

On logarithmic extensions of local scale-invariance

Malte Henkel^{a,b}

^a Departamento de Física da Universidade de Aveiro,
Campus Universitário de São Tiago, P – 3810-193 Aveiro, Portugal

^b Groupe de Physique Statistique, Département de Physique de la Matière et des Matériaux,¹
Institut Jean Lamour,² CNRS - Nancy Université - UPVM,
B.P. 70239, F – 54506 Vandœuvre lès Nancy Cedex, France

Abstract

The known logarithmic extensions of conformal and Schrödinger-invariance assume translation-invariance in their spatial and temporal coordinates. Therefore, they cannot be applied directly to slow far-from-equilibrium relaxations, where time-translation-invariance no longer holds. Here, the logarithmic extension of ageing-invariance, that is local dynamical scaling without the assumption of time-translation-invariance, is presented. Co-variant two-point functions are derived. Their form is compared to transfer-matrix renormalisation group data for the two-time autoresponse function of the $1D$ critical contact process, which is in the directed percolation universality class.

arXiv:1009.4139v1 [hep-th] 21 Sep 2010

¹permanent address

²Laboratoire associé au CNRS UMR 7198

1 Motivation and background

Dynamical scaling naturally arises in various many-body systems far from equilibrium. A paradigmatic example are ageing phenomena, which may arise in systems quenched, from some initial state, either (i) into a coexistence phase with more than one stable equilibrium state or else (ii) onto a critical point of the stationary state [10, 15, 39]. From a phenomenological point of view, ageing can be defined through the properties of (i) slow, non-exponential relaxation, (ii) breaking of time-translation-invariance and (iii) dynamical scaling. Drawing on the analogy with equilibrium critical phenomena, where scale-invariance can under rather weak conditions be extended to conformal invariance [66, 8], in recent years it has been attempted to carry out an analogous extension of simple dynamical scaling, characterised by a dynamical exponent z , to a new form of *local scale-invariance* (LSI). One of the most simple predictions of that theory is the form of the linear two-time autoresponse function [33, 35, 36]

$$R(t, s) = \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{r})} \right|_{h=0} = s^{-1-a} f_R \left(\frac{t}{s} \right), \quad f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \Theta(y-1) \quad (1.1)$$

which measures the linear response of the order-parameter $\phi(t, \mathbf{r})$ with respect to its canonically conjugated external field $h(s, \mathbf{r})$. The autoresponse exponent λ_R and the ageing exponents a, a' are universal non-equilibrium exponents.¹ The causality condition $t > s$ is explicitly included. The foundations and extensive tests of (1.1) are reviewed in detail in [39].

In the case of a degenerate vacuum state, conformal invariance (of equilibrium phase transitions) can be generalised to *logarithmic conformal invariance* [26, 23, 67], with interesting applications to disordered systems [13], percolation [21, 51] or sand-pile models [65]. For reviews, see [20, 24]. Here, we shall be interested in possible logarithmic extensions of local scale-invariance and in the corresponding generalisations of (1.1).

Logarithmic conformal invariance in $2D$ can be heuristically introduced [26, 67] by replacing, in the left-handed chiral conformal generators $\ell_n = -z^{n+1} \partial_z - (n+1)z^n \Delta$, the conformal weight Δ by a matrix. Non-trivial results are only obtained if that matrix has a Jordan form, so that one writes, in the most simple case

$$\ell_n = -z^{n+1} \partial_z - (n+1)z^n \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix} \quad (1.2)$$

Then the quasi-primary scaling operators on which the ℓ_n act have two components, which we shall denote as $\Psi := \begin{pmatrix} \psi \\ \phi \end{pmatrix}$. The generators (1.2) satisfy the commutation relations $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$ with $n, m \in \mathbb{Z}$. Similarly, the right-handed generators $\bar{\ell}_n$ are obtained by replacing $z \mapsto \bar{z}$ and $\Delta \mapsto \bar{\Delta}$. A simple example of an invariant equation can be written as $\mathcal{S}\Psi = 0$, with the Schrödinger operator

$$\mathcal{S} := \begin{pmatrix} 0 & \partial_z \partial_{\bar{z}} \\ 0 & 0 \end{pmatrix} \quad (1.3)$$

and because of $[\mathcal{S}, \ell_n] = -(n+1)z^n \mathcal{S} - (n+1)nz^{n+1} \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix} \partial_{\bar{z}}$, one has a dynamic symmetry of $\mathcal{S}\Psi = 0$, if the conformal weights $\Delta = \bar{\Delta} = 0$ are chosen.

¹In magnets, mean-field theory suggests that generically $a = a'$ for quenches to $T < T_c$ and $a \neq a'$ for $T = T_c$ [39].

Of particular importance are the consequences for the form of the two-point functions of quasi-primary operators, for which only co-variance under the finite-dimensional sub-algebra $\langle \ell_{\pm 1,0} \rangle \cong \mathfrak{sl}(2, \mathbb{R})$ is needed [26, 67] (we suppress the dependence on \bar{z}_i , but see [16]). Set

$$F := \langle \phi_1(z_1)\phi_2(z_2) \rangle \quad , \quad G := \langle \phi_1(z_1)\psi_2(z_2) \rangle \quad , \quad H := \langle \psi_1(z_1)\psi_2(z_2) \rangle \quad (1.4)$$

Translation-invariance implies that $F = F(z)$, $G = G(z)$ and $H = H(z)$ with $z = z_1 - z_2$. Combination of dilation- and special co-variance applied to F, G leads to $\Delta := \Delta_1 = \Delta_2$ and $F(z) = 0$. Finally, consideration of $H(z)$ leads to

$$G(z) = G(-z) = G_0|z|^{-2\Delta} \quad , \quad H(z) = (H_0 - 2G_0 \ln |z|)|z|^{-2\Delta} \quad (1.5)$$

where G_0, H_0 are normalisation constants. We emphasise here the *symmetric* form of the two-point functions, which does follow from the three co-variance conditions.

