frac{i\varphi_1 + \varphi_2}{\sqrt{2}}, \frac{\varphi_0 - i\varphi_3}{\sqrt{2}} \right), \quad (1.18)$$

and the same for χ .

Section II proves equivalence of two functional-integral formulae for the S-matrix in the Coulomb gauge. Section III studies the behaviour of the S-matrix under shift of integration variables. Section IV contains a detailed proof of invariance of the S-matrix under the shifted gauge transformations (1.9)–(1.17). Concluding remarks and open problems are presented in Sec. V.

II. EQUIVALENT FUNCTIONAL-INTEGRAL FORMULAE FOR THE S-MATRIX IN THE COULOMB GAUGE

We start from Eq. (1.4) and recall that, if φ and ψ are commuting and anti-commuting (Grassmann) complex scalar fields respectively, one has the following functional integral results:

$$\int d[\varphi^*]d[\varphi]e^{-\int\varphi^*B\varphi dx} = (\det B)^{-1}, \quad (2.1)$$

$$\int d[\psi^*]d[\psi]e^{-\int\psi^*B\psi dx} = (\det B), \quad (2.2)$$

where $dx \equiv d^4x$, and Wick rotation is performed when use is made of (2.1) and (2.2).

Equation (1.4) can be therefore expressed as (φ and φ^* being independent in the Euclidean regime)

$$S = S_0 \times \int d[\varphi^*]d[\varphi]e^{\int(D_\mu\varphi^*)(D^\mu\varphi)dx} \times \dots, \quad (2.3)$$

where

$$S_0 \equiv \int d[\mu]e^{i\int L_{YM}dx}\delta(\partial_j A_j). \quad (2.4)$$

Note now that

$$D_\mu\varphi^*D^\mu\varphi = D_\mu(\varphi^*D^\mu\varphi) - \varphi^*D_\mu D^\mu\varphi, \quad (2.5)$$

and the first term, being a total derivative, gives vanishing contribution by using the Stokes theorem and imposing suitable boundary conditions. Then

$$\begin{aligned}
S &= S_0 \times \int d[\varphi^*]d[\varphi]e^{\int (D_\mu\varphi^*)(D^\mu\varphi)dx} \times \int d[\chi^*]d[\chi]e^{-\int (D_\mu\chi^*)(D^\mu\chi)dx} \\
&\quad \times \int d[b^*]d[e]e^{\int (D_\mu b^*)(D^\mu e)dx} \times \int d[b]d[e]e^{\int (D_\mu e^*)(D^\mu b)dx} \\
&= S_0 \times \int d[\varphi^*]d[\varphi]e^{-\int \varphi^* D_\mu D^\mu \varphi dx} \times \int d[\chi^*]d[\chi]e^{\int \chi^* D_\mu D^\mu \chi dx} \\
&\quad \times \int d[b^*]d[e]e^{-\int b^* D_\mu D^\mu e dx} \times \int d[e^*]d[b]e^{-\int e^* D_\mu D^\mu b dx} \\
&= S_0 \times |\det D^2|^{-1} \times |\det D^2|^{-1} \times |\det D^2| \times |\det D^2| \\
&= S_0.
\end{aligned} \tag{2.6}$$

Note that the independence of the fields χ and χ^* can be exploited to consider the rotation $\chi \rightarrow i\chi$, $\chi^* \rightarrow i\chi^*$.

III. BEHAVIOUR OF THE S-MATRIX UNDER SHIFT OF INTEGRATION VARIABLES

In Eq. (1.4) we now perform the shift of integration variables described by (1.5)–(1.7), and then point out that

$$\begin{aligned}
(D_\mu\varphi^*)(D^\mu\varphi) &\rightarrow ((D_\mu\varphi^*) + g^{-1}(D_\mu\hat{a}^*)) (D^\mu\varphi + g^{-1}D^\mu\hat{a}) \\
&= (D_\mu\varphi^*) D^\mu\varphi + g^{-1}(D_\mu\hat{a}^*) D^\mu\varphi + g^{-1}(D_\mu\varphi^*) (D^\mu\hat{a}) \\
&\quad + g^{-2}(D_\mu\hat{a}^*) D^\mu\hat{a}.
\end{aligned} \tag{3.1}$$

Thus, if $g^{-2} \rightarrow 0$, one finds

$$(D_\mu\varphi^*)(D^\mu\varphi) \rightarrow (D_\mu\varphi^*)(D^\mu\varphi) + g^{-1}(D_\mu\hat{a}^*)(D^\mu\varphi) + g^{-1}(D_\mu\varphi^*)(D^\mu\hat{a}), \tag{3.2}$$

$$(D_\mu\chi^*)(D^\mu\chi) \rightarrow (D_\mu\chi^*)(D^\mu\chi) - g^{-1}(D_\mu\hat{a}^*)(D^\mu\chi) - g^{-1}(D_\mu\chi^*)(D^\mu\hat{a}), \tag{3.3}$$

and the new action defined in Eq. (1.4) transforms as

$$\begin{aligned}
S \rightarrow \tilde{S} &= \int \exp\{i \int [L_{YM} + (D_\mu\varphi^*)(D^\mu\varphi) + g^{-1}(D_\mu\hat{a}^*)(D^\mu\varphi) + g^{-1}(D_\mu\varphi^*)(D^\mu\hat{a}) \\
&\quad - (D_\mu\chi^*)(D^\mu\chi) + g^{-1}(D_\mu\hat{a}^*)(D^\mu\chi) + g^{-1}(D_\mu\chi^*)(D^\mu\hat{a}) \\
&\quad + (D_\mu b^*)(D^\mu e) + (D_\mu e^*)(D^\mu b)] dx \} \delta(\partial_j A_j) d\gamma.
\end{aligned} \tag{3.4}$$

