Calibration of Transparency Risks: a Note[*]

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Abstract

The aim of this research is to give a simple framework to evaluate/quantize the *transparency* of a firm. We assume that the process of the firm value is only observable once in a while but is strongly correlated with the stock price which is observable and tradable. This hybrid type structure make the transparency "observable". The implication of the present study is that the depth of the shock to the market caused by the precise accounting information does reflect the degree of transparency. Furthermore, it can be quantized resorting to the calibration method.

Keywords. Transparency, Credit Risk, Calibration, Merton Model

1 Introduction

1.1 Credit Risk Modeling: Literature

The models of credit risks are often classified into two groups by the degree of details in modeling how the default occurs. In the *reduced form* models

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such as Jarrow-Turnbull's [4] and Duffie-Singleton's [2], the default is an exogenous event, and its probabilistic nature is directly modeled in terms of the default probability or the hazard rate, etc. On the other hand, if the default is endogenous/set to be a consequence of some economic activities, the model is called *structural*. In most structural models including Merton [8] and Leland-Toft [7], the default occurs when the firm value process reaches a boundary.

As Jarrow and Protter [3] pointed out, the two approaches can be unified by introducing asymmetry between the the manager's perspective and the market's. The pioneering work in this hybrid approach was done by Duffie and Lando [1]. They assumed a default structure of Leland-Toft type which is not directly observable to the market. This filtering approach of Duffie-Land was generalized into the continuous time framework by Kusuoka [6] and Nakagawa [9].

1.2   Hybrid Models and Transparency Risk

It is noteworthy that in the hybrid approach the degree of accounting transparency is modeled implicitly by the difference of the manager's perspective and the market's one. In fact, it is claimed in [1, p.634 l.30–32] that *the shape of the term structure of credit spreads may indeed play a useful empirical role in estimating the degree of transparency of a firm*, and in [9, p.134 l.13–15] it is pointed out that *the credit spread can be explained by the hazard rate plus some fluctuation caused by the difference* between the manager's filtration and the market's one. That is to say, we have the following decomposition:

$$\text{credit risk} = \text{default risk} + \text{transparency risk}. \tag{1.1}$$

With this view, we can say that in the structural approach only default risk is modeled while in the reduced form approach credit risk is treated without decomposing. The transparency risk is made visible only when one construct a hybrid type model.

1.3   Our Motivation/Calibration

The importance of the accounting transparency is now widely recognized among the managers as well as the investors. Of course the Sarbanes-Oxley (SOx) Act of 2002 in U.S.A, followed by J-SOx of September 2007 in Japan,

was a cornerstone but they are basically aimed to reduce the transparency risks from the investor's perspective. The decomposition (1.1), however, suggests that the managers can reduce the credit spread by promoting transparency. It says there can be positive incentives for the manager to reduce the transparency risks. This observation is empirically supported by the study of Yu [10].

The problem is then how we could know the degree of transparency. The formula of Duffie-Lando or the one by Nakagawa is too complicated to calibrate it to the market values. We need simpler formulas but nonetheless it should be based on a hybrid type model.

1.4 Structure of Transparency / Slightly Incomplete Market

Motivated by the above demand, in the present paper we will construct a simple hybrid type model out of the classical Merton's structural model. Structure of transparency is modeled by "ρ-coupling" of Wiener processes. Roughly speaking, the filtration of manager's and market's are generated by two 1-dimensional Wiener processes W and W' starting from 0 respectively. Here

$$\langle W, W' \rangle_t = \rho t \tag{1.2}$$

for a constant $\rho \in (0,1)$, which is set the unique parameter describing the transparency.

We will work on the continuous time framework but the full accounting information (or the firm value) is supposed to be available at the discrete set of dates $t_1, \ldots, t_n, \ldots$. These discrete filtration make the market "slightly incomplete" and thus we need to be careful about the consistency with the no-arbitrage framework.