Recently, ‘non-relativistic’ versions of logarithmic conformal invariance have been studied [41]. Besides the consideration of dynamics in statistical physics referred to above, such studies can also be motivated from the analysis of dynamical symmetries in non-linear hydrodynamical equations [61, 58, 42, 29, 60], or from studies of non-relativistic versions of the AdS/CFT correspondence [52, 5, 69, 53, 22, 47, 28]. Two distinct non-semi-simple Lie algebras have been considered:

1. the *Schrödinger algebra* $\mathfrak{sch}(d)$, identified in 1881 by Lie as maximal dynamical symmetry of the free diffusion equation in $d = 1$ dimensions. Jacobi had observed already in the 1840s that the elements of $\mathfrak{sch}(d)$ generate dynamical symmetries of free motion. We write the generators compactly as follows

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \frac{n+1}{2}t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - \frac{\mathcal{M}}{2}(n+1)nt^{n-1}\mathbf{r}^2 - \frac{n+1}{2}xt^n \\ Y_m^{(j)} &= -t^{m+1/2}\partial_j - (m + \frac{1}{2})t^{m-1/2}r_j \\ M_n &= -t^n \mathcal{M} \\ R_n^{(jk)} &= -t^n(r_j\partial_k - r_k\partial_j) \end{aligned} \quad (1.6)$$

where \mathcal{M} is a dimensionful constant, x a scaling dimension, $\partial_j = \partial/\partial r_j$ and $j, k = 1, \dots, d$. Then $\mathfrak{sch}(d) = \langle X_{\pm 1,0}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(j,k)} \rangle_{j,k=1,\dots,d}$ is a dynamical symmetry of the free Schrödinger equation $\mathcal{S}\phi = (2\mathcal{M}\partial_t - \nabla_{\mathbf{r}}^2)\phi = 0$, provided $x = d/2$, see [45, 27, 57, 43], and also of Euler’s hydrodynamical equations [61]. An infinite-dimensional extension is $\langle X_n, Y_m^{(j)}, M_n, R_n^{(jk)} \rangle_{n \in \mathbb{Z}, m \in \mathbb{Z} + \frac{1}{2}, j, k = 1, \dots, d}$ [31].

2. The Schrödinger algebra is *not* the non-relativistic limit of the conformal algebra. Rather, from the corresponding contraction one obtains the *conformal Galilei algebra* $\text{CGA}(d)$ [30], which was re-discovered independently several times afterwards [32, 56, 35, 2, 50]. The generators may be written as follows [14]

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - (n+1)t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - n(n+1)t^{n-1}\boldsymbol{\gamma} \cdot \mathbf{r} - x(n+1)t^n \\ Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n \gamma_j \\ R_n^{(jk)} &= -t^n(r_j\partial_k - r_k\partial_j) - t^n(\gamma_j\partial_{\gamma_k} - \gamma_k\partial_{\gamma_j}) \end{aligned} \quad (1.7)$$

where $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ is a vector of dimensionful constants and x is again a scaling dimension. The algebra $\text{CGA}(d) = \langle X_{\pm 1,0}, Y_{\pm 1,0}^{(j)}, R_0^{(jk)} \rangle_{j,k=1,\dots,d}$ does arise as a (conditional) dynamical symmetry in certain non-linear systems, distinct from the equations of non-relativistic incompressible fluid dynamics [73, 14].² The infinite-dimensional extension $\langle X_n, Y_n^{(j)}, R_n^{(jk)} \rangle_{n \in \mathbb{Z}, j,k=1,\dots,d}$ is straightforward.

For both algebras $\mathfrak{sch}(d)$ and $\text{CGA}(d)$, the non-vanishing commutators are given by

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m^{(j)}] = \left(\frac{n}{z} - m\right) Y_{n+m}^{(j)}, \quad [R_0^{(jk)}, Y_m^{(\ell)}] = \delta^{j,\ell} Y_m^{(k)} - \delta^{k,\ell} Y_m^{(j)} \quad (1.8)$$

where the dynamical exponent $z = 2$ for the representation (1.6) and $z = 1$ for the representation (1.7). For the Schrödinger algebra, one has in addition $[Y_{1/2}^{(j)}, Y_{-1/2}^{(k)}] = \delta^{j,k} M_0$.

The algebras $\mathfrak{sch}(d)$ and $\text{CGA}(d)$ arise, besides the conformal algebra, as the only possible finite-dimensional Lie algebras in two classification schemes of non-relativistic space-time transformations, with a fixed dynamical exponent z , namely: (i) either as generalised conformal transformations [17] or (ii) as local scale-transformations which are conformal in time [34].

Now, using the same heuristic device as for logarithmic conformal invariance and replacing in the generators X_n in (1.6,1.7) the scaling dimension by a Jordan matrix

$$x \mapsto \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \quad (1.9)$$

both *logarithmic Schrödinger-invariance* and *logarithmic conformal galilean invariance* can be defined [41]. Adapting the definition (1.4), one now has $F = F(t, \mathbf{r})$, $G = G(t, \mathbf{r})$ and $H = H(t, \mathbf{r})$, with $t := t_1 - t_2$ and $\mathbf{r} := \mathbf{r}_1 - \mathbf{r}_2$ because of temporal and spatial translation-invariance. Since the conformal properties involve the time coordinate only, the practical calculation is analogous to the one of logarithmic conformal invariance outlined above (alternatively, one may use the formalism of nilpotent variables [54, 41]). In particular, one obtains $x := x_1 = x_2$ and $F = 0$. Generalising the results of Hosseiny and Rouhani [41] to d spatial dimensions, the non-vanishing two-point functions read as follows: for the case of logarithmic Schrödinger invariance

$$G = G_0 |t|^{-x} \exp \left[-\frac{\mathcal{M} \mathbf{r}^2}{2t} \right], \quad H = (H_0 - G_0 \ln |t|) |t|^{-x} \exp \left[-\frac{\mathcal{M} \mathbf{r}^2}{2t} \right] \quad (1.10)$$

subject to the constraint [6] $\mathcal{M} := \mathcal{M}_1 = -\mathcal{M}_2$.³ For the case of logarithmic conformal galilean invariance

$$G = G_0 |t|^{-2x} \exp \left[-2\frac{\boldsymbol{\gamma} \cdot \mathbf{r}}{t} \right], \quad H = (H_0 - 2G_0 \ln |t|) |t|^{-2x} \exp \left[-2\frac{\boldsymbol{\gamma} \cdot \mathbf{r}}{t} \right] \quad (1.11)$$

²The generator X_0 leads to the space-time dilatations $t \mapsto \lambda^z t$, $\mathbf{r} \mapsto \lambda \mathbf{r}$, where the dynamical exponent z takes the value $z = 2$ for the representation (1.6) of $\mathfrak{sch}(d)$ and $z = 1$ for the representation (1.7) of $\text{CGA}(d)$. We point out that there exist representations of $\text{CGA}(d)$ with $z = 2$ [35]. From this, one can show that $\mathfrak{age}(1) \subset \text{CGA}(1)$ as well.