IV. INVARIANCE OF THE S-MATRIX UNDER SHIFTED GAUGE TRANSFORMATIONS

We are aiming to show that the transformed action in Eq. (3.4) is invariant under the shifted gauge transformations (1.9)–(1.17). To begin, note that the covariant derivative in the spinor representation reads as

$$D^\mu \varphi = \partial^\mu \varphi - ig\sigma^a A^{a\mu} \varphi, \quad (4.1)$$

$$D_\mu \varphi^* = \partial_\mu \varphi^* + ig\varphi^* \sigma^a A_\mu^a, \quad (4.2)$$

$$\begin{aligned} (D_\mu \varphi^*) (D^\mu \varphi) &= (\partial_\mu \varphi^* + ig\varphi^* \sigma^a A_\mu^a) (\partial^\mu \varphi - ig\sigma^a A^{a\mu} \varphi) \\ &= \partial_\mu \varphi^* \partial^\mu \varphi + ig(\varphi^* \sigma^a \partial^\mu \varphi) A_\mu^a - ig(\partial_\mu \varphi^* \sigma^a \varphi) A^{a\mu} \\ &\quad + g^2 (\varphi^* \sigma^a \sigma^b \varphi) A_\mu^a A^{b\mu}. \end{aligned} \quad (4.3)$$

The Pauli matrices, φ and φ^* are all Hermitian in the Minkowskian regime, therefore

$$\varphi^* \sigma^a \partial^\mu \varphi = (\varphi^* \sigma^a \partial^\mu \varphi)^\dagger = \partial^\mu \varphi^* \sigma^a \varphi. \quad (4.4)$$

In the Euclidean regime, however, which is necessary to obtain well defined functional integrals, φ and φ^* become independent (see also comment before (2.3)), not related by any conjugation, despite being denoted in the same way. Moreover, we exploit the identities

$$\varphi^* \sigma^a \sigma^b \varphi = \varphi^* \mathbf{I} \delta^{ab} \varphi + i\varepsilon^{abc} \varphi^* \sigma^c \varphi, \quad (4.5)$$

$$\varepsilon^{abc} A_\mu^a A^{b\mu} = 0, \quad (4.6)$$

and hence

$$(D_\mu \varphi^*) (D^\mu \varphi) = (\partial_\mu \varphi^*) (\partial^\mu \varphi) + g^2 \varphi^* \varphi A_\mu^a A^{a\mu}. \quad (4.7)$$

We notice also that

$$\partial_\mu \varphi^* \partial^\mu \varphi = (\partial_\mu \varphi^a) (\partial^\mu \varphi^a) + (\partial_\mu \varphi^0) (\partial^\mu \varphi^0), \quad (4.8)$$

$$\varphi^* \varphi = \varphi^a \varphi^a + \varphi^0 \varphi^0, \quad (4.9)$$

which implies

$$(D_\mu \varphi^*) (D^\mu \varphi) = \partial_\mu \varphi^a \partial^\mu \varphi^a + \partial_\mu \varphi^0 \partial^\mu \varphi^0 + g^2 (\varphi^a \varphi^a + \varphi^0 \varphi^0) A_\mu^a A^{a\mu}, \quad (4.10)$$

and similarly

$$(D_\mu \chi^*) (D^\mu \chi) = \partial_\mu \chi^a \partial^\mu \chi^a + \partial_\mu \chi^0 \partial^\mu \chi^0 + g^2 (\chi^a \chi^a + \chi^0 \chi^0) A_\mu^a A^{a\mu}. \quad (4.11)$$

Let us now explicitly express the other terms in the action Eq. (3.4), i.e.

$$\begin{aligned} g^{-1}(D_\mu \hat{a})^* (D^\mu \varphi) &= g^{-1}(\partial_\mu \hat{a}^* + ig \hat{a}^* \sigma^a A_\mu^a)(\partial^\mu \varphi - ig \sigma^a A^{a\mu} \varphi) \\ &= i \hat{a}^* \sigma^a A_\mu^a (\partial^\mu \varphi - ig \sigma^a A^{a\mu} \varphi) \\ &= i (\hat{a}^* \sigma^a \partial^\mu \varphi) A_\mu^a + g (\hat{a}^* \sigma^a \sigma^b \varphi) A_\mu^a A^{b\mu} \\ &= i (\hat{a}^* \sigma^a \partial^\mu \varphi) A_\mu^a + g (\hat{a}^* \varphi) A_\mu^a A^{a\mu}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} g^{-1}(D_\mu \varphi^*) (D^\mu \hat{a}) &= g^{-1}(\partial_\mu \varphi^* + ig \varphi^* \sigma^a A_\mu^a)(\partial^\mu \hat{a} - ig \sigma^a A^{a\mu} \hat{a}) \\ &= -i (\partial_\mu \varphi^* \sigma^a \hat{a}) A^{a\mu} + g (\varphi^* \sigma^a \sigma^b \hat{a}) A_\mu^a A^{b\mu} \\ &= -i (\partial_\mu \varphi^* \sigma^a \hat{a}) A^{a\mu} + g (\varphi^* \hat{a}) A_\mu^a A^{a\mu}. \end{aligned} \quad (4.13)$$

