

Eigenvectors of open XXZ and ASEP models for a class of non-diagonal boundary conditions

N. Crampé^a, E. Ragoucy^b and D. Simon^c

^a *LPTA, CNRS and Université Montpellier II,*

Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

E-mail: ncrampe@um2.fr

^b *LAPTH, CNRS and Université de Savoie*

9 chemin de Bellevue, BP 110, 74941, Annecy-Le-Vieux Cedex, France

E-mail: ragoucy@lapp.in2p3.fr

^c *LPMA, Université Pierre et Marie Curie,*

Case Courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, France

E-mail: damien.simon@upmc.fr

Abstract

We present a generalization of the coordinate Bethe ansatz that allows us to solve integrable open XXZ and ASEP models with non-diagonal boundary matrices, provided their parameters obey some relations. These relations extend the ones already known in the literature in the context of algebraic or functional Bethe ansatz. The eigenvectors are represented as sums over cosets of the BC_n Weyl group.

1 Introduction

In the framework of integrable systems, the study of open spin chains with integrable boundaries have been developed a long time ago [1, 2]. There, it has been shown that the model is integrable provided the two matrices characterizing the boundaries obey some algebraic relations (the so-called reflection equation). However, although the integrability has been proven, the explicit resolution (eigenvalues and eigenvectors of the Hamiltonian) of the models is not known in its full generality. In fact, for a long time, only the case of diagonal boundary matrices was solved, for different kind of models, e.g. open XXX [3] and its $su(N)$ [4] or $su(N|M)$ [5, 6] generalizations, open XXZ [7] and its generalizations [8, 9, 10, 11], using different versions of the Bethe ansatz (analytical, algebraic or functional). However, the classifications of boundary matrices (obeying a reflection equation) [12, 13, 14, 15] clearly shows that non-diagonal solutions do exist, although explicit solutions for the eigenvalue problem were not known. The problem laid essentially in the construction of a reference state (a particular Hamiltonian eigenvector) that allows to initiate the procedure. Indeed, when the boundary matrices were not diagonal (or at least not simultaneously diagonalizable), the existence of this reference state was not ensured.

Recently, different approaches have been developed to overcome these difficulties, such as gauge transformations that allow to go to a diagonal basis [16], or fusion relations for TQ relations that do not need the existence of a reference state [17, 18]. In all these cases, the boundary matrices can be non-diagonal, but the parameters entering their definition need to satisfy some constraints. Note also the original approach [19] that uses another presentation of the reflection algebra, called q -Dolan-Grady relation, as well as the ‘generalized’ functional ansatz developed in [20]. Both avoid the use of constraints.

The Asymmetric Simple Exclusion Process (ASEP) is an out-of-equilibrium statistical physics representation of the Temperley-Lieb algebra [21], on which the XXZ Hamiltonian is based. Many exact probabilistic results [22, 23, 24, 25], that are not necessarily based on the integrability of the model, have been obtained in the past fifteen years and shed some new light on integrable results. The ASEP notations, as described in [26], are presented in this paper and a specific section 5.1 is dedicated to the comparison of our results with old results on the ASEP, such as the matrix ansatz [22, 27, 28, 29].

In this paper, we present a construction based on the coordinate Bethe ansatz [30] for open XXZ and ASEP models, allowing the use of non-diagonal boundary matrices. As for the other approaches, the boundary matrices need to obey some constraints, but the ones we find are more general than those already known.

We present now the structure of the paper: section 2 defines the models and introduces the notations needed for the next sections; section 3 describes the different sets of reference states and the sets of exceptional points they lead to. Section 4 gives the structure of the coordinate Bethe ansatz as a combination of the reference states and explains the role of the BC_n Weyl group and gives in details the Bethe equations that correspond to the two sets of specific points. Finally, we conclude on two open problems: section 5.1 presents a short discussion of the relation between the specific points and other previous results about the exclusion process such as the matrix ansatz; a

discussion on the completeness of the spectrum and of the eigenvectors is tackled in section 5.2.

2 XXZ and ASEP models with non-diagonal boundaries

The Markov transition matrix for the open ASEP model is given by

$$W = \widehat{K}_1 + K_L + \sum_{j=1}^{L-1} w_{j,j+1}, \quad (2.1)$$

where the indices indicate the spaces in which the following matrices act non trivially

$$w = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{K} = \begin{pmatrix} -\alpha & \gamma e^{-s} \\ \alpha e^s & -\gamma \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}. \quad (2.2)$$

It is well-established [31, 28, 26] that this ASEP model is related by a similarity transformation to the following integrable open XXZ model

$$H = \widehat{B}_1 + B_L - \frac{1}{2} \sum_{j=1}^{L-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \cos \eta \sigma_j^z \sigma_{j+1}^z), \quad (2.3)$$

where σ are the usual Pauli matrices and

$$\widehat{B} = \frac{\sin \eta}{\cos \omega_- + \cos \delta_-} \begin{pmatrix} \frac{i}{2}(\cos \omega_- - \cos \delta_-) - \sin \omega_- & e^{-i\theta_1} \\ e^{i\theta_1} & -\frac{i}{2}(\cos \omega_- - \cos \delta_-) - \sin \omega_- \end{pmatrix}, \quad (2.4)$$

$$B = \frac{\sin \eta}{\cos \omega_+ + \cos \delta_+} \begin{pmatrix} -\frac{i}{2}(\cos \omega_+ - \cos \delta_+) - \sin \omega_+ & e^{-i\theta_2} \\ e^{i\theta_2} & \frac{i}{2}(\cos \omega_+ - \cos \delta_+) - \sin \omega_+ \end{pmatrix}. \quad (2.5)$$

The explicit form of the transformation can be found in e.g. [26]. We reproduce it here for completeness:

$$W = -\sqrt{pq} U^{-1} H U \quad \text{with} \quad U = \otimes_{j=1}^L \begin{pmatrix} 1 & 0 \\ 0 & \xi \left(\sqrt{\frac{q}{p}} \right)^{j-1} \end{pmatrix}, \quad (2.6)$$

$$\sqrt{\frac{\alpha}{\gamma}} = -ie^{i\omega_-}, \quad \sqrt{\frac{\beta}{\delta}} = -ie^{i\omega_+}, \quad \sqrt{\frac{p}{q}} = -e^{i\eta} \quad (2.7)$$

$$\xi \sqrt{\frac{\alpha}{\gamma}} e^s = e^{i\theta_1} \quad \text{and} \quad \xi \sqrt{\frac{\delta}{\beta}} \left(\sqrt{\frac{q}{p}} \right)^{L-1} = e^{i\theta_2}, \quad (2.8)$$

where ξ is an arbitrary (gauge) parameter that disappears in all the following computations.

Throughout the paper we will stick to this ASEP notation, keeping in mind that by this similarity transformation we treat also the XXZ model with non-diagonal boundaries. However, in the ASEP model, all the parameters must be positive (transition rates) while there is not such constraint in the XXZ spin chain. To be as general as possible, we will not assume this constraint in the paper.

The vectors of the canonical basis used for the previous definitions are indexed by the spin value \uparrow or \downarrow for the XXZ-spin chain and corresponds to the number of particle $\tau_i \in \{0, 1\}$ of each site in the ASEP (see figure 1).