³In order to keep the physical convention of non-negative masses $\mathcal{M} \geq 0$, one may introduce a ‘complex conjugate’ ϕ^* to the scaling field ϕ , with $\mathcal{M}^* = -\mathcal{M}$. In dynamics, co-variant two-point functions are interpreted as response functions, written as $R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle$ in the context of Janssen-de Dominicis theory, where the response field $\tilde{\phi}$ has a mass $\tilde{\mathcal{M}} = -\mathcal{M}$, see e.g. [15, 39] for details.

Furthermore, the physical relevant equations are *stochastic* Langevin equations, whose noise terms do break any interesting extended dynamical scale-invariance. However, one may identify a ‘deterministic part’ which may be Schrödinger-invariant, such that the predictions (1.10) remain valid even in the presence of noise [64]. This was rediscovered recently under name of ‘time-dependent deformation of Schrödinger geometry’ [55].

together with the constraint $\gamma := \gamma_1 = \gamma_2$. Here, G_0, H_0 are again normalisation constants.⁴

From the comparison of the results (1.10,1.11) with the form (1.6) of logarithmic conformal invariance, we see that logarithmic corrections to scaling are systematically present. As we shall show, this feature is a consequence of the assumption of time-translation-invariance, since the time-translation operator $X_{-1} = -\partial_t$ is contained in both algebras. On the other hand, from the point of view of non-equilibrium statistical physics, neither the Schrödinger nor the conformal Galilei algebra is a satisfactory choice for a dynamical symmetry, since time-translation-invariance can only hold true at a stationary state and hence eqs. (1.6,1.7) can only be valid in situations such as *equilibrium* critical dynamics. For non-equilibrium systems, it is more natural to leave out time-translations from the algebra altogether. An enormous variety of physical situations with a natural dynamical scaling is known to exist, although the associated stationary state(s), towards which the system is relaxing to, need not be scale-invariant [39]. We then arrive at the so-called *ageing algebra* $\mathbf{age}(d) := \langle X_{0,1}, Y_{\pm 1/2}^{(j)}, M_0, R_0^{(jk)} \rangle_{j,k=1,\dots,d} \subset \mathfrak{sch}(d)$. We shall study the consequences of a logarithmic extension of ageing invariance.

In section 2, we shall write down the generators of logarithmic ageing invariance and shall find the co-variant two-point functions in section 3. In section 4, we discuss some applications. In particular, we shall show that the scaling of the two-time autoresponse function in 1D critical directed percolation is well described in terms of logarithmic ageing invariance. We conclude in section 5.

2 Logarithmic extension of the ageing algebra $\mathbf{age}(d)$

For definiteness, we consider the ageing algebra $\mathbf{age}(d) \subset \mathfrak{sch}(d)$ of the Schrödinger algebra. The generators of the representation (1.6) can in general be taken over, but with the important exception

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n \mathbf{r} \cdot \nabla_{\mathbf{r}} - \frac{\mathcal{M}}{2}(n+1)nt^{n-1}\mathbf{r}^2 - \frac{n+1}{2}xt^n - (n+1)n\xi t^n \quad (2.1)$$

where now $n \geq 0$ and (1.8) remains valid. In contrast to the representation (1.6), we now have *two distinct* scaling dimensions x and ξ , with important consequences on the form of the co-variant two-point functions [64, 36], see also below.⁵ To simplify the discussion, we shall concentrate from now on the temporal part $\langle \Psi(t_1, \mathbf{r})\Psi(t_2, \mathbf{r}) \rangle$, the form of which is described by the two generators $X_{0,1}$, with the commutator $[X_1, X_0] = X_1$. At the end, the spatial part is easily added.

We construct the logarithmic extension of $\mathbf{age}(d)$, analogously to section 1, by considering two scaling operators, with *both* scaling dimensions x and ξ identical, and replacing

$$x \mapsto \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}, \quad \xi \mapsto \begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix} \quad (2.2)$$

in eq. (2.1), the other generators (1.6) being kept unchanged. Without restriction of generality, one can always achieve either a diagonal form (with $x' = 0$) or a Jordan form (with $x' = 1$) of

⁴There is a so-called ‘exotic’ central extension of CGA(2) [49], but the extension of the known two-point functions [3, 4, 50] to the logarithmic version has not yet been attempted.

⁵If one assumes time-translation-invariance, the commutator $[X_1, X_{-1}] = 2X_0$ leads to $\xi = 0$ and one is back to (1.6).

the first matrix, but for the moment it is not yet clear if the second matrix in (2.2) will have any particular structure. Setting $\mathbf{r} = \mathbf{0}$, we have from (2.1) the two generators

$$X_0 = -t\partial_t - \frac{1}{2} \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}, \quad X_1 = -t^2\partial_t - t \begin{pmatrix} x + \xi & x' + \xi' \\ \xi'' & x + \xi \end{pmatrix} \quad (2.3)$$

and we find $[X_1, X_0] = X_1 + \frac{1}{2}t x' \xi'' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \stackrel{!}{=} X_1$. The condition $x' \xi'' \stackrel{!}{=} 0$ follows and we must distinguish two cases.

1. $x' = 0$. The first matrix in (2.2) is diagonal. In this situation, there are two distinct possibilities: (i) either, the matrix $\begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix} \rightarrow \begin{pmatrix} \xi_+ & 0 \\ 0 & \xi_- \end{pmatrix}$ is diagonalisable. We then have a pair of quasi-primary operators, with scaling dimensions (x, ξ_+) and (x, ξ_-) . This reduces to the standard form of non-logarithmic ageing invariance [36]. Or else, (ii), the matrix $\begin{pmatrix} \xi & \xi' \\ \xi'' & \xi \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\xi} & 1 \\ 0 & \bar{\xi} \end{pmatrix}$ reduces to a Jordan form. This is a special case of the situation considered below.
2. $\xi'' = 0$. Both matrices in (2.2) reduce simultaneously to a Jordan form. While one can always normalise such that either $x' = 1$ or else $x' = 0$, there is no obvious normalisation for ξ' . This is the main case which we shall study in the remainder of this paper.

In conclusion: *without restriction on the generality, we can set $\xi'' = 0$ in eqs. (2.2,2.3).*

For illustration and completeness, we give an example of a logarithmically invariant Schrödinger equation. Consider the Schrödinger operator

$$\mathcal{S} := \left(2\mathcal{M}\partial_t - \nabla_{\mathbf{r}}^2 + \frac{2\mathcal{M}}{t} \left(x + \xi - \frac{d}{2} \right) \right) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.4)$$

Using (2.3) with the spatial parts restored, we have $[\mathcal{S}, X_0] = -\mathcal{S}$ and $[\mathcal{S}, X_1] = -2t\mathcal{S}$ and furthermore, \mathcal{S} commutes with all other generators of $\mathfrak{age}(d)$. Therefore, the elements of $\mathfrak{age}(d)$ map any solution of $\mathcal{S} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to another solution of the same equation.