At this stage, the Hermiticity condition

$$(\hat{a}^* \sigma^a \partial^\mu \varphi) = (\hat{a}^* \sigma^a \partial^\mu \varphi)^\dagger = (\partial_\mu \varphi^* \sigma^a \hat{a}) \quad (4.14)$$

and the previous formulae lead to

$$\begin{aligned} g^{-1}(D_\mu \hat{a})^* (D^\mu \varphi) + g^{-1}(D_\mu \varphi^*) (D^\mu \hat{a}) &= g (\hat{a}^* \varphi) A_\mu^a A^{a\mu} + g (\varphi^* \hat{a}) A_\mu^a A^{a\mu} \\ &= g A_\mu^a A^{a\mu} (\hat{a}^* \varphi + \varphi^* \hat{a}) = ag \varphi^0 A_\mu^a A^{a\mu}, \end{aligned}$$

which implies

$$g^{-1}(D_\mu \hat{a})^* (D^\mu \varphi) + g^{-1}(D_\mu \varphi^*) (D^\mu \hat{a}) = ag \varphi^0 A_\mu^a A^{a\mu}. \quad (4.15)$$

In the same way one gets

$$(D_\mu b^*) (D^\mu e) + (D_\mu e^*) (D^\mu b) = \partial_\mu b^* \partial^\mu e + \partial_\mu e^* \partial^\mu b + g^2 (b^* e + e^* b) A_\mu^a A^{a\mu}, \quad (4.16)$$

or, equivalently,

$$(D_\mu b^*) (D^\mu e) + (D_\mu e^*) (D^\mu b) = \partial_\mu b^0 \partial^\mu e^0 + \partial_\mu e^i \partial^\mu b^i + g^2 (b^0 e^0 + b^i e^i) A_\mu^a A^{a\mu}. \quad (4.17)$$

By virtue of (4.10), (4.11), (4.15) and (4.17) the action in (3.4) takes the form

$$L_{\text{tot}} = L_{YM} + (D_\mu \varphi^*) (D^\mu \varphi) + g^{-1}(D_\mu \hat{a}^*) (D^\mu \varphi) + g^{-1}(D_\mu \varphi^*) (D^\mu \hat{a})$$

$$\begin{aligned}
& -(D_\mu \chi^*)(D^\mu \chi) + g^{-1}(D_\mu \hat{a}^*)(D^\mu \chi) + g^{-1}(D_\mu \chi^*)(D^\mu \hat{a}) \\
& + (D_\mu b^*)(D^\mu e) + (D_\mu e^*)(D^\mu b) \\
= & -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} \\
& + (\partial_\mu \varphi^a \partial^\mu \varphi^a + \partial_\mu \varphi^0 \partial^\mu \varphi^0) + g^2 (\varphi^a \varphi^a + \varphi^0 \varphi^0) A_\mu^a A^{a\mu} \\
& + ag\varphi^0 A_\mu^a A^{a\mu} \\
& - (\partial_\mu \chi^a \partial^\mu \chi^a + \partial_\mu \chi^0 \partial^\mu \chi^0) - g^2 (\chi^a \chi^a + \chi^0 \chi^0) A_\mu^a A^{a\mu} \\
& + ag\chi^0 A_\mu^a A^{a\mu} \\
& + (\partial_\mu b^0 \partial^\mu e^0 + \partial^\mu b^a \partial_\mu e^a) + g^2 (b^0 e^0 + b^a e^a) A_\mu^c A^{c\mu}. \tag{4.18}
\end{aligned}$$

Now we just list the equations of motion for all fields in the action pertaining to the Lagrangian (4.18), i.e. $\partial_\mu \frac{\partial L_{\text{tot}}}{\partial(\partial_\mu A_\nu^a)} - \frac{\partial L_{\text{tot}}}{\partial(A_\nu^a)} = 0 \rightarrow$

$$D_\mu F^{c\mu\nu} = -2g^2 A^{c\nu} (\varphi^a \varphi^a + \chi^a \chi^a + b^a e^a + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0) - 2agA^{c\nu} (\varphi^0 + \chi^0), \tag{4.19}$$

$$\begin{aligned}
\partial_\mu \frac{\partial L_{\text{tot}}}{\partial(\partial_\mu \varphi^a)} - \frac{\partial L_{\text{tot}}}{\partial(\varphi^a)} = 0 \rightarrow \\
\partial^2 \varphi^a = g^2 \varphi^a A_\mu^b A^{b\mu}, \tag{4.20}
\end{aligned}$$

$$\partial^2 \varphi^0 = \left(g^2 \varphi^0 + \frac{ag}{2} \right) A_\mu^a A^{a\mu}, \tag{4.21}$$

$$\partial^2 \chi^a = g^2 \chi^a A_\mu^b A^{b\mu}, \tag{4.22}$$

$$\partial^2 \chi^0 = \left(g^2 \chi^0 - \frac{ag}{2} \right) A_\mu^a A^{a\mu}, \tag{4.23}$$

$$\partial^2 e^a = g^2 e^a A_\mu^b A^{b\mu}, \tag{4.24}$$

$$\partial^2 e^0 = g^2 e^0 A_\mu^b A^{b\mu}, \tag{4.25}$$

$$\partial^2 b^a = g^2 b^a A_\mu^c A^{c\mu}, \tag{4.26}$$

$$\partial^2 b^0 = g^2 b^0 A_\mu^c A^{c\mu}. \tag{4.27}$$

The variation of the action obtained from Eq. (4.18) is

$$\begin{aligned}
\delta L_{\text{tot}} = & \delta \left(-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} \right) \\
& + 2 (\partial_\mu \delta \varphi^a \partial^\mu \varphi^a + \partial_\mu \delta \varphi^0 \partial^\mu \varphi^0) + 2g^2 (\delta \varphi^a \varphi^a + \delta \varphi^0 \varphi^0) A_\mu^a A^{a\mu} \\
& + 2g^2 (\varphi^a \varphi^a + \varphi^0 \varphi^0) \delta A_\mu^a A^{a\mu} \\
& + ag\delta \varphi^0 A_\mu^a A^{a\mu} + 2ag\varphi^0 \delta A_\mu^a A^{a\mu}
\end{aligned}$$