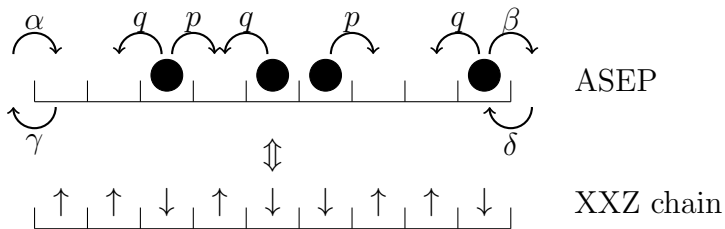


Figure 1: Asymmetric exclusion process for a system of size L with two boundaries and mapping to the XXZ spin chain.

3 Basis vectors for the coordinate Bethe ansatz

For a chain of L sites with periodic or diagonal boundary conditions, it is easy to find one eigenstate, called pseudo-vacuum. It is usually chosen as L spins up (resp. L empty sites for the ASEP). Then, excited states are constructed by adding some excitations that flip a given number n of spins (resp. add n particles in the ASEP). The conventional Bethe methods (coordinate, algebraic or analytical) allow us to compute the linear combinations between the $\binom{L}{n}$ excited states that diagonalize the Hamiltonian.

For an open chain with non-diagonal boundary (which is the case we want to deal with, see equations (2.1) and (2.3)), the first problem consists in finding the pseudo-vacuum. Different approaches have been elaborated to overcome this problem as explained in the introduction. However, the coordinate Bethe ansatz, which is the historical method introduced by H. Bethe [30], has never been successfully applied. To treat this problem, we must generalize the ansatz. A first step in this direction have been done in [21]. The general strategy we adopt here can be summarized as follows:

- we do not choose anymore all spins up for the vacuum and spins down for the excitations. We take general vectors (see their explicit construction below),
- Hamiltonian eigenfunctions will be constructed as linear combinations between states with n excitations *together with* m ($< n$) excitations,
- the explicit forms of the excitations and of the vacuum depend on the total number of excitations in the state we consider.

More physically, one allows that excitations may be destroyed or created by the boundaries with the restriction that one cannot create more than n excitations. Without any loss of generality, we can always choose that one of the boundaries preserves the number of excitations. In this paper, we choose the right-hand-side one.

Finally, we found two different ways to fulfill these conditions. In both cases, some constraints between the parameters of the model appear (as it is already the case in the approaches [16, 26] to non-diagonal boundaries). We will show that we can solve the problem (within the framework

of coordinate Bethe ansatz) for all the sets of constraints that have already been produced in the literature as well as for some new sets.

The following subsection introduces some notations that are standard for the ASEP and that will be used throughout the paper. Then, in the next two subsections 3.2 and 3.3, we will present the two different sets of vectors with the associated constraints. Finally, in section 4, we present their particular linear combinations diagonalizing the Hamiltonian.

3.1 Boundary operators and a duality

Two particular sets of 2-dimensional vectors are relevant in the study of the boudary dynamics. The first set corresponds to vectors that diagonalize the boundary operators. We choose the following arbitrary normalizations, that will become clearer later:

$$\widehat{K} \begin{pmatrix} 1 \\ e^s/c_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ e^s/c_1 \end{pmatrix}, \quad K \begin{pmatrix} 1 \\ c_L \end{pmatrix} = \lambda_L \begin{pmatrix} 1 \\ c_L \end{pmatrix}. \quad (3.1)$$

There are two solutions for the first equation (resp. the second) given by $c_1 = c_{\pm}(\alpha, \gamma)$ and $\lambda_1 = \lambda_{\pm}(\alpha, \gamma)$ (resp. $c_L = c_{\pm}(\beta, \delta)$ and $\lambda_L = \lambda_{\pm}(\beta, \delta)$) where $c_{\pm}(u, v)$ and $\lambda_{\pm}(u, v)$ are the roots of the two functions:

$$P_{u,v}(X) = uX + (u - v) - v/X \quad \text{i.e.} \quad c_+(u, v) = v/u \text{ and } c_-(u, v) = -1, \quad (3.2)$$

$$Q_{u,v}(X) = X^2 + X(u + v) \quad \text{i.e.} \quad \lambda_+(u, v) = 0 \text{ and } \lambda_-(u, v) = -u - v. \quad (3.3)$$

These notations allow for a convenient way of parametrizing both boundaries at the same time.

The second set of relevant vectors satisfies the diagonal relations:

$$\left[\widehat{K} - \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \right] \begin{pmatrix} 1 \\ e^s/c_1^* \end{pmatrix} = \lambda_1^* \begin{pmatrix} 1 \\ e^s/c_1^* \end{pmatrix}, \quad \left[K + \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \right] \begin{pmatrix} 1 \\ c_L^* \end{pmatrix} = (\lambda_L^* + p + q) \begin{pmatrix} 1 \\ c_L^* \end{pmatrix}. \quad (3.4)$$

As previously, the coefficients c_1^* and λ_1^* (resp. c_L^* and λ_L^*) take the two possible values $c_{\pm}^*(\alpha, \gamma)$ and $\lambda_{\pm}^*(\alpha, \gamma)$ (resp. $c_{\pm}^*(\beta, \delta)$ and $\lambda_{\pm}^*(\beta, \delta)$), which are the respective zeroes of

$$P_{u,v}^*(X) = uX + (u - v + q - p) - v/X = P_{u,v}(X) + (q - p), \quad (3.5)$$

$$Q_{u,v}^*(X) = X^2 + X(u + v + p + q) + (qp + pu + qv). \quad (3.6)$$

The explicit values of $c_{\pm}^*(u, v)$ ¹ and $\lambda_{\pm}^*(u, v)$ are given by:

$$c_{\pm}^*(u, v) = \frac{p - q + v - u \pm \sqrt{(p - q + v - u)^2 + 4uv}}{2u}, \quad (3.7)$$

$$\lambda_{\pm}^*(u, v) = \frac{-p - q - v - u \pm \sqrt{(p - q + v - u)^2 + 4uv}}{2}. \quad (3.8)$$

¹These functions are sometimes called $\kappa_{\pm}(u, v)$. We change this notation to be consistent throughout the paper.

The definitions (3.4) suggest the definition of the new operators:

$$\widehat{K}^* = \widehat{K} - \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}, \quad K^* = K + \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}, \quad (3.9a)$$

$$w^* = w + \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \otimes I - I \otimes \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p & p & 0 \\ 0 & q & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.9b)$$

where I is the two by two identity matrix. One verifies that the matrix W given in (2.1) has the following second representation:

$$W = \widehat{K}_1^* + K_L^* + \sum_{i=1}^{L-1} w_{i,i+1}^*. \quad (3.10)$$

This representation will be essential in the definitions of the new exceptional points.