3 Two-point functions

Consider the following two-point functions, built from the components of quasi-primary operators of logarithmic ageing symmetry

$$\begin{aligned} F &= F(t_1, t_2) &:= &\langle \phi_1(t_1)\phi_2(t_2) \rangle \\ G_{12} &= G_{12}(t_1, t_2) &:= &\langle \phi_1(t_1)\psi_2(t_2) \rangle \\ G_{21} &= G_{21}(t_1, t_2) &:= &\langle \psi_1(t_1)\phi_2(t_2) \rangle \\ H &= H(t_1, t_2) &:= &\langle \psi_1(t_1)\psi_2(t_2) \rangle \end{aligned} \quad (3.1)$$

Their co-variance under the representation (2.3), with $\xi'' = 0$, is expressed by the conditions $\hat{X}_{0,1}F \stackrel{!}{=} 0, \dots$, where $\hat{X}_{0,1}$ stands for the extension of (2.3) to two-body operators. This leads

to the following system of eight equations for a set of four functions in two variables.

$$\begin{aligned}
\left[t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2}(x_1 + x_2) \right] F(t_1, t_2) &= 0 \\
\left[t_1^2 \partial_1 + t_2^2 \partial_2 + (x_1 + \xi_1)t_1 + (x_2 + \xi_2)t_2 \right] F(t_1, t_2) &= 0 \\
\left[t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2}(x_1 + x_2) \right] G_{12}(t_1, t_2) + \frac{x'_2}{2} F(t_1, t_2) &= 0 \\
\left[t_1^2 \partial_1 + t_2^2 \partial_2 + (x_1 + \xi_1)t_1 + (x_2 + \xi_2)t_2 \right] G_{12}(t_1, t_2) + (x'_2 + \xi'_2)t_2 F(t_1, t_2) &= 0 \\
\left[t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2}(x_1 + x_2) \right] G_{21}(t_1, t_2) + \frac{x'_1}{2} F(t_1, t_2) &= 0 \\
\left[t_1^2 \partial_1 + t_2^2 \partial_2 + (x_1 + \xi_1)t_1 + (x_2 + \xi_2)t_2 \right] G_{21}(t_1, t_2) + (x'_1 + \xi'_1)t_1 F(t_1, t_2) &= 0 \quad (3.2) \\
\left[t_1 \partial_1 + t_2 \partial_2 + \frac{1}{2}(x_1 + x_2) \right] H(t_1, t_2) + \frac{x'_1}{2} G_{12}(t_1, t_2) + \frac{x'_2}{2} G_{21}(t_1, t_2) &= 0 \\
\left[t_1^2 \partial_1 + t_2^2 \partial_2 + (x_1 + \xi_1)t_1 + (x_2 + \xi_2)t_2 \right] H(t_1, t_2) \\
+ (x'_1 + \xi'_1)t_1 G_{12}(t_1, t_2) + (x'_2 + \xi'_2)t_2 G_{21}(t_1, t_2) &= 0
\end{aligned}$$

where $\partial_i = \partial/\partial t_i$. We expect an unique solution, up to normalisations. It is convenient to solve the system (3.2) via the ansatz, with $y := t_1/t_2$

$$\begin{aligned}
F(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} f(y) \\
G_{12}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \sum_{j \in \mathbb{Z}} \ln^j t_2 \cdot g_{12,j}(y) \\
G_{21}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \sum_{j \in \mathbb{Z}} \ln^j t_2 \cdot g_{21,j}(y) \quad (3.3) \\
H(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \sum_{j \in \mathbb{Z}} \ln^j t_2 \cdot h_j(y)
\end{aligned}$$

1. The function F does not contain any logarithmic contributions and its scaling function satisfies the equation $f'(y) = 0$, hence

$$f(y) = f_0 = \text{cste.} \quad (3.4)$$

This reproduces the well-known form of non-logarithmic local scaling [36].

Comparing this with the usual form (1.1) of standard LSI with $z = 2$, the ageing exponents a, a', λ_R are related to the scaling dimensions as follows:

$$a = \frac{1}{2}(x_1 + x_2) - 1 \quad , \quad a' - a = \xi_1 + \xi_2 \quad , \quad \lambda_R = 2(x_1 + \xi_1) \quad (3.5)$$

For example, the exactly solvable 1D kinetic Ising model with Glauber dynamics at zero temperature [25] satisfies (1.1) with the values $a = 0, a' - a = -\frac{1}{2}, \lambda_R = 1, z = 2$. Further examples of systems with $a' - a \neq 0$ are given by the non-equilibrium critical dynamics of the kinetic Ising model with Glauber dynamics, both for $d = 2$ and $d = 3$ [36, 39].

2. Next, we turn to the function G_{12} . Co-variance under X_0 leads to the condition

$$\left(g_{12,1}(y) + \frac{1}{2}x'_2 f(y) \right) + \sum_{j \neq 0} (j+1) \ln^j t_2 \cdot g_{12,j+1}(y) = 0 \quad (3.6)$$

which must hold true for all times t_2 . This implies

$$g_{12,1}(y) = -\frac{1}{2}x'_2 f(y) \quad , \quad g_{12,j}(y) = 0 \quad ; \quad \forall j \neq 0, 1 \quad (3.7)$$

In order to simplify the notation for later use, we set

$$g_{12}(y) := g_{12,0}(y) \quad , \quad \gamma_{12}(y) := g_{12,1}(y) = -\frac{1}{2}x'_2 f(y) \quad (3.8)$$

and these two give the only non-vanishing contributions in the ansatz (3.2). Furthermore, the last remaining function g_{12} is found from the co-variance under X_1 , which gives

$$\sum_{j \in \mathbb{Z}} \ln^j t_2 \left(y(y-1)g'_{12,j}(y) + (j+1)g_{12,j+1}(y) \right) + (x'_2 + \xi'_2)f(y) = 0 \quad (3.9)$$

for all times t_2 . Combining the resulting two equations for g_{12} and γ_{12} with (3.8) leads to

$$y(y-1)g'_{12}(y) + \left(\frac{x'_2}{2} + \xi'_2 \right) f(y) = 0 \quad (3.10)$$

3. The function G_{21} is treated similarly. We find

$$g_{21}(y) := g_{21,0}(y) \quad , \quad \gamma_{21}(y) := g_{21,1}(y) = -\frac{1}{2}x'_1 f(y) \quad , \quad g_{21,j}(y) = 0 \quad ; \quad \text{for all } j \neq 0, 1 \quad (3.11)$$

and the differential equation

$$y(y-1)g'_{21}(y) + (x'_1 + \xi'_1) y f(y) - \frac{1}{2}x'_1 f(y) = 0 \quad (3.12)$$