$$\begin{aligned}
& -2 (\partial_\mu \delta \chi^a \partial^\mu \chi^a + \partial_\mu \delta \chi^0 \partial^\mu \chi^0) - 2g^2 (\delta \chi^a \chi^a + \delta \chi^0 \chi^0) A_\mu^a A^{a\mu} \\
& \quad - 2g^2 (\chi^a \chi^a + \chi^0 \chi^0) \delta A_\mu^a A^{a\mu} \\
& + ag \delta \chi^0 A_\mu^a A^{a\mu} + 2ag \chi^0 \delta A_\mu^a A^{a\mu} \\
& + (\partial_\mu \delta b^0 \partial^\mu e^0 + \partial_\mu b^0 \partial^\mu \delta e^0 + \partial^\mu b^d \partial_\mu \delta e^d + \partial^\mu \delta b^d \partial_\mu e^d) \\
& \quad + g^2 (\delta b^0 e^0 + b^0 \delta e^0 + \delta b^i e^i + b^i \delta e^i) A_\mu^a A^{a\mu} \\
& \quad + 2g^2 (b^0 e^0 + b^i e^i) \delta A_\mu^a A^{a\mu}. \tag{4.28}
\end{aligned}$$

Variation of the gauge-field Lagrangian L_{YM} , separately, is as follows:

$$\delta L_{YM} = -\frac{1}{4} \delta (F_{\mu\nu}^a F^{\mu\nu a}) = -\frac{1}{2} F^{\alpha\mu\nu} \delta F_{\mu\nu}^a, \tag{4.29}$$

$$\delta F_{\mu\nu}^a = \partial_\mu \delta A_\nu^a - \partial_\nu \delta A_\mu^a + g \varepsilon^{abc} \delta A_\mu^b A_\nu^c + g \varepsilon^{abc} A_\mu^b \delta A_\nu^c, \tag{4.30}$$

$$\begin{aligned}
\implies F^{\alpha\mu\nu} \delta F_{\mu\nu}^a &= F^{\alpha\mu\nu} (\partial_\mu \delta A_\nu^a - \partial_\nu \delta A_\mu^a + g \varepsilon^{abc} \delta A_\mu^b A_\nu^c + g \varepsilon^{abc} A_\mu^b \delta A_\nu^c) \\
&= 2F^{\alpha\mu\nu} (\partial_\mu \delta A_\nu^a + g \varepsilon^{abc} A_\mu^b \delta A_\nu^c), \tag{4.31}
\end{aligned}$$

$$\delta A_\nu^a = \partial_\nu \eta^a - g \varepsilon^{abc} A_\nu^b \eta^c, \tag{4.32}$$

$$\begin{aligned}
\delta L_{YM} &= -F^{\alpha\mu\nu} (\partial_\mu (\partial_\nu \eta^a - g \varepsilon^{abc} A_\nu^b \eta^c) + g \varepsilon^{abc} A_\mu^b (\partial_\nu \eta^c - g \varepsilon^{cde} A_\nu^d \eta^e)) \\
&= -F^{\alpha\mu\nu} (\partial_\mu \partial_\nu \eta^a - g \varepsilon^{abc} \partial_\mu (A_\nu^b \eta^c) \\
& \quad + g \varepsilon^{abc} A_\mu^b \partial_\nu \eta^c - g^2 \varepsilon^{abc} \varepsilon^{cde} A_\mu^b A_\nu^d \eta^e). \tag{4.33}
\end{aligned}$$

In light of the identities

$$F^{\alpha\mu\nu} \partial_\mu \partial_\nu \eta^a = 0, \tag{4.34}$$

$$\varepsilon^{abc} \varepsilon^{cde} = \varepsilon^{cab} \varepsilon^{cde} = \delta^{ae} \delta^{bd} - \delta^{ad} \delta^{be}, \tag{4.35}$$

one finds eventually

$$\delta L_{YM} = g \varepsilon^{abc} F^{\alpha\mu\nu} \partial_\mu (A_\nu^b \eta^c) - g \varepsilon^{abc} F^{\alpha\mu\nu} A_\mu^b \partial_\nu \eta^c + g^2 F^{\alpha\mu\nu} A_\mu^b A_\nu^a \eta^b, \tag{4.36}$$

Thus, Eqs. (1.9)–(1.17) and (3.4) yield

$$\begin{aligned}
\delta L_{\text{tot}} &= g \varepsilon^{abc} F^{\alpha\mu\nu} \partial_\mu (A_\nu^b \eta^c) - g \varepsilon^{abc} F^{\alpha\mu\nu} A_\mu^b \partial_\nu \eta^c + g^2 F^{\alpha\mu\nu} A_\mu^b A_\nu^a \eta^b \\
& \quad + 2\partial_\mu \left(-\frac{a\eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \varphi^b \eta^c - \frac{g}{2} \varphi^0 \eta^a \right) \partial^\mu \varphi^a + 2\partial_\mu \left(\frac{g}{2} \varphi^a \eta^a \right) \partial^\mu \varphi^0
\end{aligned}$$