3.2 First choice

We first consider the matrix W as written in eq. (2.1). Let us define the family of vectors:

$$|\omega(u)\rangle_i = \begin{pmatrix} 1 \\ u(p/q)^{i-1} \end{pmatrix}, \quad (3.11)$$

$$|V\rangle_i = \begin{pmatrix} q-p \\ 0 \end{pmatrix}. \quad (3.12)$$

where u is still an arbitrary parameter. From these elementary vectors, we build the tensor product over the sites i to j and write it as

$$|\Omega(u)\rangle_i^j = |\omega(u)\rangle_i |\omega(u)\rangle_{i+1} \dots |\omega(u)\rangle_j. \quad (3.13)$$

We now fix an integer n and introduce the state with $n - m$ excitations at the ordered positions $1 \leq x_{m+1} < \dots < x_n \leq L$ defined as the $(\mathbb{C}^2)^{\otimes L}$ -vector:

$$\begin{aligned} |x_{m+1}, \dots, x_n\rangle &= \left(\sqrt{\frac{q}{p}} \right)^{(x_{m+1}-1)+\dots+(x_n-1)} \times \\ &\times |\Omega(u_{m+1})\rangle_1^{x_{m+1}-1} |\omega(v_{m+1})\rangle_{x_{m+1}} |\Omega(u_{m+2})\rangle_{x_{m+1}+1}^{x_{m+2}-1} |\omega(v_{m+2})\rangle_{x_{m+2}} \dots |\omega(v_n)\rangle_{x_n} |\Omega(u_{n+1})\rangle_{x_{n+1}}^L. \end{aligned} \quad (3.14)$$

The overall factor $(\sqrt{\frac{q}{p}})$ is introduced only in order to normalize the Bethe roots. The coefficients u_m and v_m are related through the recursion relation:

$$u_{m+1} = \frac{q}{p} u_m, \quad v_{m+1} = \frac{q}{p} v_m, \quad (3.15)$$

and the initial coefficients u_1 and v_1 are still arbitrary. These vectors correspond to states where m excitations have left the system (through the left boundary). To clarify the notation, the state with no excitation corresponds to $m = n$ and is given by

$$|\emptyset\rangle = |\Omega(u_{n+1})\rangle_1^L, \quad (3.16)$$

while the state with one excitation ($m = n - 1$) reads

$$|x_n\rangle = \left(\sqrt{\frac{q}{p}}\right)^{x_n-1} |\Omega(u_n)\rangle_1^{x_n-1} |\omega(v_n)\rangle_{x_n} |\Omega(u_{n+1})\rangle_{x_{n+1}}^L. \quad (3.17)$$

The above states are product states and are linearly independent. We denote by \mathcal{B}_n the set of these $\sum_{k=0}^n \binom{L}{k}$ independent vectors. We also need the vectors $|x_{m+1}, \dots, \bar{x}_\alpha, \dots, x_n\rangle$ deduced from $|x_{m+1}, \dots, x_n\rangle$ by replacing the vector $|\omega(u)\rangle$ in position x_α by $|V\rangle_{x_\alpha}$.

These vectors have been chosen such that we get for the Hamiltonian bulk part:

$$w_{\beta, \beta+1} |x_{m+1}, \dots, x_n\rangle = \begin{cases} 0 & \text{if } \beta, \beta + 1 \neq x_{m+1}, \dots, x_n; \\ \sqrt{pq} | \dots, x_j - 1, \dots \rangle - q | \dots, x_j, \dots \rangle + | \dots, \bar{x}_j, \dots \rangle & \text{if } \beta + 1 = x_j, \beta \neq x_{j-1}; \\ \sqrt{pq} | \dots, x_j + 1, \dots \rangle - p | \dots, x_j, \dots \rangle - | \dots, \bar{x}_j, \dots \rangle & \text{if } \beta = x_j, \beta + 1 \neq x_{j+1}; \\ | \dots, x_j, \bar{x}_{j+1}, \dots \rangle - | \dots, \bar{x}_j, x_{j+1}, \dots \rangle & \text{if } \beta = x_j, \beta + 1 = x_{j+1}. \end{cases} \quad (3.18)$$

Remark the additional states appearing on the right hand side involving vector $|V\rangle$. Because of the alternating signs, they cancel each other in the Hamiltonian bulk part, but the first and last terms (boundary terms). It is interesting to remark that the same "telescopic" trick is also used in the proof of the matrix ansatz in section 5.1 and in the definitions (3.9).

The first and last terms $-|V\rangle_1$ and $|V\rangle_L$ of the telescopic sum must be absorbed by the boundary operators and this justifies the study of the operator (3.4).

We now require that the left boundary diagonalizes $|\omega(u_1)\rangle_1$ and that the right boundary diagonalizes $|\omega(u_{n+1})\rangle_L$, hence excitations are created neither on the right, nor on the left, when n particles are already in the bulk. Thus, the action of the boundary operators must be

$$\widehat{K}_1 |x_{m+1}, \dots, x_n\rangle = \begin{cases} \Lambda_1^{(m)} |x_{m+1}, \dots, x_n\rangle + C_1^{(m)} |1, x_{m+1}, \dots, x_n\rangle & \text{if } x_{m+1} > 1, \\ \widetilde{\Lambda}_1^{(m)} |1, x_{m+2}, \dots, x_n\rangle + D_1^{(m)} |x_{m+2}, \dots, x_n\rangle + |\bar{1}, x_{m+2}, \dots, x_n\rangle & \text{if } x_{m+1} = 1, \end{cases} \quad (3.19)$$

$$K_L |x_{m+1}, \dots, x_n\rangle = \begin{cases} \Lambda_L |x_{m+1}, \dots, x_n\rangle & \text{if } x_n < L, \\ \widetilde{\Lambda}_L |x_{m+1}, \dots, x_{n-1}, L\rangle - |x_{m+1}, \dots, x_{n-1}, \bar{L}\rangle & \text{if } x_n = L, \end{cases} \quad (3.20)$$

with the constraint

$$C_1^{(0)} = 0. \quad (3.21)$$

We introduce also the more compact form $\Lambda_1 = \Lambda_1^{(0)}$ since this value will appear in the energy.

The constraints (3.21) and (3.20) lead to the following trivial identifications with the parameters introduced in section 3.1:

$$u_1 = e^s / c_\epsilon(\alpha, \gamma), \quad \Lambda_1 = \lambda_\epsilon(\alpha, \gamma), \quad (3.22)$$

$$(p/q)^{L-1} u_{n+1} = c_{\epsilon'}(\beta, \delta), \quad \Lambda_L = \lambda_{\epsilon'}(\beta, \delta), \quad (3.23)$$

$$(p/q)^{L-1} v_n = c_{\epsilon''}^*(\beta, \delta), \quad \widetilde{\Lambda}_L = \lambda_{\epsilon''}^*(\beta, \delta) + q, \quad (3.24)$$

where $\epsilon, \epsilon', \epsilon'' \in \{+, -\}$ are arbitrary. The compatibility relation of the previous equalities with the recursion relations (3.15) lead to four possible solutions summarized in Table 1. They correspond to the different choices of the signs ϵ, ϵ' in the relation:

$$c_\epsilon(\alpha, \gamma)c_{\epsilon'}(\beta, \delta) = e^s \left(\frac{p}{q}\right)^{L-1-n}. \quad (3.25)$$

The third sign ϵ'' does not appear in this condition, nor in the energy, nor in the Bethe equations: it just corresponds to the full reflection of the excitations on the right boundary².

Λ_1	Λ_L	$c_\epsilon(\alpha, \gamma)$	$c_{\epsilon'}(\beta, \delta)$	Constraints
0	0	$c_+(\alpha, \gamma)$	$c_+(\beta, \delta)$	$\frac{\alpha\beta}{\gamma\delta} e^s \left(\frac{p}{q}\right)^{L-1-n} = 1$
$-\alpha - \gamma$	$-\beta - \delta$	$c_-(\alpha, \gamma)$	$c_-(\beta, \delta)$	$e^s \left(\frac{p}{q}\right)^{L-1-n} = 1$
$-\alpha - \gamma$	0	$c_-(\alpha, \gamma)$	$c_+(\beta, \delta)$	$-\frac{\beta}{\delta} e^s \left(\frac{p}{q}\right)^{L-1-n} = 1$
0	$-\beta - \delta$	$c_+(\alpha, \gamma)$	$c_-(\beta, \delta)$	$-\frac{\alpha}{\gamma} e^s \left(\frac{p}{q}\right)^{L-1-n} = 1$

Table 1: Different possible values for the parameters and the constraints imposed by (3.25).