1.5 Main Results

Under the simple hybrid model assumptions described roughly in the above, the present study will show that

- with a proper choice of the state price density, the market value of a firm is obtained by the image of the Ornstein-Uhlenbeck semigroup,

- the credit spread formula is explicitly obtained and it doesn't depend explicitly on the transparency parameter at the dates $t_1, \ldots, t_n, \ldots$,

- and by this property the parameter can be calibrated to the market value.

1.6   Organization of the Rest of the Paper

In section 2.1, we shall give the setting and the market model on which we will be working. The first result on the consistency of our market model is given as Theorem 1. In section 2.2, presented is a result on an economic property of our market model. Under these settings, we will obtain a generic formula (2.3) in section 2.3. The formula is applied to evaluation of the credit risk and transparency risk in sections 3.1 and 3.2. Proofs for those mathematical results are given in sections 4.1, 4.2, and 4.3.

2   The Formulas

2.1   The Firm Value and the Market

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which a one dimensional Wiener process $\{W_t\}$ can be defined, and $\{\mathcal{F}_t\}$ be the natural filtration of W. We assume Merton's economy in [8]; i.e, the firm value is assumed to be

$$V_t := V_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t), \tag{2.1}$$

where $\mu \in \mathbf{R}$ and $\sigma \geq 0$.

Let $\mathbf{T} := \{t_k : k \in \mathbf{Z}_+\} \subset \mathbf{R}_+$, where $t_0 = 0$ and $t_k < t_{k+1}$ for each $k \in \mathbf{Z}_+$, be the dates of the accounting report; namely we suppose that the firm value V is observable only at each $t_k \in \mathbf{T}$, and during the interval (t_k, t_{k+1}) the market can only "guess" the firm value. Let

$$\mathcal{G}_t := \mathcal{F}_t' \vee \sigma(V_s : s \in \mathbf{T}, s \leq t),$$

where $\mathcal{F}_t'$ is the natural filtration of W' which is introduced in Introduction. Note that the joint law of W and W' is completely determined by (1.2).

To be consistent with no-arbitrage framework, we set the state price density of the market by

$$Z_t = \exp(\theta W_t' - \frac{1}{2}\theta^2 t - rt),$$

where r is the constant interest rate and

$$\theta := -\frac{\mu - r}{\sigma\rho}.$$

It should be noted that $Z_t e^{rt}$ is a martingale with respect to the filtration $\{\mathcal{G}\}$ and therefore a probability measure $\mathbf{Q}$ on $\{\mathcal{G}\}$ is defined by

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{G}_t} = Z_t e^{rt}.$$

Our first result is the following.

Theorem 1. *The "filtered" firm value process $V_t' := \mathbf{E}[V_t|\mathcal{G}_t]$ defines the no-arbitrage price with respect to the state price density Z defined above. Namely, for $t_k \le t < t_{k+1}$, we have*

$$V_t' = Z_t^{-1}\mathbf{E}[Z_{t_{k+1}}V_{t_{k+1}}|\mathcal{G}_t] = e^{-r(t_{k+1}-t)}\mathbf{E}^{\mathbf{Q}}[V_{t_{k+1}}|\mathcal{G}_t].$$

In particular, $V_t' e^{-rt}$ is a $\mathbf{Q}$-$\mathcal{G}$- martingale.

A proof of Theorem 1 will be given in section 4.1.

Remark 2. The market value of the firm V' is explicitly given as

$$\begin{aligned} V_t' = V_{t_k}\exp((\mu - \frac{(\sigma\rho)^2}{2})(t - t_k) \\ + \sigma\rho(W_t' - W_{t_k}')), \end{aligned} \tag{2.2}$$

which is the image of Ornstein-Uhlenbeck Semigroup T_u defined by setting $u = \log\rho$. At the dates t_k, the market value of the firm V_{t_k}' coincides with the firm value V_{t_k}. When $\rho = 1$, the market fully observe the firm value even during the period (t_k, t_{k+1}).