$$\begin{aligned}
& +2g^2 A_\mu^d A^{d\mu} \left(-\frac{a\eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \varphi^b \eta^c - \frac{g}{2} \varphi^0 \eta^a \right) \varphi^a + 2g^2 A_\mu^b A^{b\mu} \left(\frac{g}{2} \varphi^a \eta^a \right) \varphi^0 \\
& +2g^2 (\varphi^d \varphi^d + \varphi^0 \varphi^0) (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& +ag \left(\frac{g}{2} \varphi^a \eta^a \right) A_\mu^b A^{b\mu} + 2ag \varphi^0 (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& -2\partial_\mu \left(\frac{a\eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \chi^b \eta^c - \frac{g}{2} \chi^0 \eta^a \right) \partial^\mu \chi^a - 2\partial_\mu \left(\frac{g}{2} \chi^a \eta^a \right) \partial^\mu \chi^0 \\
& -2g^2 \left(\frac{a\eta^a}{2} - \frac{g}{2} \varepsilon^{abc} \chi^b \eta^c - \frac{g}{2} \chi^0 \eta^a \right) \chi^a A_\mu^d A^{d\mu} - 2g^2 \left(\frac{g}{2} \chi^a \eta^a \right) \chi^0 A_\mu^b A^{b\mu} \\
& +2g^2 (\chi^d \chi^d + \chi^0 \chi^0) (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& +ag \left(\frac{g}{2} \chi^a \eta^a \right) A_\mu^b A^{b\mu} + 2ag \chi^0 (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& +\partial_\mu \left(\frac{g}{2} b^a \eta^a \right) \partial^\mu e^0 + \partial_\mu b^0 \partial^\mu \left(\frac{g}{2} e^a \eta^a \right) + \partial^\mu b^a \partial_\mu \left(-\frac{g}{2} \varepsilon^{adc} e^d \eta^c \right) \\
& +\partial^\mu \left(-\frac{g}{2} \varepsilon^{adc} b^d \eta^c - \frac{g}{2} b^0 \eta^a \right) \partial_\mu e^a \\
& +g^2 A_\mu^b A^{b\mu} \left(\frac{g}{2} b^a \eta^a \right) e^0 + g^2 A_\mu^b A^{b\mu} b^0 \left(\frac{g}{2} e^a \eta^a \right) \\
& +g^2 A_\mu^b A^{b\mu} \left(-\frac{g}{2} \varepsilon^{adc} b^d \eta^c - \frac{g}{2} b^0 \eta^a \right) e^a + g^2 A_\mu^b A^{b\mu} b^a \left(-\frac{g}{2} \varepsilon^{adc} e^d \eta^c \right) \\
& +2g^2 (b^0 e^0 + b^d e^d) (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu}. \tag{4.37}
\end{aligned}$$

This is a huge Lagrangian variation that we should show is identically vanishing! For this purpose, we write in detail all terms in the form

$$\begin{aligned}
\delta L_{\text{tot}} &= g \varepsilon^{abc} F^{a\mu\nu} \partial_\mu (A_\nu^b \eta^c) - g \varepsilon^{abc} F^{a\mu\nu} A_\mu^b \partial_\nu \eta^c + g^2 F^{a\mu\nu} A_\mu^b A_\nu^a \eta^b \\
& - \underbrace{a \partial_\mu \eta^a \partial^\mu \varphi^a}_{\text{}} - \underbrace{g \varepsilon^{abc} \partial_\mu (\varphi^b \eta^c) \partial^\mu \varphi^a}_{\text{}} - \underbrace{g \partial_\mu (\varphi^0 \eta^a) \partial^\mu \varphi^a}_{\text{}} + \underbrace{g \partial_\mu (\varphi^a \eta^a) \partial^\mu \varphi^0}_{\text{}} \\
& - \underbrace{a g^2 A_\mu^b A^{b\mu} \varphi^a \eta^a}_{\text{}} - \underbrace{g^3 \varepsilon^{abc} A_\mu^d A^{d\mu} \varphi^a \varphi^b \eta^c}_{\text{}} - \underbrace{g^3 A_\mu^b A^{b\mu} \varphi^a \eta^a \varphi^0}_{\text{}} + \underbrace{g^3 A_\mu^b A^{b\mu} \varphi^a \eta^a \varphi^0}_{\text{}} \\
& +2g^2 (\varphi^d \varphi^d + \varphi^0 \varphi^0) (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& + \underbrace{\frac{ag^2}{2} A_\mu^b A^{b\mu} \varphi^a \eta^a}_{\text{}} + 2ag A^{a\mu} \partial_\mu \eta^a \varphi^0 - \underbrace{2ag^2 \varepsilon^{abc} A^{a\mu} A_\mu^b \eta^c \varphi^0}_{\text{}} \\
& - \underbrace{a \partial_\mu \eta^a \partial^\mu \chi^a}_{\text{}} + \underbrace{g \varepsilon^{abc} \partial_\mu (\chi^b \eta^c) \partial^\mu \chi^a}_{\text{}} + \underbrace{g \partial_\mu (\chi^0 \eta^a) \partial^\mu \chi^a}_{\text{}} - \underbrace{g \partial_\mu (\chi^a \eta^a) \partial^\mu \chi^0}_{\text{}} \\
& - \underbrace{a g^2 \chi^a \eta^a A_\mu^b A^{b\mu}}_{\text{}} + \underbrace{g^3 \varepsilon^{abc} \chi^a \chi^b \eta^c A_\mu^d A^{d\mu}}_{\text{}} + \underbrace{g^3 \chi^a \eta^a \chi^0 A_\mu^d A^{d\mu}}_{\text{}} - \underbrace{g^3 \chi^a \eta^a \chi^0 A_\mu^b A^{b\mu}}_{\text{}} \\
& +2g^2 (\chi^d \chi^d + \chi^0 \chi^0) (\partial_\mu \eta^a - g \varepsilon^{abc} A_\mu^b \eta^c) A^{a\mu} \\
& + \underbrace{\frac{ag^2}{2} A_\mu^b A^{b\mu} \chi^a \eta^a}_{\text{}} + 2ag \chi^0 A^{a\mu} \partial_\mu \eta^a - \underbrace{2ag^2 \varepsilon^{abc} A^{a\mu} A_\mu^b \eta^c \chi^0}_{\text{}} \\
& + \underbrace{\frac{g}{2} \partial_\mu (b^a \eta^a) \partial^\mu e^0}_{\text{}} + \underbrace{\frac{g}{2} \partial_\mu b^0 \partial^\mu (e^a \eta^a)}_{\text{}} - \underbrace{\frac{g}{2} \varepsilon^{adc} \partial^\mu b^a \partial_\mu (e^d \eta^c)}_{\text{}}
\end{aligned}$$