We then obtain the following values of the coefficients:

$$C_1^{(m)} = \frac{u_2}{v_1 - u_2} P_{\alpha, \gamma}(e^s/u_{m+1}), \quad (3.26)$$

$$D_1^{(m-1)} = -\frac{v_1}{v_1 - u_2} P_{\alpha, \gamma}^*(e^s/v_m), \quad (3.27)$$

$$\Lambda_1^{(m)} = \frac{(q/p)^m(\Lambda_1 + \alpha)v_1 - (p/q)^m(\Lambda_1 + \gamma)u_2 + \gamma u_2 - \alpha v_1}{v_1 - u_2}, \quad (3.28)$$

$$\tilde{\Lambda}_1^{(m-1)} = \frac{(p/q)^m(\Lambda_1 + \gamma)u_2 - (q/p)^m v_1(\Lambda_1 + \alpha) - \gamma v_1 + (\alpha + q - p)u_2}{v_1 - u_2}. \quad (3.29)$$

We also remind that for ASEP probabilistic models, all parameters have to be positive, so that only the two first lines of table 1 have to be considered in this case. These two constraints can be recasted into a single one

$$\left(\frac{\alpha\beta}{\gamma\delta} e^s - \left(\frac{q}{p}\right)^{L-1-n}\right) \left(e^s - \left(\frac{q}{p}\right)^{L-1+n}\right) = 0. \quad (3.30)$$

Then, using the correspondence (2.8), it takes the form for XXZ model for even L :

$$\cos(\theta_1 - \theta_2) = \cos(\omega_+ + \omega_- - m\eta), \quad (3.31)$$

that is just the constraint given in [17, 16, 26] for some integer m . Doing the same with the last two constraints, we get

$$\cos(\theta_1 - \theta_2) = \cos(\omega_+ - \omega_- - n\eta). \quad (3.32)$$

²Of course, we could have also chosen a full reflection on the left boundary instead.

3.3 Second choice

Using the same technique as in the previous subsection, we introduce a basis of states that is suitable to the diagonalization of the alternative representation (3.10) of the matrix W . The vector $|V\rangle_i$ has the same definition as before but we introduce the family of vectors

$$|\omega^*(u)\rangle_i = \begin{pmatrix} 1 \\ u \end{pmatrix}. \quad (3.33)$$

We define similarly the product state $|\Omega^*(u)\rangle_i^j$ as in (3.13) by replacing $|\omega(u)\rangle$ by $|\omega^*(u)\rangle$ and we introduce the new state with the new excitations in ordered positions $1 \leq y_{m+1} < \dots < y_n \leq n$:

$$|y_{m+1}, \dots, y_n\rangle^* = \left(\sqrt{\frac{p}{q}} \right)^{(y_{m+1}-1)+\dots+(y_n-1)} |\Omega^*(u_{m+1}^*)\rangle_1^{y_{m+1}-1} |\omega^*(v_{m+1}^*)\rangle_{y_{m+1}} \dots |\omega^*(v_n^*)\rangle_{y_n} |\Omega^*(u_{n+1}^*)\rangle_{y_{n+1}}^L \quad (3.34)$$

where the new coefficients u_i^* and v_i^* satisfy the recursion relation

$$u_{m+1}^* = (p/q)u_m^*, \quad v_{m+1}^* = (p/q)v_m^*. \quad (3.35)$$

These vectors have been chosen such that we get for the Hamiltonian bulk part

$$w_{\beta, \beta+1}^* |y_{m+1}, \dots, y_n\rangle^* = \begin{cases} 0 & \text{if } \beta, \beta+1 \neq y_{m+1}, \dots, y_n; \\ \sqrt{pq} |..y_j - 1.. \rangle^* - p |..y_j.. \rangle^* - |..\bar{y}_j.. \rangle^* & \text{if } \beta+1 = y_j; \beta \neq y_{j-1}; \\ \sqrt{pq} |..y_j + 1.. \rangle^* - q |.., y_j, .. \rangle^* + |..\bar{y}_j, .. \rangle^* & \text{if } \beta = y_j; \beta+1 \neq y_{j+1}; \\ |..\bar{\beta}, .. \rangle^* - |..\bar{\beta}+1, .. \rangle^* & \text{if } \beta = y_j; \beta+1 = y_{j+1}. \end{cases} \quad (3.36)$$

As for the previous choice, there are additional states $|V\rangle_1$ and $|V\rangle_L$ that survive the telescopic sum $\sum_{i=1}^{L-1} w_{i, i+1}^*$ and that must be absorbed by the boundary operators. We impose the dynamics:

$$\widehat{K}_1^* |x_{m+1}, \dots\rangle^* = \begin{cases} \Lambda_1^{*,(m)} |x_{m+1}, \dots\rangle^* + C_1^{*,(m)} |1, x_{m+1}, \dots\rangle^* & \text{if } x_{m+1} > 1, \\ \widetilde{\Lambda}_1^{*,(m)} |1, x_{m+2}, \dots\rangle^* + D_1^{*,(m)} |x_{m+2}, \dots\rangle^* - |\bar{1}, x_{m+2}, \dots\rangle^* & \text{if } x_{m+1} = 1, \end{cases} \quad (3.37)$$

$$K_L^* |x_{m+1}, \dots, x_n\rangle^* = \begin{cases} \Lambda_L^* |x_{m+1}, \dots, x_n\rangle^* & \text{if } x_n < L, \\ \widetilde{\Lambda}_L^* |x_{m+1}, \dots, x_{n-1}, L\rangle^* + |x_{m+1}, \dots, x_{n-1}, \bar{L}\rangle^* & \text{if } x_n = L, \end{cases} \quad (3.38)$$

with the additional closure constraint

$$C_1^{*,(0)} = 0. \quad (3.39)$$

We introduce again the more compact form $\Lambda_1^* = \Lambda_1^{*,(0)}$ since it is this value that will appear in the energy. From the constraints (3.38) and (3.39) and the notations introduced in section 3.1, we identify directly the value of most parameters:

$$u_1^* = e^s / c_\epsilon^*(\alpha, \gamma), \quad \Lambda_1^* = \lambda_\epsilon^*(\alpha, \gamma), \quad (3.40)$$

$$u_{n+1}^* = c_{\epsilon'}^*(\beta, \delta), \quad \Lambda_L^* = \lambda_{\epsilon'}^*(\beta, \delta) + p + q \quad (3.41)$$

$$v_n^* = c_{\epsilon''}(\beta, \delta), \quad \widetilde{\Lambda}_L^* = p + \lambda_{\epsilon''}(\beta, \delta). \quad (3.42)$$

As in the previous subsection, these values are compatible with the recursion relation (3.35) if and only if the parameters of the model satisfy the relation

$$c_\epsilon^*(\alpha, \gamma)c_{\epsilon'}^*(\beta, \delta) = e^s \left(\frac{p}{q}\right)^n, \quad (3.43)$$

for some signs ϵ and ϵ' and some integer n , which then fixes the number of excitations. Similar exceptional points already appeared in the literature in the context of the matrix ansatz described in section 5.1. Table 2 summarizes the different possibilities.