4. Finally, dilatation-covariance of the function H leads to $h_j(y) = 0$ for all $j \neq 0, 1, 2$ and

$$\begin{aligned} h_1(y) &= -\frac{1}{2}(x'_1 g_{12}(y) + x'_2 g_{21}(y)) \\ h_2(y) &= \frac{1}{4}x'_1 x'_2 f(y) \end{aligned} \quad (3.13)$$

The last remaining function $h_0(y)$ is found from co-variance under X_1 which leads to

$$y(y-1)h'_0(y) + \left((x'_1 + \xi'_1) y - \frac{1}{2}x'_1 \right) g_{12}(y) + \left(\frac{1}{2}x'_2 + \xi'_2 \right) g_{21}(y) = 0 \quad (3.14)$$

Using (3.4), the equations (3.10,3.12,3.14) are readily solved and we find

$$\begin{aligned} g_{12}(y) &= g_{12,0} + \left(\frac{x'_2}{2} + \xi'_2 \right) f_0 \ln \left| \frac{y}{y-1} \right| \\ g_{21}(y) &= g_{21,0} - \left(\frac{x'_1}{2} + \xi'_1 \right) f_0 \ln |y-1| - \frac{x'_1}{2} f_0 \ln |y| \\ h_0(y) &= h_0 - \left[\left(\frac{x'_1}{2} + \xi'_1 \right) g_{21,0} + \left(\frac{x'_2}{2} + \xi'_2 \right) g_{12,0} \right] \ln |y-1| - \left[\frac{x'_1}{2} g_{21,0} - \left(\frac{x'_2}{2} + \xi'_2 \right) g_{12,0} \right] \ln |y| \\ &\quad + \frac{1}{2} f_0 \left[\left(\left(\frac{x'_1}{2} + \xi'_1 \right) \ln |y-1| + \frac{x'_1}{2} \ln |y| \right)^2 - \left(\frac{x'_2}{2} + \xi'_2 \right)^2 \ln^2 \left| \frac{y}{y-1} \right| \right] \end{aligned} \quad (3.15)$$

where $f_0, g_{12,0}, g_{21,0}, h_0$ are normalisation constants. We summarise our results:

$$\begin{aligned}
F(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} f_0 \\
G_{12}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \left(g_{12}(y) + \ln t_2 \cdot \gamma_{12}(y) \right) \\
G_{21}(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \left(g_{21}(y) + \ln t_2 \cdot \gamma_{21}(y) \right) \\
H(t_1, t_2) &= t_2^{-(x_1+x_2)/2} y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \\
&\quad \times \left(h_0(y) + \ln t_2 \cdot h_1(y) + \ln^2 t_2 \cdot h_2(y) \right)
\end{aligned} \tag{3.16}$$

where the scaling functions, depending only on $y = t_1/t_2$, are given by eqs. (3.8,3.11,3.13,3.15).

Although the algebra $\mathfrak{age}(d)$ was written down for a dynamic exponent $z = 2$, the space-independent part of the two-point functions is essentially independent of this feature. The change $(x, x', \xi, \xi') \mapsto ((2/z)x, (2/z)x', (2/z)\xi, (2/z)\xi')$ in eq. (3.16) produces the form valid for an arbitrary dynamical exponent z .

Since for $z = 2$, the space-dependent part of the generators is not affected by the passage to the logarithmic theory via the substitution (2.2), we recover the same space-dependence as for the non-logarithmic theory with $z = 2$. For example,

$$\begin{aligned}
F(t_1, t_2; \mathbf{r}_1, \mathbf{r}_2) &= \delta(\mathcal{M}_1 + \mathcal{M}_2) \Theta(t_1 - t_2) t_2^{-(x_1+x_2)/2} f_0 \\
&\quad \times y^{\xi_2+(x_2-x_1)/2} (y-1)^{-(x_1+x_2)/2-\xi_1-\xi_2} \exp \left[-\frac{\mathcal{M}_1 (\mathbf{r}_1 - \mathbf{r}_2)^2}{2 t_1 - t_2} \right]
\end{aligned} \tag{3.17}$$

where we also included the causality condition $t_1 > t_2$, expressed by the Heaviside function Θ , which can be derived using the methods of [35]. Similar forms hold true for G_{12}, G_{21}, H .

Comparison with the result (1.10) of logarithmic Schrödinger-invariance shows:

1. logarithmic contributions, either as corrections to the scaling behaviour via additional powers of $\ln t_2$, or else in the scaling functions themselves, may be described independently in terms of the parameter sets (x'_1, x'_2) and (ξ'_1, ξ'_2) .

In particular, one may choose to introduce the logarithmic structure only through a single one of the two generators X_0 and X_1 .

2. If one sets $x'_1 = x'_2 = 0$, the scaling functions g_{12}, g_{21} and h_0 contain logarithmic terms, although there is no predicted logarithmic breaking of scaling, in contrast to what occurs in logarithmic conformal invariance or logarithmic Schrödinger invariance.
3. The constraint $F = 0$ of both logarithmic conformal invariance and logarithmic Schrödinger invariance is no longer required.
4. If time-translation-invariance is assumed, one has $\xi_1 = \xi_2 = \xi'_1 = \xi'_2 = 0$, $x_1 = x_2$ and $f_0 = 0$. The functional form of eqs. (3.16,3.17) then reduces to the Schrödinger-invariant forms of eq. (1.10).

4 Applications

4.1 Directed percolation

It is well-understood that critical $2D$ percolation can be described in terms of conformal invariance [46]. Notably, Cardy [12] and Watts [72] used conformal invariance to derive their celebrated formulæ for the crossing probabilities. More recently, it has been shown that a precise formulation of the conformal invariance methods required in their derivations actually leads to a logarithmic conformal field theory [51]. Since *directed* percolation is in many respects quite analogous to ordinary percolation, we raise the question:

can one describe dynamical scaling properties of critical directed percolation in terms of logarithmic ageing invariance ?