2.2 An Economic Interpretation of the State Price Density Z

Since we set Z to be a state price density,

$$C(X)_t = Z_t^{-1}\mathbf{E}[Z_{t_{k+1}}X|\mathcal{G}_t] = e^{-r(t_k-t)}\mathbf{E}^{\mathbf{Q}}[X|\mathcal{G}_t].$$

gives a fair value at time $t(< t_k)$ of the cash flow X at time $t_k \in \mathbf{T}$. The following proposition could give an economic interpretation of the process $C(X)$.

Proposition 3. *The value $C(X)$ is the replication cost (within the market $\mathcal{G}$) of*

$$K^* := \mathbf{E}[X|\mathcal{G}_{t_k-}],$$

which is minimizer of

$$\inf\{||K_T - X||_{L^2} : K_T \in L^2(\mathcal{G}_{t_k-})\},$$

where

$$\mathcal{G}_{t-} := \vee_{s<t}\mathcal{G}_s.$$

Namely we have, for $t < t_k$

$$C(X)_t = e^{-r(t_k-t)}\mathbf{E}^{\mathbf{Q}}[K^*|\mathcal{G}_t] = e^{-r(t_k-t)}\mathbf{E}^{\mathbf{Q}}[X|\mathcal{G}_t].$$

A proof will be given in section 4.2.

2.3 The Generic Formulas

In this section we will obtain explicit formulas for $C(X)$ with $X = f(V_{t_n})$ for a bounded Borel function

$$f : \mathbf{R} \to \mathbf{R}.$$

Now, let

$$\tau^{\sigma,\rho}(s) := -\frac{\sigma^2}{2}(t_n - t_k)$$
$$+ \frac{(\sigma\rho)^2}{2}(s - t_k) + r(t_n - s),$$
$$\nu^{\sigma,\rho}(s) := \sigma\sqrt{(t_n - t_k) - \rho^2(s - t_k)}.$$

Theorem 4. *We have the following explicit formulas for the value $C(f(V_{t_n}))$:*

$$C(f(V_{t_n}))_t$$

$$= e^{-r(t_n - t)} \int_{-\infty}^{\infty} f(V_t' \exp\{\tau^{\sigma,\rho}(t) + z\nu^{\sigma,\rho}(t)\}) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz, \qquad (2.3)$$

$$(t_k \leq t < t_{k+1} \leq t_n).$$

In particular, the expression does not contain ρ at $t = t_k$:

$$C(f(V_{t_n}))_{t_k} = e^{-r(t_n - t_k)} \frac{1}{\sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty} f(V_{t_{k-1}} \exp(-\frac{\sigma^2}{2}(t_n - t_k) + r(t_n - t_k) + \sigma\sqrt{t_n - t_k}z)) e^{-\frac{z^2}{2}} dz. \qquad (2.4)$$

A proof will be given in section 4.3.

3 Evaluation of Transparency

3.1 Default Structure and the Market Value of the Debt

As we have stated in Introduction, for the structure of the default we rely the Merton's classical framework, but actually we need a deeper consideration on the default structure and the firm value.

Let $\{\delta_t : t \in \mathbf{T}\}$ be the debt structure of a firm. Here we assume that the maturity of each debt is always in $\mathbf{T}$. We also assume that the default occurs only when and whenever the firm value V_{t_k} is less than δ_{t_k}. In particular, the default occurs only at the dates of accounting report $\mathbf{T}$. Just like Merton's model, we can set the pay-off of the debt to be

$$\min(\delta_{t_k}, V_{t_k -}),$$

and therefore the market value of the debt is given by

$$D_t^{\rho, t_k} := e^{-r(t_k - t)} \mathbf{E}^{\mathbf{Q}}[\min(\delta_{t_k}, V_{t_k -}) \prod_{l < k} 1_{\{V_{t_l -} > \delta_{t_l}\}} | \mathcal{G}_t], \quad (t < t_k).$$

In this debt structure, it may be natural to assume

$$V_{t_k} = V_{t_k -} - \delta_{t_k},$$

which is somehow inconsistent with (2.1). To avoid this inconsistency, we only consider the debt with nearest maturity: $\delta_{t_k} > 0$ and $\delta_{t_l} = 0$ for $l < k$. under these structural assumptions, we may consider the market value of the firm as the total market value of the issued stock.