$\frac{\Lambda_1^* + \Lambda_L^*}{2}$	ϵ	ϵ'	Constraints
$\frac{-\alpha-\gamma-\beta-\delta+\sqrt{(p-q+\gamma-\alpha)^2+4\gamma\alpha}+\sqrt{(p-q+\delta-\beta)^2+4\beta\delta}}{2}$	+	+	$c_+^*(\alpha, \gamma)c_+^*(\beta, \delta) = e^s \left(\frac{p}{q}\right)^n$
$\frac{-\alpha-\gamma-\beta-\delta-\sqrt{(p-q+\gamma-\alpha)^2+4\gamma\alpha}-\sqrt{(p-q+\delta-\beta)^2+4\beta\delta}}{2}$	-	-	$c_-^*(\alpha, \gamma)c_-^*(\beta, \delta) = e^s \left(\frac{p}{q}\right)^n$
$\frac{-\alpha-\gamma-\beta-\delta+\sqrt{(p-q+\gamma-\alpha)^2+4\gamma\alpha}-\sqrt{(p-q+\delta-\beta)^2+4\beta\delta}}{2}$	+	-	$c_+^*(\alpha, \gamma)c_-^*(\beta, \delta) = e^s \left(\frac{p}{q}\right)^n$
$\frac{-\alpha-\gamma-\beta-\delta-\sqrt{(p-q+\gamma-\alpha)^2+4\gamma\alpha}+\sqrt{(p-q+\delta-\beta)^2+4\beta\delta}}{2}$	-	+	$c_-^*(\alpha, \gamma)c_+^*(\beta, \delta) = e^s \left(\frac{p}{q}\right)^n$

Table 2: Different possible values for the parameters and the constraints imposed by (3.25).

The values of all the coefficients are then given by:

$$C_1^{*,(m)} = u_2^* \frac{P_{\alpha,\gamma}^*(e^s/u_{m+1}^*)}{v_1^* - u_2^*}, \quad (3.44)$$

$$D_1^{*,(m-1)} = -v_1^* \frac{P_{\alpha,\gamma}^*(e^s/v_m^*)}{v_1^* - u_2^*}, \quad (3.45)$$

$$\Lambda_1^{*,(m)} = \frac{(p/q)^m v_1^* (\Lambda_1^* + \alpha + q) - (q/p)^m u_2^* (\Lambda_1^* + \gamma + p) + (\gamma + p) u_2^* - (\alpha + q) v_1^*}{v_1^* - u_2^*}, \quad (3.46)$$

$$\tilde{\Lambda}_1^{*,(m-1)} = \frac{(q/p)^m u_2^* (\Lambda_1^* + \gamma + p) - (p/q)^m (\Lambda_1^* + \alpha + q) v_1^* + (\alpha + p) u_2^* - (\gamma + p) v_1^*}{v_1^* - u_2^*}. \quad (3.47)$$

In the language of the XXZ spin chain, we should introduce the additional angles Ω_\pm defined by

$$2 \cos \Omega_\pm = \cos \delta_\pm - \cos \omega_\pm, \quad (3.48)$$

and the condition (3.43) takes the following simple form for even L :

$$\cos(\theta_1 - \theta_2) = \cos(\Omega_- \pm \Omega_+ - \eta m), \quad (3.49)$$

for some even integer m between $-L$ and L . Up to our knowledge, this condition is new. The associated Bethe equations are presented in section 4.2.

4 Coordinate Bethe Ansatz

4.1 The first set of specific points

We are now in position to propose an ansatz for the eigenfunction

$$\Phi_n = \sum_{m=0}^n \sum_{x_{m+1} < \dots < x_n} \sum_{g \in G_m} A_g^{(m)} e^{i\mathbf{k}_g^{(m)} \cdot \mathbf{x}^{(m)}} |x_{m+1}, \dots, x_n\rangle, \quad (4.1)$$

where G_m is a full set of representatives of the following coset BC_n/BC_m (see appendix A), the vectors $|x_{m+1}, \dots, x_n\rangle$ are either given by (3.14) or by (3.34) and we introduce the notation $\mathbf{k}^{(m)}$ for the following truncated vector

$$\mathbf{k}^{(m)} = (k_{m+1}, \dots, k_n). \quad (4.2)$$

For this definition to be consistent, the coefficients $A_g^{(m)}$ do not have to depend on the choice of the representative i.e.

$$A_{gh}^{(m)} = A_g^{(m)} \quad \text{for any } h \in BC_m. \quad (4.3)$$

The coefficients $A_g^{(m)}$ are complex numbers to be determined such that Φ_n is an eigenfunction of H i.e. such that the following equation holds

$$H\Phi_n = E\Phi_n. \quad (4.4)$$

Due to the results of section 3, $H\mathcal{B}_n \subset \mathcal{B}_n$. Then, we project equation (4.4) on the different independent vectors of \mathcal{B}_n to get constraints on the coefficients $A_g^{(m)}$. This projection depends on the two choices done for the vectors $|x_{m+1}, \dots, x_n\rangle$. However, we chose the coefficients in such a way that one can be deduced from the other easily by adding a star to most quantities. Therefore, we write only the projections for the choice of section 3.2 and only the ones leading to independent relations (one can check that the remaining ones do not lead to new relations).

On $|x_1, \dots, x_n\rangle$ for (x_1, \dots, x_n) generic (i.e. $1 < x_1, x_n < L$ and $1 + x_j < x_{j+1}$)

As in the usual coordinate Bethe ansatz [30], this projection provides the energy:

$$E = \Lambda_1 + \Lambda_L + \sum_{j=1}^n \lambda(e^{ik_j}) \quad \text{where} \quad \lambda(x) = \sqrt{pq} \left(x + \frac{1}{x} \right) - p - q = \frac{\sqrt{pq}}{x} \left(x - \sqrt{\frac{p}{q}} \right) \left(x - \sqrt{\frac{q}{p}} \right). \quad (4.5)$$

Let us remark that, up to the boundary terms Λ_1 and Λ_L , the energy takes the same form as in the periodic case.

On $|x_1, \dots, x_n\rangle$ with $x_{j+1} = 1 + x_j$ (and $x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_n$ generic)

This projection is also a usual one and provides the scattering matrix between excitations. It is given by a relation between $A_g^{(0)}$ and $A_{g\sigma_j}^{(0)}$ where σ_j is the permutation of j and $j+1$ (see appendix A). Namely, we get

$$A_{g\sigma_j}^{(0)} = S(e^{ik_{gj}}, e^{ik_{g(j+1)}}) A_g^{(0)}, \quad (4.6)$$

with

$$S(u, v) = -\frac{a(u, v)}{a(v, u)} \quad \text{where} \quad a(x, y) = \frac{i}{xy - 1} \left(\left(\sqrt{\frac{q}{p}} + \sqrt{\frac{p}{q}} \right) y - xy - 1 \right). \quad (4.7)$$

The normalisation chosen for the function $a(x, y)$ is for further simplifications. As expected, this relation is similar to the periodic case since the boundaries are not involved in this process.