The directed percolation universality class can be realised in many different ways, with often-used examples being either the contact process or else Reggeon field theory, and very precise estimates of the location of the critical point and the critical exponents are known, see [40, 59, 38] and references therein, and in agreement with extensive recent experiments [70]. In the contact process, a response function can be defined by considering the response of the time-dependent particule concentration with respect to a time-dependent particule-production rate. The relaxation from an initial state is in many respects quite analogous to what is seen in systems with an equilibrium stationary state [18, 68, 7]. In figure 1, we show data of the autoresponse function $R(t, s) = s^{-1-a} f_R(t/s)$ of $1D$ critical directed percolation, realised here by the contact process. The initial state contains uncorrelated particles at a finite density. The data are obtained from the transfer matrix renormalisation group (TMRG) which are considerably more precise than data obtained from a Monte Carlo simulation, subject to stochastic uncertainties [18, 19]. Aspects of local scaling can be emphasised by plotting the function

$$h_R(y) := f_R(y) y^{\lambda_R/z} (1 - y^{-1})^{1+a} \quad (4.1)$$

over against $y = t/s$, with the exponents taken from [38]. We observe an excellent collapse of the data when y is large enough, but we also see that finite-time corrections to dynamical scaling arise when $y \rightarrow 1$, the precise form of which depends on the waiting time s .

Starting from large values of y , and proceeding towards $y \rightarrow 1$, a description of the data in terms of local scale-invariance increasingly needs to take finer points into account. First, in its most simple form, one would naïvely assume $a = a'$, when (1.1) predicts a horizontal line in this plot. Indeed, this describes the data down to $y = t/s \approx 3 - 4$, but fails when y becomes smaller. We had tried earlier [36] to take these deviations into account by admitting that a and a' can be different. This is equivalent to the assumption that $\xi + \tilde{\xi} \neq 0$ and describes the data well down to $t/s \approx 1.1$. However, further systematic deviations exist when t/s is yet closer to unity. For the values of s used in figure 1, it is clear that one is still in the dynamical scaling regime and an explanation in terms of a more general form of the scaling function should be sought.

We now try to explain the TMRG data in figure 1 in terms of logarithmic ageing invariance (extended to an arbitrary dynamical exponent z as outlined above). We make the working hypothesis $R(t, s) = \langle \psi(t) \tilde{\psi}(s) \rangle$, where the two scaling operators ψ and $\tilde{\psi}$ are described by the

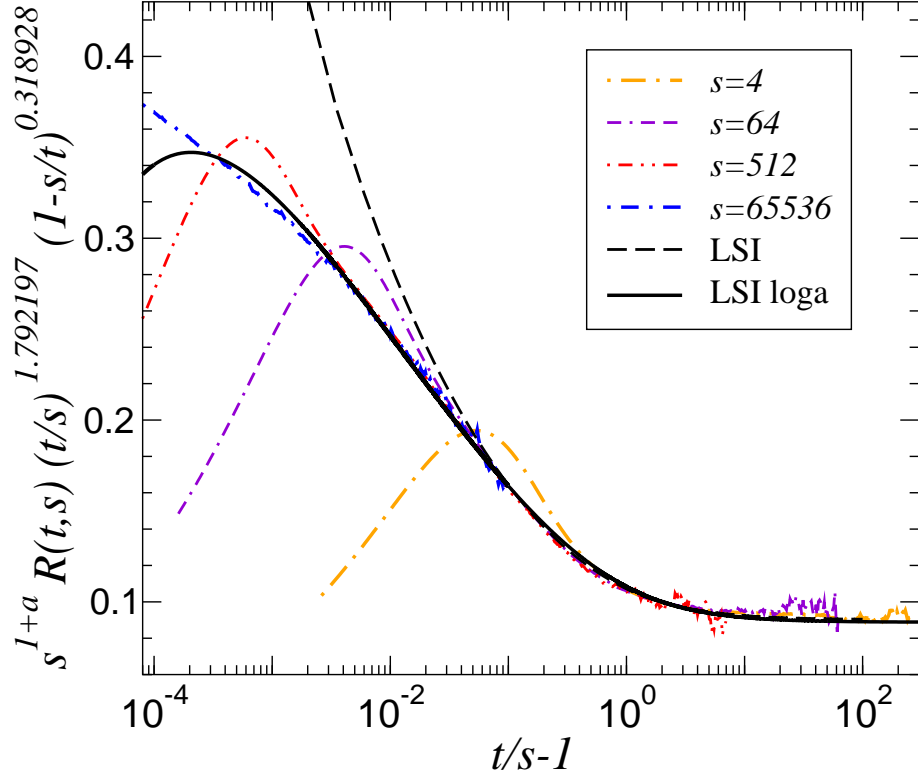


Figure 1: Scaling of the autoresponse $R(t, s) = s^{-1-a} f_R(t/s)$ of the 1D critical contact process, as a function of $y = t/s$, for several values of the waiting time s , and indicated by the dash-dotted lines. The dashed line labelled ‘LSI’ gives the prediction for the scaling function $h_R(y) = f_R(y) y^{\lambda_R/z} (1 - 1/y)^{1+a}$ as obtained from standard, non logarithmic local scale-invariance (1.1), with $a' - a = 0.26$. The full curve labelled ‘LSI loga’ is the prediction (4.3) of logarithmic local scale-invariance with $\xi' = 0$, as described in the text.

logarithmically extended scaling dimensions

$$\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}, \quad \begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{x} & \tilde{x}' \\ 0 & \tilde{x} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\xi} & \tilde{\xi}' \\ 0 & \tilde{\xi} \end{pmatrix} \quad (4.2)$$

In principle, one might have logarithmic corrections to scaling, according to eq. (3.16). Because of the excellent scaling behaviour seen in figure 1, we conclude that logarithmic corrections are absent in the data. Hence the two functions $h_{1,2}(y)$ must vanish. Because of eq. (3.13), this means that $x' = \tilde{x}' = 0$. Then logarithmic ageing invariance (3.15) predicts

$$\begin{aligned} h_R(y) = & \left(1 - \frac{1}{y}\right)^{a-a'} \left(h_0 - g_{12,0} \tilde{\xi}' \ln(1 - 1/y) - \frac{1}{2} f_0 \tilde{\xi}'^2 \ln^2(1 - 1/y) \right. \\ & \left. - g_{21,0} \xi' \ln(y - 1) + \frac{1}{2} f_0 \xi'^2 \ln^2(y - 1) \right) \end{aligned} \quad (4.3)$$

Since for y sufficiently large, the numerically observed scaling function $h_R(y)$ becomes essentially constant, we conclude that there are no leading logarithmic contributions in the $y \rightarrow \infty$ asymptotic behaviour, hence $\xi' = 0$ and the second line in (4.3) vanishes. Hence we arrive at the following phenomenological form $h_R(y) = h_0 (1 - 1/y)^{a-a'} (1 - A \ln(1 - 1/y) - B \ln^2(1 - 1/y))$,

with the normalisation constant h_0 and where the universal parameters A, B and the exponent $a - a'$ must be determined from the data. The full curve in figure 1 shows to what extent this describes the available data, with the chosen values

$$a' - a \simeq 0.174 \quad , \quad A \simeq 0.13 \quad , \quad B \simeq 0.0168 \quad , \quad h_0 \simeq 0.0888 \quad (4.4)$$

Indeed, the chosen form gives a full account of the TMRG data, down to $t/s - 1 \approx 2 \cdot 10^{-3}$, about two orders of magnitude smaller than for non-logarithmic local scale-invariance. This is about the size of the region where dynamical scaling has been confirmed through the collapse of data with different values of s . However, we also point out that the functional form of $h_R(y)$ may depend quite sensitively on the values of the several parameters such that error bars in $a' - a, A, B$ are of the order of at least 25%.

