

Theory of market fluctuations

S.V. Panyukov

P.N. Lebedev Physics Institute, Russian academy of Science, Leninskiy pr., 53, Moscow, 117924, Russia

We propose coalescent mechanism of economic grow because of redistribution of external resources. It leads to Zipf distribution of firms over their sizes, turning to stretched exponent because of size-dependent effects, and predicts exponential distribution of income between individuals.

We present new approach to describe fluctuations on the market, based on separation of hot (short-time) and cold (long-time) degrees of freedoms, which predicts tent-like distribution of fluctuations with stable tail exponent $\mu = 3$ ($\mu = 2$ for news). The theory predicts observable asymmetry of the distribution, and its size dependence. For financial markets the theory explains first time “market mill” patterns, conditional distribution, “D-smile”, z-shaped response, “conditional double dynamics”, the skewness and so on.

We propose a set of Langeven equations for the market, and derive equations for multifractal random walk model. We find logarithmic dependence of price shift on the volume, and volatility patterns after jumps. We calculate correlation functions and Hurst exponents at different time scales. We show, that price experiences fractional Brownian motion with chaotically switching of sub- and super-diffusion, and calculate corresponding probabilities, response functions, and risks.

PACS numbers: 05.40, 81.15.Aa, 89.65.Gh

Contents

I. Introduction	1	4. Time and size dependence of fluctuations	20
II. Firms, cities and income distributions	2	C. Nonlinear dynamics of fluctuations	21
A. Is there thermodynamics of the market?	2	1. Correlation functions: multifractality	21
B. Mean field theory	3	2. Volume statistics	22
1. Zipf distribution	3	3. Langeven equations and market entropy	22
2. Stretched exponent	4	4. Response functions	23
3. Income distribution	5	5. Stock and news jumps	24
C. Fluctuation theory	6	6. Virtual trading time	26
1. Cold and hot degrees of freedom	6	7. Brownian motion, sub- and super-diffusion	27
2. Double Gaussian model	6	8. Fluctuation corrections	28
3. Asymmetry of PDF	7	D. Universality of fluctuations	28
4. Fat tails	7	V. Conclusion	28
D. Main results	9	References	29
III. Financial market	9	A. Entropy formulation	31
A. Cold and hot degrees of freedom	10	B. Solution of coalescence equations	32
B. Markovian model	11	C. Macroeconomic interpretation	32
C. Effective market model	11	D. PDF of Double Gaussian model	33
D. Double Gaussian model	12	E. PDF of volatility fluctuations	33
1. Market MILL, ACOR and COR stocks	13		
2. Univariate PDF	14		
3. Conditional response	14		
4. Conditional double dynamics	15		
5. Skewness	16		
E. Results and restrictions	16		
IV. Multiscale dynamics of the market	16	I. INTRODUCTION	
A. Renormalization group transformation	18		
1. Ultrametricity and restricted ergodicity	18		
2. Recurrence relation	19		
B. Amplitude of fluctuations	19		
1. Excess of volatility	19		
2. Cross-over time and Hurst exponents	20		
3. Parameters of Double Gaussian model	20		

First question behind any research is why do we need it? There are no unique approach in econophysics, and the number of different approaches grows exponentially with time. How can we decide, which of them is “correct”, if by construction, any one well describes empirical facts?

The answer is simple: in no way. All of them are equivalent at regions of their applicability. But these regions are very different, and only several theories able to

describe a large variety of empirical facts. Extrapolating, one can assume that only one theory can predict all important phenomena: Market mill patterns, multifractality of fluctuations, volatility patterns, different Hurst exponents above and below a time τ_x , and many other facts. Some of them can be described in different ways, but not the hookup of all facts.

What criteria should satisfy such theory? At first sight, it is mathematical rigor. The most striking example is the Flory approach in polymer physics, which is absolutely “wrong” mathematically, but extremely well describing all known situations. All multiple attempts to (im)prove it were failed. We conclude, that the rigor of the theory is usually “inversely proportional” to intuition.

Well, what kind of the theory should not be? If any new fact or their series need to introduce additional terms or ideas into the theory, the later can be considered as a collection of facts, arbitrary ordered according to the test of the author. We think, the real theory must predict in future yet unknown facts (at present this criterium is equivalent to extremely wide region of its applicability), to be as rigor as possible, and use minimum of initial assumptions.