On $|\mathbf{1}, \mathbf{x}_{m+1} \dots, \mathbf{x}_n\rangle$ (x_{m+1}, \dots, x_n generic and $m \geq 1$)

This relation is a new one and we must take into account that the left boundary can create one particle at the site 1. We finally get, for any $g \in G_m$,

$$\begin{aligned} & \left(\tilde{\Lambda}_1^{(m-1)} - \Lambda_1 - p + \sum_{j=1}^m (p + q - \sqrt{pq}(e^{ik_{gj}} + e^{-ik_{gj}})) \right) \sum_{h \in H_m} A_{gh}^{(m-1)} e^{ik_{ghm}} \\ & + \sqrt{pq} \sum_{h \in H_m} A_{gh}^{(m-1)} e^{2ik_{ghm}} + C_1^{(m)} A_g^{(m)} = 0, \end{aligned} \quad (4.8)$$

where $H_m = BC_m/BC_{m-1}$. To obtain this relation, we have used the following property

$$\sum_{g \in G_{m-1}} A_g e^{ik_g^{(m)}} e^{ik_g^{(m)} \mathbf{x}^{(m)}} = \sum_{g \in G_m} e^{ik_g^{(m)} \mathbf{x}^{(m)}} \sum_{h \in H_m} A_{gh} e^{ik_{gh}^{(m)}}. \quad (4.9)$$

Let us stress that equation (4.8) is invariant by the choice of representative of G_m .

On $|\mathbf{x}_{m+1} \dots, \mathbf{x}_n\rangle$ (x_{m+1}, \dots, x_n generic and $m \geq 1$)

This projection provides a second relation between the coefficient from the level $m - 1$ and m since we must take into account that the left boundary can destroy a particle present on the site 1. We obtain the following constraint, for any $g \in G_m$,

$$D_1^{(m-1)} \sum_{h \in H_m} A_{gh}^{(m-1)} e^{ik_{gh}^{(m)}} + (\Lambda_1^{(m)} - \Lambda_1 + \sum_{j=1}^m (p + q - \sqrt{pq}(e^{ik_{gj}} + e^{-ik_{gj}}))) A_g^{(m)} = 0. \quad (4.10)$$

From (4.8) and (4.10), we may express all the $A_g^{(m)}$ ($m \geq 1$) in terms of $A_g^{(0)}$ thanks to the following recursive relations defined for $m \geq 1$

$$A_g^{(m)} = T^{(m)}(e^{ik_{g1}}, \dots, e^{ik_{gm}}) A_g^{(m-1)}, \quad (4.11)$$

with the following definitions:

$$T^{(m)}(x_1, \dots, x_m) = \frac{D_1^{(m-1)}}{p_1(x_m) V_1(x_m)} \frac{x_m^2 - 1}{\prod_{j=1}^{m-1} a(x_m, x_j) a(x_j, 1/x_m)}, \quad (4.12)$$

$$V_1(Z) = \lambda(Z) + (\Lambda_1 + \gamma) \left(1 - \frac{1}{Z} \sqrt{\frac{p}{q}} \right) + (\Lambda_1 + \alpha) \left(1 - \frac{1}{Z} \sqrt{\frac{q}{p}} \right), \quad (4.13)$$

$$p_1(Z) = Z + r, \quad (4.14)$$

$$r = \frac{1}{\sqrt{pq}} \frac{pu_2 - qv_1}{v_1 - u_2} = \frac{1}{\sqrt{pq}} \frac{pu_{m+1} - qv_m}{v_m - u_{m+1}}, \quad (4.15)$$

where r describes the difference between $|\omega(v_k)\rangle$ and the vacuum states $|\omega(u_k)\rangle$ and $|\omega(u_{k+1})\rangle$. It is independent from k .

The proof that (4.11) is a solution of both equations (4.8) and (4.10) is postponed to appendix B and relies on a residue computation. The integrability of the model plays a role at this place, since there are a priori too many constraints but not all of them are independent.

Let us emphasize that this recurrence relation (4.11) is the main result of this article. Indeed, firstly, it proves that the ansatz (4.1) and, in particular, the choice of cosets used to define it, is the appropriate choice. Secondly, it gives a very simple way to construct eigenfunctions, and we believe that it may be used for further computations.

A consequence of (4.11) (for $m = 1$) and $A_{gr_1}^{(1)} = A_g^{(1)}$, is

$$A_{gr_1}^{(0)} = \frac{T^{(1)}(e^{ik_{g1}})}{T^{(1)}(e^{-ik_{g1}})} A_g^{(0)}. \quad (4.16)$$

This relation, together with (4.6), allow us to express $A_g^{(0)}$ for any $g \in BC_n$ in terms of $A_1^{(0)}$ (where the subscript 1 stands for the unit of BC_n group). Finally, using recursively (4.11), we can express all the coefficients $A_g^{(m)}$ in terms of only $A_1^{(0)}$. This last coefficient is usually chosen such that the eigenfunction Φ_n be normed.

On $|\mathbf{x}_1 \dots, \mathbf{x}_{n-1}, \mathbf{L}\rangle$ ($x_1 \dots, x_{n-1}$ generic)

This last constraint consists in the quantization of the excitations moments since the system is in a finite volume. In the context of the coordinate Bethe ansatz, this quantization leads to the so-called Bethe equations, explicitly given by, for $1 \leq j \leq n$,

$$\prod_{\substack{\ell=1 \\ \ell \neq j}}^n S(e^{ik_\ell}, e^{ik_j}) S(e^{-ik_j}, e^{ik_\ell}) = e^{2iLk_j} \frac{V_1(e^{ik_j}) V_L(e^{ik_j})}{V_1(e^{-ik_j}) V_L(e^{-ik_j})}, \quad (4.17)$$

$$V_L(x) = \lambda(x) + (\Lambda_L + \beta) \left(1 - \frac{1}{x} \sqrt{\frac{q}{p}}\right) + (\Lambda_L + \delta) \left(1 - \frac{1}{x} \sqrt{\frac{p}{q}}\right). \quad (4.18)$$

We remind that the parameters have to obey one of the relations given in table 1.

4.2 The new second set of specific points

When condition (3.43) is satisfied, the same construction as in the previous section works, up to easy modifications in the equations. We introduce a new function $V_1^*(Z)$ and a new parameter r^* , which

replaces r , defined by:

$$V_1^*(Z) = \lambda(Z) + (\Lambda_1^* + \alpha + q) \left(1 - \frac{1}{Z} \sqrt{\frac{p}{q}}\right) + (\Lambda_1^* + \gamma + p) \left(1 - \frac{1}{Z} \sqrt{\frac{q}{p}}\right), \quad (4.19)$$

$$V_L^*(Z) = \lambda(Z) + (\Lambda_L^* + \alpha + q) \left(1 - \frac{1}{Z} \sqrt{\frac{p}{q}}\right) + (\Lambda_L^* + \gamma + p) \left(1 - \frac{1}{Z} \sqrt{\frac{q}{p}}\right), \quad (4.20)$$

$$r^* = \frac{1}{\sqrt{pq}} \frac{pv_1^* - qu_2^*}{u_2^* - v_1^*}. \quad (4.21)$$

The same proof as the one presented in appendix A is valid and we obtain the Bethe equations for $j \in \{1, \dots, n\}$:

$$e^{2iLk_j} \frac{V_1^*(e^{ik_j})V_L^*(e^{ik_j})}{V_1^*(e^{-ik_j})V_L^*(e^{-ik_j})} = \prod_{\substack{l=1 \\ l \neq j}}^n S(e^{ik_l}, e^{ik_j})S(e^{-ik_j}, e^{ik_l}), \quad (4.22)$$

with the parameters obeying now one of the relations described in table 2.