In particular, the TMRG data suggest that the logarithmic nature of the scaling operator should merely enter via the response field, and only through the consideration of the ‘special’ transformation X_1 , since $x' = \tilde{x} = \xi = 0$ and $\tilde{\xi}' \neq 0$ is the only quantity which describes the departure from the standard non-logarithmic scaling. We observe that the present estimate $a' - a = 0.17(5)$ is considerably more small than our earlier estimate $a' - a \approx 0.27$. This is the first time that a theory could be formulated which describes the autoresponse in the entire range of the scaling variable, $2 \cdot 10^{-3} \lesssim y - 1 \leq \infty$. We point out that existing field-theoretical methods based on the ε -expansion [7] obtain reliable results only in the opposite case $y \gg 1$, notably on non-equilibrium exponents and universal amplitudes.

4.2 Logarithmic scaling forms

In the ageing of several magnetic systems, such as the 2D XY model quenched from a fully disordered initial state to a temperature $T < T_{KT}$ below the Kosterlitz-Thouless transition temperature [11, 9, 1] or fully frustrated spin systems quenched onto their critical point [71, 44], the following phenomenological scaling behaviour

$$R(t, s) = s^{-1-a} f_R \left(\frac{t}{\ln t} \frac{\ln s}{s} \right) \quad (4.5)$$

has been found to describe the simulational data well. Could this scaling form be explained within the context of logarithmic ageing invariance? *Hélas*, this question has to be answered in the negative. If one fixes $y = t/s$ and expands the quotient $\ln s / \ln t = \ln s / (\ln y + \ln s)$ for $s \rightarrow \infty$, eq. (4.5) leads to the following generic scaling behaviour

$$R(t, s) = s^{-1-a} \sum_{k, \ell} f_{k, \ell} y^k \left(\frac{\ln y}{\ln s} \right)^\ell \quad (4.6)$$

Comparison with the explicit scaling forms derived in section 3 shows that there arise only combinations of the form $\ln^n y \cdot \ln^m s$ or $\ln^n(y - 1) \cdot \ln^m s$, where the integers n, m must satisfy $0 \leq n + m \leq 2$. This is incompatible with (4.6).

In conclusion, the logarithmic scaling form (4.5) cannot be understood in terms of logarithmic ageing invariance, as presently formulated.

5 Conclusions

We have discussed the extension of dynamical scaling towards local scale-invariance in the case when the physical scaling operator acquires a partner with the same scaling dimension. Since in far-from-equilibrium relaxation, time-translation-invariance does not hold, one cannot appeal directly to the known cases of logarithmic conformal and Schrödinger-invariance. Indeed, analogously to the non-logarithmic case, the doublets of scaling operators are described by *pairs* of Jordan matrices of scaling dimensions. When computing the co-variant two-point functions, the absence of time-translation-invariance allows, independently, to include logarithmic corrections to scaling and also non-trivial modification of the scaling functions, see eq. (3.15,3.16). This generalises the forms found from logarithmic conformal or Schrödinger-invariance [41].

Motivated by the fact that important properties of ordinary $2D$ critical percolation can be understood in terms of *logarithmic* conformal invariance [51], we have re-analysed the autoresponse $R(t, s)$ of critical $1D$ directed percolation in terms of our logarithmic extension of local scale-invariance. The available data suggest that at least this observable behaves as if directed percolation were described by logarithmic local scale-invariance (with an obvious generalisation to $z \neq 2$). Of course, further independent tests of this possibility are required.

Since logarithmic conformal invariance also arises in disordered systems at equilibrium, it would be of interest to see whether logarithmic local scale-invariance could help in improving the understanding of the relaxation processes of disordered systems far from equilibrium, see e.g. [63, 37, 48, 62].

Acknowledgement: I thank T. Enns for the TMRG data, M. Pleimling for useful correspondence and the Departamento de Física da Universidade de Aveiro for warm hospitality.

References

- [1] S. Abriet and D. Karevski, Eur. Phys. J. **B37**, 47 (2004).
- [2] A. Bagchi and I. Mandal, Phys. Lett. **B675**, 393 (2009).
- [3] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, arXiv:0912.1090.
- [4] A. Bagchi and R. Gopakumar, J. High-energy Phys. 0907:307 (2009).
- [5] K. Balasubramanian and J. McGreivy, Phys. Rev. Lett. **101**, 061601 (2008).
- [6] V. Bargman, Ann. of Math. **56**, 1 (1954).
- [7] F. Baumann and A. Gambassi, J. Stat. Mech. P01002 (2007).
- [8] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).
- [9] L. Berthier, P. Holdsworth and M. Selitto, J. Phys. **A34**, 1805 (2001).
- [10] A.J. Bray, Adv. Phys. **43**, 357 (1994).
- [11] A.J. Bray, A.J. Briant and D.K. Jarvis, Phys. Rev. Lett. **84**, 001503 (2000).