We do not know other criteria of the “validity” of the theory, and this is the reason why the theory *must* describe all known trustable facts. Present paper can be considered as an attempt to follow this criteria. Only one main idea lays in the basis of our theory of market fluctuations: we assume, that they can be described as random walk motion at all time scales. In the case of financial market, it is random trading at all time horizons from seconds to tenths years.

Our theory can be considered as an attempt to make a step from numerous descriptive approaches toward a physical Langeven formulation of the “econophysical” problem. This is why we emphasize analogies with other branches of physics, which may confuse econo-physicists otherwise. Although we show, that multi-time random trading allows to explain most of market dynamics, it may be extended later in many directions.

As a strategy line, for each problem we try to construct a simplified model of such multi-time random motion, capturing the most of physics. As the result, we left with several parts of the whole puzzle, strongly inter-correlated with each other. It is the reason of unusual length of this paper, which can not be cut into several independent small parts.

II. FIRMS, CITIES AND INCOME DISTRIBUTIONS

A. Is there thermodynamics of the market?

Econophysics studies physical problems in economics, and most of its results were obtained from analogy with thermodynamics. One of classical problems of econo-

physics, the firm grow, is usually described by the model of stochastic firm growing¹. In order to explain empirically observed Zipf distribution of firm sizes² it is proposed to introduce the lower reflecting boundary in the space of firm sizes, which stabilizes the distribution to a power law³. Unfortunately, this explanation is inconsistent for firms of one or several employers, well described by the same empirical Zipf distribution.

Different models of internal structure of firms were proposed for the stochastic mechanism of firm growing. Hierarchical tree-like model of firm was studied in Refs.^{4,5}. A model of equiprobable distribution of all partitions of a firm was introduced in Ref.⁶. Both models neglect the effect of competition between different firms. The random exchange of resources between firms was taken into consideration in “saving” models⁷. In Refs.⁸⁻¹¹ the process of stochastic firm grow and loss was considered by analogy with scattering processes in liquids and gases. The distribution of firms over their sizes in different countries was studied in Ref.¹².

The theory of firms is usually called microeconomics, and from economical point of view it is hard to consider the stochasticity as the moving force of economic grow. While in thermodynamics the stochasticity originates from interaction with a huge “thermostat”, there are no such thermostat for the market, which subsists only because of activity of its direct participants.

This puzzle forces us to develop a “mean field” theory of firm growing, neglecting any fluctuation processes. We show, that the moving force of evolution on the market are not thermal-like excitations, but the supply of external resources, which are (re-)distributed between different firms. Exhaustion of the resource kills this (part of the) market, while appearance of a new resource gives rise to a new market. The process of firm growing and mergence is similar to coalescence of droplets of a new phase, when stochasticity plays only minor role.

In section IIB we show that the coalescence theory predicts Pareto power law for the distribution of firm sizes. We propose self-similar tree-like model of firms in section IIB2. This model is solved in Appendix B, and we show, that it explains empirically observable time dependence of the Pareto exponent for the world income.

The formal resemblance of observable exponential distribution of the income between individuals to Boltzmann statistics was used in Ref.¹³ to justify the applicability of methods of equilibrium thermodynamics. But how can all sectors of country economics and services always be in thermal equilibrium? In section IIB3 we propose an alternative explanation, based on unified tax policy in the whole country: the coalescent approach predicts, as a by-product, the exponential income distribution, even without invention of thermal equilibrium. This distribution is valid for the majority of the population, and statistical fluctuations are only responsible for power tails of its upper part (1-3%).

Countries with different financial policy have different “effective temperature” of the distribution, which can

be equilibrated only after unification of their financial policies, even without establishment of a “heat death” – global thermal equilibrium. Although one may consider the perpetual trade deficit of US as consequence of the fundamental second law of thermodynamics¹³, it would be more natural to explain it by financial policy, directed on attraction of resources to the country.

Econophysics is not only one field, deceptively resembling thermodynamics, we have to mention also a sand, turbulence and other macroscopic systems, which form complex dissipative structures in the response on some external forces. Although such “open systems” can not be characterized by thermodynamic potentials, the process of dissipation is accompanied by the rise of information entropy. We calculate the entropy of the market and show, that it can only increase with time, since the market irreversibly absorbs external information (there is deep analogy with physics of decoherence, discussed in Conclusion).

“Thermodynamic-type” models predict asymptotically Gaussian distribution of firm grow rates, while the real distribution has tent-like shape. In order to reproduce it, in Ref.¹⁴ an artificial potential was introduced in diffusion equation, restoring the firm size to a certain reference value, at which the grow rate abruptly changes its sign. In this paper we elaborate a new approach to study dynamics of temporal dissipative structures on the market, which do not use these artificial assumptions.