5 Conclusions

We discuss in this section two open problems, which need further investigation.

5.1 Connection with the matrix ansatz

For $s = 0$, the matrix W is stochastic (Markov transition matrix of the exclusion process) and hence has a known ground state eigenvalue $E = 0$ with a simple left eigenvector, whose components are all equal to 1 (conservation of total probability). Nevertheless, the corresponding right eigenvector is non-trivial since the matrix is not Hermitian and describes the stationary properties of the asymmetric exclusion process. The structure of this specific eigenvector was elucidated first in [22] and then studied algebraically in more details in [28, 27].

As already explained, the state space is 2^L dimensional and the canonical basis can be indexed by the values of the occupation $\tau_i \in \{0, 1\}$ (resp. spin $s_i \in \{-1, 1\}$ in the XXZ language) on each site i . The matrix ansatz states that the ground state of the ASEP with $s = 0$ has components given by:

$$\langle \tau_1 \tau_2 \dots \tau_L | \Phi \rangle = \langle \langle V_1 | \prod_{1 \leq i \leq n}^{\rightarrow} (\tau_i D + (1 - \tau_i) E) | V_2 \rangle \rangle, \quad (5.1)$$

where the arrow means that the product have to be build from left to right when the index i increases. One has for example $\langle 00 \dots 00 | \Phi \rangle = \langle \langle V_1 | E^L | V_2 \rangle \rangle$. The non-commuting matrices D and E act on an abstract auxiliary vector space \mathcal{V} . The vector $|V_1\rangle$ is in this space \mathcal{V} , whereas the vector $\langle\langle V_2|$ is in its dual. Φ is an eigenvector of W for $E = 0$ when $s = 0$ if the two matrices D et E and the two boundary vectors satisfy the commutation rules [22]:

$$pDE - qED = D + E, \quad (5.2a)$$

$$\langle \langle V_1 | (\gamma D - \alpha E) = -\langle \langle V_1 |, \quad (5.2b)$$

$$(\beta D - \delta E) | V_2 \rangle \rangle = | V_2 \rangle \rangle. \quad (5.2c)$$

with the condition $s = 0$. These three relations allow to determine recursively all the components of the eigenvector Φ and do not need an explicit representation of the algebra. In particular, the matrix ansatz gives an easy access to classical correlation functions with standard transfer matrix techniques.

The connection with the present Bethe ansatz approach comes from the algebraic study of the algebra generated by D and E , as performed in [28, 27], from computations that arise in the combinatorics of some so-called *staircase tableaux* [29], as well as from the physical interpretation [23, 25, 24] of the matrix ansatz in terms of so-called "shock" product state as introduced in section 3.

The first remark is that the matrix ansatz described here *fails* for some values of the parameters $p, q, \alpha, \beta, \gamma, \delta$. Failure happens when the recursion relation on the size L induced by (5.2) leads to $\langle\langle V_2|V_1\rangle\rangle = 0$ and thus more generally to a null vector Φ , as explained in [28]. This failure condition for a system of size L is the existence of an integer $n \in \{0, 1, \dots, L - 1\}$ such that

$$\frac{\alpha\beta}{\gamma\delta} \left(\frac{p}{q}\right)^n = 1. \quad (5.3)$$

This relation is precisely one of the two cases of (3.30) when $s = 0$, for which the present Bethe ansatz approach works.

The second remark is that the operators D and E do not have generically finite dimensional representations. Detailed studies shows that such finite-dimensional representations may exist at some specific points [27, 28]. It appears that an n -dimensional representation exists if one among the four cases of the condition (3.43) is satisfied for some integer n and plus/minus signs, i.e.

$$c_+^*(\alpha, \gamma)c_+^*(\beta, \delta) = \left(\frac{p}{q}\right)^n. \quad (5.4)$$

In [27], the finite representation is used to study how the ground state evolves through the phase transition that occurs in the open ASEP along the manifold (5.4): the present paper gives also the excited states on the same manifold and the Bethe ansatz equations we obtain in this case now allow one to study how the gap behaves near this phase transition. Once again, atypical results appear in both the Bethe ansatz approach and the matrix ansatz approach for the same specific parameters.

A third remark relies on the observation that the present construction does not allow to express the ground state described by the matrix ansatz in terms of a coordinate Bethe ansatz, precisely because one fails when the second works. This mismatch leads us to think that the matrix ansatz state may be used as a new reference state to build the missing eigenvectors if one could manage to add excitations on it. However, up to our knowledge, no deep understanding of the relation between these two approaches exists and has been exploited.

A last remark deals with the combinatorics work [29]. Besides the fact that quantities such as (3.30) and (3.43) appear in numerators or denominators in their computations and thus either simplifies or invalidates the matrix ansatz, their rewriting of the matrix ansatz in terms of *staircase tableaux* sheds a new light on the precise structure of this ansatz, and puts it in a form that is more suitable to comparison with Bethe ansatz because of the relation of these tableaux with permutation tableaux. In particular, the number of so-called *staircase tableaux* is $n!4^n$, whereas the cardinal of

BC_n is precisely $n!2^n$. Understanding both the similarities and the mismatch may lead to a better understanding of the Bethe ansatz.

5.2 Completeness of the spectrum

As usual for Bethe Ansatz methods, the delicate point to check is the completeness of the spectrum. Indeed, in many cases, completeness is not proved but expected to be true: numerics for small size systems [32], enumeration of the number of roots in the thermodynamic limit [33].

In the present case, numerical checks of the completeness for small sizes have been considered in [34]: the full spectrum is shown to be described by two sets of Bethe equations. Our approach for the right eigenvectors gives only one set of Bethe equations for each specific point and thus only one part of the spectrum and one part of the right eigenvectors. The same construction can be done for left eigenvectors (or right eigenvectors of the adjoint). In this case, one obtains the other part of the spectrum and one part of the left eigenvectors, as discussed in [21].

Let us stress that the operators we are considering may not be hermitian and thus left and right eigenvectors may not be simply related. Although the full spectrum is known, it would be interesting to build the full set of right and left eigenvectors.

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A Weyl group BC_n and coset

The Weyl group BC_n is generated by the set $\{r_1, \sigma_1, \dots, \sigma_{n-1}\}$ with the following constraints:

$$\sigma_j^2 = 1 = r_1^2 \quad , \quad \sigma_1 r_1 \sigma_1 r_1 = r_1 \sigma_1 r_1 \sigma_1 \quad , \quad \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} . \quad (\text{A.1})$$

It acts on a vector $\mathbf{k} = (k_1, \dots, k_n)$ of \mathbb{C}^n :

$$\mathbf{k}_{r_1} = (-k_1, k_2, \dots, k_n) \quad , \quad \mathbf{k}_{\sigma_j} = (k_1, \dots, k_{j+1}, k_j, \dots, k_n) . \quad (\text{A.2})$$

The subgroup generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$ is just the symmetric group. We now consider its subgroups generated by $\{r_1, \sigma_1, \dots, \sigma_{m-1}\}$, $m \leq n$, which we identify with BC_m .