- [12] J.L. Cardy, J. Phys. **A25**, L201 (1992).
- [13] J.-S. Caux, I.I. Kogan and A.M. Tselik, Nucl. Phys. **B466**, 444 (1996).
- [14] R. Cherniha and M. Henkel, J. Math. Anal. Appl. **369**, 120 (2010).
- [15] L.F. Cugliandolo, in *Slow Relaxation and non equilibrium dynamics in condensed matter*, Les Houches Session 77 July 2002, J-L Barrat, J Dalibard, J Kurchan, M V Feigel'man eds, Springer(Heidelberg 2003) (cond-mat/0210312).
- [16] A.-L. Do and M. Flohr, Nucl. Phys. **B802**, 475 (2008).
- [17] C. Duval and P.A. Horváthy, J. Phys. A: Math. Theor. **42**, 465206 (2009).
- [18] T. Enss, M. Henkel, A. Picone and U. Schollwöck, J. Phys. **A37**, 10479 (2004).
- [19] T. Enss, private communication (2006).
- [20] M. Flohr, Int. J. Mod. Phys. **A18**, 4497 (2003).
- [21] M. Flohr and A. Müller-Lohmann, J. Stat. Mech. P12004 (2005).
- [22] C.A. Fuertes and S. Moroz, Phys. Rev. **D79**, 106004 (2009).
- [23] M.R. Gaberdiel and H.G. Kausch, Phys. Lett. **B386**, 131 (1996); Nucl. Phys. **B538**, 631 (1999).
- [24] M.R. Gaberdiel, Int. J. Mod. Phys. **A18**, 4593 (2003).
- [25] C. Godrèche and J.M. Luck, J. Phys. A. Math. Gen. **33**, 1151 (2000).
- [26] V. Gurarie, Nucl. Phys. **B410**, 535 (1993).
- [27] C. R. Hagen, Phys. Rev. **D5**, 377 (1972).
- [28] S.A. Hartnoll, Class. Quant. Grav. **26**, 224002 (2009).
- [29] M. Hassaïne and P.A. Horváthy, Ann. of Phys. **282**, 218 (2000); Phys. Lett. **A279**, 215 (2001).
- [30] P. Havas and J. Plebanki, J. Math. Phys. **19**, 482 (1978).
- [31] M. Henkel, J. Stat. Phys. **75**, 1023 (1994).
- [32] M. Henkel, Phys. Rev. Lett. **78**, 1940 (1997).
- [33] M. Henkel, M. Pleimling, C. Godrèche and J.-M. Luck, Phys. Rev. Lett. **87**, 265701 (2001).
- [34] M. Henkel, Nucl. Phys. **B641**, 405 (2002).
- [35] M. Henkel and J. Unterberger, Nucl. Phys. **B660**, 407 (2003).
- [36] M. Henkel, T. Enss and M. Pleimling, J. Phys. A Math. Gen. **39**, L589 (2006).

- [37] M. Henkel and M. Pleimling, Phys. Rev. **B78**, 224419 (2008).
- [38] M. Henkel, H. Hinrichsen and S. Lübeck, *Non-equilibrium phase transitions vol. 1: absorbing phase transitions*, Springer (Heidelberg 2009).
- [39] M. Henkel and M. Pleimling, *Non-equilibrium phase transitions vol. 2: ageing and dynamical scaling far from equilibrium*, Springer (Heidelberg 2010).
- [40] H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
- [41] A. Hosseiny and S. Rouhani, J. Math. Phys at press (2010) - [arXiv:1001.1036](https://arxiv.org/abs/1001.1036)
- [42] E.V. Ivashkevich, J. Phys. A: Math. Gen. **30**, L525 (1997).
- [43] R. Jackiw, Physics Today **25**, 23 (1972).
- [44] M. Karsai, J.-Ch. Anglès d’Auriac and F. Iglói, J. Stat. Mech. P07044 (2009).
- [45] H.A. Kastrup, Nucl. Phys. **B7**, 545 (1968).
- [46] R. Langlands, P. Pouloit and Y. Saint-Aubin, Bull. Am. Math. Soc. **30**, 1 (1994).
- [47] R.G. Leigh and N.N. Hoang, J. High-energy Phys. 0911:010 (2009); and 1003:027 (2010).
- [48] M.P.O. Loureiro, J.J. Arenzon, L.F. Cugliandolo and M. Picco, Phys. Rev. **E81**, 021129 (2010).
- [49] J. Lukierski, P.C. Stichel and W.J. Zakrewski, Phys. Lett. **A357**, 1 (2006); Phys. Lett. **B650**, 203 (2007).
- [50] D. Martelli and Y. Tachikawa, J. High-energy Phys. 1005:091 (2010).
- [51] P. Mathieu and D. Ridout, Phys. Lett. **B657**, 120 (2007); and Nucl. Phys. **B801**, 268 (2008).
- [52] J.M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
- [53] D. Minic and M. Pleimling, Phys. Rev. **E78**, 061108 (2008); and [arXiv:1007.3970](https://arxiv.org/abs/1007.3970).
- [54] S. Moghimi-Araghi, S. Rouhani and M. Saadat, Nucl. Phys. **B599**, 531 (2000).
- [55] Yu. Nakayama, J. High-energy Phys. **04**, 102 (2010).
- [56] J. Negro, M.A. del Olmo and A. Rodríguez-Marco, J. Math. Phys. **38**, 3786 (1997).
- [57] U. Niederer, Helv. Phys. Acta **45**, 802 (1972).
- [58] U. Niederer, Helv. Phys. Acta **51**, 220 (1978).
- [59] G. Ódor, Rev. Mod. Phys. **76**, 663 (2004).
- [60] L. O’Raifeartaigh and V.V. Sreedhar, Ann. of Phys. **293**, 215 (2001).
- [61] L.V. Ovsiannikov, *The Group Analysis of Differential Equations*, Academic Press (London 1980).

- [62] H. Park and M. Pleimling, Phys. Rev. **B** at press (2010) – [arXiv:1009.1677](#).
- [63] R. Paul, S. Puri and H. Rieger, Europhys. Lett. **68**, 881 (2004); Phys. Rev. **E71**, 061109 (2005).
- [64] A. Picone and M. Henkel, Nucl. Phys. **B688** 217 (2004).
- [65] V.S. Pogosyan, S.Y. Grigorev, V.B. Priezzhev and P. Ruelle, Phys. Lett. **B659**, 768 (2008); and J. Stat. Mech. P07025 (2010).
- [66] A.M. Polyakov, Sov. Phys. JETP Lett. **12**, 381 (1970).
- [67] M.R. Rahimi Tabar, A. Aghamohammadi and M. Khorrami, Nucl. Phys. **B497**, 555 (1997).
- [68] J.J. Ramasco, M. Henkel, M.A. Santos, C. da Silva Santos, J. Phys. **A37**, 10497 (2004).
- [69] D.T. Son, Phys. Rev. **D78**, 106005 (2008).
- [70] K.A. Takeuchi, M. Kuroda, H. Chaté and M. Sano, Phys. Rev. Lett. **99**, 234503 (2009) – err. **103**, 089901(E) (2009); and Phys. Rev. **E80**, 051116 (2009).
- [71] J.-C. Walter and C. Chatelain, J. Stat. Mech. P07005 (2008); and P10017 (2009).
- [72] G.M.T. Watts, J. Phys. **A29**, L363 (1996).
- [73] P.-M. Zhang and P.A. Horváthy, Eur. Phys. J. **C65**, 607 (2010).