Furthermore, since default will never occur before the nearest maturity t_k, we obtain the following explicit formulas using Theorem 4:

$$
\begin{aligned}
D_t^{\rho,t_k} = V_t' &\Phi(\alpha(t)) \\
&- \delta_{t_k} e^{-r(t_k - t)} \Phi(\alpha(t) + \nu^{\sigma,\rho}(t)) \\
&+ \delta_{t_k} e^{-r(t_k - t)}, \quad (t_{k-1} \le t < t_k)
\end{aligned}
\tag{3.1}
$$

where

$$
\Phi(y) := \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2} dx,
$$

$$
\alpha(t) := \frac{\log \frac{\delta_{t_k}}{V_t'} - \tau^{\sigma,\rho}(t)}{\nu^{\sigma,\rho}(t)},
$$

and when $t \in \mathbf{T}$ in particular,

$$
\begin{aligned}
D_t^{\rho,t_k} = V_t' &\Phi(\beta) \\
&- \delta_{t_k} e^{-r(t_k - t)} \Phi(\beta + \sigma\sqrt{t_k - t}) \\
&+ \delta_{t_k} e^{-r(t_k - t)}
\end{aligned}
\tag{3.2}
$$

where

$$
\beta := \frac{\log \frac{\delta_{t_k}}{V_{t_{k-1}}} - (r + \frac{\sigma^2}{2})(t_k - t_{k-1})}{\sigma\sqrt{t_k - t_{k-1}}}.
$$

3.2 Discussion for Calibration

Looking at the explicit formulas (3.1) and (3.2), we notice that they depend on the *unknown* parameters σ and ρ but independent of μ. Furthermore, since the formula (3.2) does not explicitly depend on ρ (of course V' does depend on ρ but this does not matter), the parameter σ can be calibrated to the market value D_t^{ρ,t_k} for $t \in \mathbf{T}$.

Once we know the parameter σ, the formula (3.1) contains only one unknown parameter ρ. Therefore, it can be calibrated to the market value D_t^{ρ,t_k} for $t \notin \mathbf{T}$.

A detailed guidance of the procedure is presented in [5].

4 Proofs and Mathematical Results

4.1 A Proof of Theorem 1

For $t_k \leq t < t_{k+1}$

$$
\begin{aligned}
V_t' &= Z_t^{-1}\mathbf{E}[Z_{t_{k+1}}V_{t_{k+1}}|\mathcal{G}_t] \\
&= Z_t^{-1}\mathbf{E}[\mathbf{E}[Z_{t_{k+1}}V_{t_{k+1}}|\mathcal{G}_{t_k-}]|\mathcal{G}_t] \\
&= Z_t^{-1}\mathbf{E}[Z_{t_{k+1}}V_{t_{k+1}}'|\mathcal{G}_t].
\end{aligned}
$$

By definition we have

$$
\begin{aligned}
Z_{t_k} &= Z_t \exp \theta(W_{t_k}' - W_t') - \frac{1}{2}\theta^2(t_k - t) - r(t_k - t), \\
V_{t_k}' &= V_t' \exp\{(\mu - \frac{(\sigma\rho)^2}{2})(t - t_k) + \sigma\rho(W_{t_k}' - W_t')\},
\end{aligned}
$$

therefore we obtain that

$$
Z_{t_k}V_{t_k}' = Z_t V_t' \exp\{(\theta + \sigma\rho)(W_{t_k}' - W_t') - \frac{1}{2}(\theta + \sigma\rho)^2(t_k - t)\}
$$