For $g \in BC_n$, we then define the class $gBC_m = \{gh; h \in BC_m\}$, called a left coset. It is known that the set of all classes gBC_n , which is called BC_n/BC_m , forms a partition of BC_n : we can thus define G_m as a full set of representatives of BC_n/BC_m , such that one has the unique decomposition $BC_n = \bigoplus_{g \in G_m} gBC_m$. We set, by convention, $G_0 \sim BC_n$ and $G_n = \{1\}$.

The action of an element g of G_m on a vector $\mathbf{k}^{(m)} = (k_{m+1}, \dots, k_n)$ of \mathbb{C}^{n-m} is given by $\mathbf{k}_g^{(m)} = (k_{g(m+1)}, \dots, k_{g(n)})$. One checks that this action does not depend on the choice of the

representative g , such that the action of BC_n/BC_m is well-defined on \mathbb{C}^{n-m} without further specifications. This definition is useful because the set $\{\mathbf{k}_g^{(m)}|g \in G_m\}$ contains one and only once the vector $(\epsilon_{i_1}k_{i_1}, \dots, \epsilon_{i_{n-m}}k_{i_{n-m}})$ for any choice $\epsilon_j = \pm 1$, $1 \leq i_j \leq n$ and $i_j \neq i_k$. For example, $\{\mathbf{k}_g^{(n-1)}|g \in G_{n-1}\} = \{(k_n), (-k_n), (k_{n-1}), (-k_{n-1}), \dots, (k_1), (-k_1)\}$.

Finally, we introduce H_m which is a full set of representatives of the coset BC_m/BC_{m-1} used in section 4 and in the next appendix to simplify computations.

B Proof of solution (4.11)

We want to prove that (4.11) implies (4.8) and (4.10).

We start with (4.10). We start by remarking that a consequence of (4.11) is

$$A_{g\sigma_\alpha}^{(m)} = A_g^{(m)} \times \begin{cases} 1 & 1 \leq \alpha \leq m-1, \\ \frac{T^{(m)}(e^{ik_{g1}}, \dots, e^{ik_{gm-1}}, e^{ik_{gm+1}})}{T^{(m)}(e^{ik_{g1}}, \dots, e^{ik_{gm}})} S(e^{ik_{gm}}, e^{ik_{gm+1}}) & \alpha = m, \\ S(e^{ik_{g\alpha}}, e^{ik_{g\alpha+1}}) & \alpha \geq m+1. \end{cases} \quad (\text{B.1})$$

Then, using again (4.11) to express now $A_g^{(m+1)}$ in terms of $A_g^{(m)}$ and using (B.1) to express $A_{gh}^{(m)}$ in terms of $A_g^{(m)}$, relation (4.10) becomes the functional relation

$$\sum_{j=1}^{m+1} \left[x_j V_1(x_j) p_1(x_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(x_j, x_\ell) a(x_\ell, \frac{1}{x_j}) + \frac{1}{x_j} V_1(\frac{1}{x_j}) p_1(\frac{1}{x_j}) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(\frac{1}{x_j}, x_\ell) a(x_\ell, x_j) - \lambda(x_j) \right] = \Lambda_1 - \Lambda_1^{(m+1)}, \quad (\text{B.2})$$

where x_j stands for $\exp(ik_{gj})$ and the functions are defined in (4.7)-(4.13).

To prove this last relation (B.2), let us introduce the following function

$$F^{(m)}(Z) = \sqrt{pq} \frac{V_1(Z) p_1(Z)}{\lambda(Z) \left(2Z - \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} \right)} \prod_{\ell=1}^{m+1} a(Z, x_\ell) a(x_\ell, \frac{1}{Z}), \quad (\text{B.3})$$

such that

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=x_j} = x_j V_1(x_j) p_1(x_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(x_j, x_\ell) a(x_\ell, \frac{1}{x_j}), \quad (\text{B.4})$$

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=1/x_j} = \frac{1}{x_j} V_1(\frac{1}{x_j}) p_1(\frac{1}{x_j}) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(\frac{1}{x_j}, x_\ell) a(x_\ell, x_j), \quad (\text{B.5})$$

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=\sqrt{\frac{p}{q}}} = \sqrt{pq} \left(\frac{q}{p}\right)^{m+1} \frac{\Lambda_1 + \alpha}{p - q} \left(\sqrt{\frac{p}{q}} + r\right), \quad (\text{B.6})$$

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=\sqrt{\frac{q}{p}}} = -\sqrt{pq} \left(\frac{p}{q}\right)^{m+1} \frac{\Lambda_1 + \gamma}{p - q} \left(\sqrt{\frac{q}{p}} + r\right), \quad (\text{B.7})$$

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=\frac{p+q}{2\sqrt{pq}}} = \frac{\sqrt{pq}}{2(p-q)} (p - q + 2\gamma - 2\alpha) \left(\frac{1}{2} \left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}\right) + r\right), \quad (\text{B.8})$$

$$\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=\infty} = -\sum_{\ell=1}^{m+1} \lambda(x_\ell) - \frac{\sqrt{pq}}{2} \left(\frac{1}{2} \left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}\right) + r\right) - \frac{1}{2}(2\Lambda_1 + \alpha + \gamma). \quad (\text{B.9})$$

Then, (B.2) is equivalent to $\sum_{\text{residue}} F^{(m)}(x) = 0$ which finishes the proof.

For (4.8), we use the same procedure and (B.2) to get

$$\begin{aligned} & \sqrt{pq} \sum_{j=1}^{m+1} \left[x_j^2 V_1(x_j) p_1(x_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(x_j, x_\ell) a(x_\ell, \frac{1}{x_j}) + \frac{1}{x_j^2} V_1(\frac{1}{x_j}) p_1(\frac{1}{x_j}) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{m+1} a(\frac{1}{x_j}, x_\ell) a(x_\ell, x_j) \right] \\ &= \left(\sum_{j=1}^{m+1} \lambda(x_j) + \Lambda_1 - \tilde{\Lambda}_1^{(m)} + p \right) \left(\sum_{j=1}^{m+1} \lambda(x_j) + \Lambda_1 - \Lambda_1^{(m+1)} \right) - C_1^{(m+1)} D_1^{(m)}. \end{aligned} \quad (\text{B.10})$$

The function to consider is now $G^{(m)}(x) = \sqrt{pq} x F^{(m)}(x)$. Its residues at the point $x = x_0$ with $x_0 \neq 0, \infty$ are simple to compute: $\operatorname{Res}(G^{(m)}(x))\Big|_{x=x_0} = \sqrt{pq} x_0 \operatorname{Res}(F^{(m)}(x))\Big|_{x=x_0}$. Since 0 is not a pole, it remains to compute the residue at infinity. It reads:

$$\begin{aligned} \operatorname{Res}(G^{(m)}(x))\Big|_{x=\infty} &= \sum_{j,k=1}^{m+1} \lambda(x_j) \lambda(x_k) + 2 \left(\operatorname{Res}(F^{(m)}(Z))\Big|_{Z=\infty} - \frac{p+q}{4} \right) \sum_{j=1}^{m+1} \lambda(x_j) \\ &\quad - \frac{\sqrt{pq}}{2} \left(r + \left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} \right) \right) \left(2\Lambda_1 + \alpha + \gamma + \frac{p+q}{2} \right) - \frac{(p-q)(\alpha-\gamma)}{4} + \frac{(p+q)^2}{8}. \end{aligned} \quad (\text{B.11})$$

Then, as above, (B.10) is equivalent to $\sum_{\text{residue}} G^{(m)}(x) = 0$.

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