$$\begin{aligned}
& -\underbrace{\frac{g}{2}\varepsilon^{adc}\partial^\mu(b^d\eta^c)\partial_\mu e^a}_{T_1} - \underbrace{\frac{g}{2}\partial^\mu(b^0\eta^a)\partial_\mu e^a}_{T_2} \\
& + \underbrace{\frac{g^3}{2}A_\mu^b A^{b\mu} b^a e^0 \eta^a}_{T_3} + \underbrace{\frac{g^3}{2}A_\mu^b A^{b\mu} b^0 e^a \eta^a}_{T_4} \\
& - \underbrace{\frac{g^3}{2}\varepsilon^{adc}A_\mu^b A^{b\mu} b^d e^a \eta^c}_{T_5} - \underbrace{\frac{g^3}{2}A_\mu^b A^{b\mu} b^0 e^a \eta^a}_{T_6} - \underbrace{\frac{g^3}{2}\varepsilon^{adc}A_\mu^b A^{b\mu} b^a e^d \eta^c}_{T_7} \\
& + 2g^2(b^0 e^0 + b^d e^d)(\partial_\mu \eta^a - g\varepsilon^{abc}A_\mu^b \eta^c)A^{a\mu}.
\end{aligned} \tag{4.38}$$

The 30 terms underlined by a curly bracket in (4.38) are hereafter denoted by T_i , with i ranging from 1 through 30. To begin, note that, by virtue of (4.20),

$$\begin{aligned}
T_1 &= -a\partial_\mu \eta^a \partial^\mu \varphi^a = -\underbrace{a\partial_\mu(\eta^a \partial^\mu \varphi^a)}_{T_1} + a\eta^a \partial_\mu \partial^\mu \varphi^a \equiv a\eta^a \partial^2 \varphi^a \\
&= ag^2 \eta^a \varphi^a A_\mu^b A^{b\mu},
\end{aligned} \tag{4.39}$$

and hence

$$T_1 + T_5 = 0. \tag{4.40}$$

Moreover, again by virtue of (4.20),

$$\begin{aligned}
T_2 &= g\varepsilon^{abc}\partial_\mu(\varphi^b \eta^c)\partial^\mu \varphi^a = g\varepsilon^{abc}\partial_\mu(\varphi^b \eta^c \partial^\mu \varphi^a) - g\varepsilon^{abc}\varphi^b \eta^c \partial_\mu \partial^\mu \varphi^a \\
&\equiv -g\varepsilon^{abc}\varphi^b \eta^c \partial^2 \varphi^a = -g^3 A_\mu^b A^{b\mu} \underbrace{\varepsilon^{abc}\varphi^b \varphi^a \eta^c}_{T_2},
\end{aligned} \tag{4.41}$$

i.e.

$$T_2 = 0, \tag{4.42}$$

and

$$\begin{aligned}
T_3 &= -g\partial_\mu(\varphi^0 \eta^a)\partial^\mu \varphi^a = -g\partial_\mu(\varphi^0 \eta^a \partial^\mu \varphi^a) + g(\varphi^0 \eta^a)\partial_\mu \partial^\mu \varphi^a \equiv g\varphi^0 \eta^a \partial^2 \varphi^a \\
&= g^3 A_\mu^b A^{b\mu} \varphi^0 \eta^a \varphi^a,
\end{aligned} \tag{4.43}$$

$$T_3 + T_7 = 0. \tag{4.44}$$

The term T_4 is studied with the help of (4.21), so that

$$\begin{aligned}
T_4 &= g\partial_\mu(\varphi^a \eta^a)\partial^\mu \varphi^0 = g\partial_\mu(\varphi^a \eta^a \partial^\mu \varphi^0) - g\varphi^a \eta^a \partial_\mu \partial^\mu \varphi^0 \equiv -g\varphi^a \eta^a \partial^2 \varphi^0 \\
&= -g\varphi^a \eta^a \left(g^2 \varphi^0 + \frac{ag}{2}\right) A_\mu^a A^{a\mu},
\end{aligned} \tag{4.45}$$

$$T_4 + T_8 + T_9 = 0. \tag{4.46}$$

Furthermore,

$$T_6 = -g^3 \underbrace{\varepsilon^{abc} \varphi^a \varphi^b}_{\text{antisymmetric}} A_\mu^d A^{d\mu} \eta^c, \quad (4.47)$$

which vanishes because the antisymmetric ε^{abc} is contracted with the symmetric product $\varphi^a \varphi^b$:

$$T_6 = 0. \quad (4.48)$$

For the same reason the term T_{10} vanishes as well,

$$T_{10} = 0. \quad (4.49)$$

Now we consider the terms which include the field χ . By virtue of (4.22), one finds

$$\begin{aligned} T_{11} &= -a \partial_\mu \eta^a \partial^\mu \chi^a = -a \partial_\mu (\eta^a \partial^\mu \chi^a) + a \eta^a \partial_\mu \partial^\mu \chi^a \equiv a \eta^a \partial^2 \chi^a \\ &= a g^2 \eta^a \chi^a A_\mu^b A^{b\mu}, \end{aligned} \quad (4.50)$$

$$T_{11} + T_{15} = 0, \quad (4.51)$$

$$\begin{aligned} T_{12} &= g \varepsilon^{abc} \partial_\mu (\chi^b \eta^c) \partial^\mu \chi^a = g \varepsilon^{abc} \partial_\mu (\chi^b \eta^c \partial^\mu \chi^a) - g \varepsilon^{abc} \chi^b \eta^c \partial_\mu \partial^\mu \chi^a \equiv -g \varepsilon^{abc} \chi^b \eta^c \partial^2 \chi^a \\ &= -g^3 \underbrace{\varepsilon^{abc} \chi^a \chi^b \eta^c}_{\text{antisymmetric}} A_\mu^d A^{d\mu}, \end{aligned} \quad (4.52)$$

i.e.

$$T_{12} = 0, \quad (4.53)$$

while

$$\begin{aligned} T_{13} &= g \partial_\mu (\chi^0 \eta^a) \partial^\mu \chi^a \equiv -g \chi^0 \eta^a \partial_\mu \partial^\mu \chi^a \\ &= -g^3 A_\mu^b A^{b\mu} \eta^a \chi^a \chi^0, \end{aligned} \quad (4.54)$$

$$\begin{aligned} T_{13} + T_{14} + T_{19} &= -g^3 A_\mu^b A^{b\mu} \eta^a \chi^a \chi^0 - 2g \partial_\mu (\chi^a \eta^a) \partial^\mu \chi^0 + \frac{a g^2}{2} A_\mu^b A^{b\mu} \chi^a \eta^a \\ &= -g^3 A_\mu^b A^{b\mu} \eta^a \chi^a \chi^0 + 2g \chi^a \eta^a \partial^2 \chi^0 + \frac{a g^2}{2} A_\mu^b A^{b\mu} \chi^a \eta^a \\ &= -g^3 A_\mu^b A^{b\mu} \eta^a \chi^a \chi^0 + g \chi^a \eta^a \left(g^2 \chi^0 - \frac{a g}{2} \right) A_\mu^b A^{b\mu} + \frac{a g^2}{2} A_\mu^b A^{b\mu} \chi^a \eta^a \\ &= 0. \end{aligned} \quad (4.55)$$

Moreover

$$T_{17} + T_{18} = 0, \quad (4.56)$$

$$T_{20} = 0, \quad (4.57)$$

and we can exploit Eq. (4.25) to find

$$\begin{aligned} T_{21} &= -\frac{g}{2} \partial_\mu (b^a \eta^a) \partial^\mu e^0 \equiv -\frac{g}{2} (b^a \eta^a) \partial_\mu \partial^\mu e^0 \\ &= -\frac{g^3}{2} A_\mu^b A^{b\mu} b^a e^0 \eta^a, \end{aligned} \quad (4.58)$$

$$T_{21} + T_{26} = 0. \quad (4.59)$$

By inspection, we also find the cancellation

$$T_{27} + T_{29} = 0. \quad (4.60)$$

Now we exploit Eq. (4.27) to find

$$\begin{aligned} T_{22} &= \frac{g}{2} \partial_\mu b^0 \partial^\mu (e^a \eta^a) \equiv -\frac{g}{2} \partial^\mu \partial_\mu b^0 (e^a \eta^a) \\ &= -\frac{g^3}{2} A_\mu^c A^{c\mu} b^0 e^a \eta^a, \end{aligned} \quad (4.61)$$

$$\begin{aligned} T_{25} &= -\frac{g}{2} \partial^\mu (b^0 \eta^a) \partial_\mu e^a \equiv \frac{g}{2} b^0 \eta^a \partial^\mu \partial_\mu e^a \\ &= \frac{g^3}{2} A_\mu^b A^{b\mu} b^0 e^a \eta^a, \end{aligned} \quad (4.62)$$

$$T_{22} + T_{25} = 0. \quad (4.63)$$

At this stage, we can also exploit Eq. (4.26) to find

$$\begin{aligned} T_{23} &= -\frac{g}{2} \varepsilon^{adc} \partial^\mu b^a \partial_\mu (e^d \eta^c) \equiv \frac{g}{2} \varepsilon^{adc} \partial_\mu \partial^\mu b^a (e^d \eta^c) \\ &= \frac{g^3}{2} \varepsilon^{adc} A_\mu^b A^{b\mu} b^a e^d \eta^c, \end{aligned} \quad (4.64)$$

while Eq. (4.24) yields

$$\begin{aligned} T_{24} &= -\frac{g}{2} \varepsilon^{adc} \partial^\mu (b^d \eta^c) \partial_\mu e^a \equiv \frac{g}{2} \varepsilon^{adc} b^d \eta^c \partial^\mu \partial_\mu e^a \\ &= \frac{g^3}{2} \varepsilon^{adc} A_\mu^b A^{b\mu} e^a b^d \eta^c = -\frac{g^3}{2} \varepsilon^{adc} A_\mu^b A^{b\mu} e^d b^a \eta^c, \end{aligned} \quad (4.65)$$