By the exponential martingale property, we obtain that

$$
V_t' = \mathbf{E}[V_t|\mathcal{G}_t].
$$

$\square$

4.2 A Proof of Proposition 3

For $t_k \leq t < t_{k+1}$

$$
\begin{aligned}
\mathbf{E}^Q[K^*|\mathcal{G}_t] &= \mathbf{E}[\frac{Z_{t_k}e^{rt_k}}{Z_t e^{rt}}K^*|\mathcal{G}_t] = \mathbf{E}[\frac{Z_{t_k}e^{rt_k}}{Z_t e^{rt}}\mathbf{E}[X|\mathcal{G}_{t_k}]|\mathcal{G}_t] \\
&= \mathbf{E}[\mathbf{E}[\frac{Z_{t_k}e^{rt_k}}{Z_t e^{rt}}X|\mathcal{G}_{t_k}]|\mathcal{G}_t] = \mathbf{E}[\frac{Z_{t_k}e^{rt_K}}{Z_t e^{rt}}X|\mathcal{G}_t] = \mathbf{E}^Q[X|\mathcal{G}_t].
\end{aligned}
$$

$\square$

4.3 A Proof of Theorem 4

We calculate $C(f(V_{t_n}))_t$ for $t_k \le t < t_{k+1} \le t_n$;

$$C(f(V_{t_n}))_t = \mathbf{E}^{\mathbf{Q}}[e^{-r(t_n-t)}f(V_{t_n})|\mathcal{G}_t],$$

where $\mathbf{E}^{\mathbf{Q}}$ denotes the expectation with respect to equivalent martingale measure $\mathbf{Q}$.

Let us decompose W as

$$W = \rho W' + \sqrt{1-\rho^2}W'',$$

where W'' is a one dimensional Wiener process independent of W'. Then we have

$$
\begin{aligned}
V'_{t_k} &= V'_t \exp\big(\sigma\rho(W'_{t_n} - W'_t) + \sigma\sqrt{1-\rho^2}(W''_{t_n} - W''_{t_k}) \\
&\quad - \frac{1}{2}\sigma^2\{(t_n - t_k) - \rho^2(t - t_k)\} + \mu(t_n - t)\big) \\
&= V'_t \exp\big(\sigma\rho\{(W'_{t_n} - W'_t) - \theta(t_n - t)\} + \sigma\sqrt{1-\rho^2}(W''_{t_n} - W''_{t_k}) \\
&\quad - \frac{1}{2}(\nu^{\sigma,\rho}(t))^2 + r(t_n - t)\big) \\
&= V'_t \exp\big(\sigma\rho\{(W'_{t_n} - W'_t) - \theta(t_n - t)\} + \sigma\sqrt{1-\rho^2}(W''_{t_n} - W''_{t_k}) + \tau^{\sigma,\rho}(t)\big).
\end{aligned}
$$

Since $\widetilde{W}' := W'_t - \theta t$ is a Wiener process under $\mathbf{Q}$, we have

$$
\begin{aligned}
C(f(V_{t_n}))_t &= e^{-r(t_n-t)} \\
&\quad \mathbf{E}^{\mathbf{Q}}[f\big(V'_t \exp\{\sigma\rho(\widetilde{W}'_{t_n} - \widetilde{W}'_t) + \sigma\sqrt{1-\rho^2}(W''_{t_n} - W''_{t_k}) + \tau^{\sigma,\rho}(t)\}\big)|\mathcal{G}_t] \\
&= \frac{e^{-r(t_n-t)}}{\sqrt{2\pi}} \int_{\mathbf{R}} f\big(V'_t \exp\{\nu^{\sigma,\rho}(t)z + \tau^{\sigma,\rho}(t)\}\big)e^{-\frac{z^2}{2}} \, dz.
\end{aligned}
$$

The last identity holds because

$$\sigma\rho(\widetilde{W}'_{t_n} - \widetilde{W}'_t) + \sigma\sqrt{1-\rho^2}(W''_{t_n} - W''_{t_k})$$

is independent of $\mathcal{G}_t$ and distributed as $N(0, (\nu^{\sigma,\rho}(t))^2)$. $\qquad\square$

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