$$T_{23} + T_{24} = 0. \quad (4.66)$$

By inspection, we also find the cancellations

$$T_{27} + T_{29} = 0, \quad (4.67)$$

$$T_{28} + T_{30} = 0. \quad (4.68)$$

We are now left, in (4.38), with the following 5 terms underlined here by a curly bracket and denoted hereafter by A, B, C, D, E :

$$\begin{aligned} \delta L_{\text{tot}} = & \underbrace{g\varepsilon^{abc} F^{a\mu\nu} \partial_\mu (A_\nu^b \eta^c)}_A - \underbrace{g\varepsilon^{abc} F^{a\mu\nu} A_\mu^b \partial_\nu \eta^c}_B + \underbrace{g^2 F^{a\mu\nu} A_\mu^b A_\nu^a \eta^b}_C \\ & + \underbrace{2g^2 A^{a\mu} \partial_\mu \eta^a (\varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0 + \varphi^d \varphi^d + \chi^d \chi^d + b^d e^d)}_D \\ & + \underbrace{2ag A^{a\mu} \partial_\mu \eta^a (\varphi^0 + \chi^0)}_E. \end{aligned} \quad (4.69)$$

Recall also that the equation of motion for the gauge field is

$$D_\mu F^{a\mu\nu} = 2g^2 A^{a\nu} (\varphi^d \varphi^d + \chi^d \chi^d + b^d e^d + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0) + 2ag A^{a\nu} (\varphi^0 + \chi^0), \quad (4.70)$$

or equivalently

$$\begin{aligned} \partial_\mu F^{a\mu\nu} + g\varepsilon^{abc} A_\mu^b F^{c\mu\nu} = & -2g^2 A^{a\nu} (\varphi^d \varphi^d + \chi^d \chi^d + b^d e^d + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0) \\ & - 2ag A^{a\nu} (\varphi^0 + \chi^0). \end{aligned} \quad (4.71)$$

The first term in Eq. (4.69) then becomes

$$\begin{aligned} A = & g\varepsilon^{aij} F^{a\mu\nu} \partial_\mu (A_\nu^i \eta^j) = g\varepsilon^{aij} [-\partial_\mu F^{a\mu\nu}] A_\nu^i \eta^j \\ = & g\varepsilon^{aij} [g\varepsilon^{abc} A_\mu^b F^{c\mu\nu} + 2g^2 A^{a\nu} (\varphi^d \varphi^d + \chi^d \chi^d + b^d e^d + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0)] \\ & + g\varepsilon^{aij} [2ag A^{a\nu} (\varphi^0 + \chi^0)] A_\nu^i \eta^j \\ = & g^2 \varepsilon^{aij} \varepsilon^{abc} F^{c\mu\nu} A_\mu^b A_\nu^i \eta^j \\ & + 2g^3 \underbrace{\varepsilon^{aij} A^{a\nu} A_\nu^i \eta^j}_{C} (\varphi^d \varphi^d + \chi^d \chi^d + b^d e^d + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0) \\ & + 2ag^2 \underbrace{\varepsilon^{aij} A^{a\nu} A_\nu^i}_{D} (\varphi^0 + \chi^0) \eta^j \\ = & -g^2 F^{a\mu\nu} A_\mu^b A_\nu^a \eta^b. \end{aligned} \quad (4.72)$$

By inspection, the terms A and C cancel each other exactly:

$$A + C = 0, \quad (4.73)$$

while the second term can be written as follows:

$$\begin{aligned} B = & -g\varepsilon^{abc} F^{a\mu\nu} A_\mu^b \partial_\nu \eta^c = (g\varepsilon^{abc} A_\mu^b F^{c\mu\nu}) \partial_\nu \eta^a \\ = & -\underbrace{\partial_\mu F^{a\mu\nu} \partial_\nu \eta^a}_B - 2g^2 A^{a\nu} \partial_\nu \eta^a (\varphi^d \varphi^d + \chi^d \chi^d + b^d e^d + \varphi^0 \varphi^0 + \chi^0 \chi^0 + b^0 e^0) \\ & - 2ag A^{a\nu} \partial_\nu \eta^a (\varphi^0 + \chi^0), \end{aligned} \quad (4.74)$$

and therefore

$$B + D + E = 0. \tag{4.75}$$

Thus, we eventually obtain the important result that the new Lagrangian defined in Eq. (3.4) is invariant under the shifted gauge transformations (1.9)–(1.17): $\delta L_{\text{tot}} = 0$.

V. CONCLUDING REMARKS AND OPEN PROBLEMS

Our paper does not contain new results but, having a purely pedagogical character, aims at helping advanced readers to become familiar with a new formalism for quantum Yang–Mills theory as proposed in Ref. [6]. In particular, the calculations of Sec. IV, which prove in detail the invariance of the S-matrix under shifted gauge transformations, are worth reading for all those who are interested in modern quantum field theory.

It now appears desirable to understand whether the Slavnov formalism can be extended to quantum gravity, since Yang–Mills theory and general relativity share the property of being type-I gauge theories in a space-of-histories formulation [1, 9]. This means that the vector fields such that the action functional is invariant under them have Lie brackets which are a linear combination of the vector fields only with structure constants (i.e. independent of the gauge fields).

The counterpart of the Coulomb gauge considered in (1.1) by the author of Ref. [6] is the Prentki gauge, studied by the authors of Ref. [10] in their pioneering work on one-loop divergences in quantum gravity. Such an extension to quantum gravity would be, to our knowledge, original, and might lead to a better understanding of this new class of gauge-fixed functional integrals.

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