In section II C we introduce new general approach to study market fluctuations. Main ideas of this approach will be first formulated for the problem of firm grow. The market is the system with multiple (quasi-) equilibrium states, characterized by extremely wide spectrum of relaxation times. By analogy with glasses, for given observation (coarse graining) time interval τ we can divide all degrees of freedom of the market into “hot” and “cold” ones, depending on their relaxation times. Hot degrees of freedom are in equilibrium, and they generate high frequency fluctuations because of uncertainty on the market, while cold degrees of freedom are not equilibrated, and evolve on times large with respect to τ . As in the case of spin-glasses, high degeneracy of quasi-equilibriums in the market is reflected in the presence of a gauge invariance. Any averages should be defined in two stages: first, the annealed averaging over hot degrees of freedom, and then quenched averaging over cold degrees of freedom.

We demonstrate, that our theory reproduces empirically observable (in general, asymmetric) tent-like distribution of firms over their grow rates. In section II C 4 we show, that this distribution has fat tail with stable exponent μ , equals to the number of essential degrees of freedom of the noise ($\mu = 3$ for Markovian statistics of hot degrees of freedom, and $\mu = 2$ for uncorrelated noise).

B. Mean field theory

Dynamics of firm growing is similar to kinetics of growing of droplets of a new phase. Large firms can absorb smaller ones, and they can grow or leave the business, by analogy with resorption and growing of droplets in the supersaturated solution. Below we use this analogy to construct a new theory, not relying on stochastic mechanisms of firm growth. Entropic and microeconomic interpretations of our theory are discussed in Appendixes A and C.

1. Zipf distribution

For definiteness sake we define the firm size as the number G of its employees. In general, it could be any resource, shared between different firms on the market. According to economic approach (analog of the mean field approach in physics) firms can hire or loose the staff only through the “reservoir” of unemployments of value $U(t)$ at time t . Diffusion processes lead to finite value $U_* > 0$ of the “natural unemployment”. “Actual unemployment” U is the sum of U_* ¹⁵ and the “market unemployment”, $\Delta(t)$:

$$U(t) = U_* + \Delta(t).$$

The equation of the resource balance can be written in the form

$$Q(t) = U(t) + \int Gf(G, t) dG, \quad (1)$$

where $Q(t)$ is the supply of external resources. The probability distribution function (PDF) $f(G, t)$ of firm sizes is determined by the continuity equation,

$$\frac{\partial f(G, t)}{\partial t} = -\frac{\partial}{\partial G} \left[\frac{dG}{dt} f(G, t) \right], \quad (2)$$

where dG/dt is the rate of ordered motion in the space of firm sizes. Diffusion contribution in Eq. (2) is negligible in coalescent regime. According to the famous Gibrat’s observation¹ the relative grow rate of the firm,

$$\frac{1}{G} \frac{dG}{dt} = r_G \quad (3)$$

do not depend on its size, G . In the case of full employment, $\Delta = 0$, the average numbers of people getting a job and leaving it are the same, and there are no source for firm grow, $r_G = 0$. At small Δ we can hold only linear term in the series expansion of the grow rate $r_G = q\Delta$ in powers of Δ with constant q .

To solve the set of equations (1) – (3) we substitute Eq. (3) with $r_G = q\Delta$ into Eq. (2), and find its general solution

$$f(G, t) = \frac{1}{G} \chi \left[\ln \frac{G}{G_0} - q \int_0^t \Delta(t') dt' \right],$$

where G_0 is the firm size at initial time $t = t_0$ and χ is arbitrary function. Substituting this solution into the balance equation (1) and introducing new variable of integration $u = \ln(G/G_0)$, we find

$$Q(t) = U(t) + G_0 \int e^u \chi \left[u - q \int_0^t \Delta(t') dt' \right] du. \quad (4)$$

Consider the case of power growing of external resources,

$$Q(t) = Q_0 t^m. \quad (5)$$

For general time dependence $Q(t)$ its logarithmic rate m is determined by expression

$$m = \frac{d \ln Q(t)}{d \ln t}. \quad (6)$$

In the case of small unemployment value, $U \ll Q$, general solution of Eq. (4) takes exponential form, $\chi(u) = \chi_0 e^{-\kappa u}$. Substituting this expression into Eq. (4) and taking into account that the distribution $f(G, t)$ can not depend on initial firm size, G_0 , we find $\kappa = 1$ and

$$Q_0 t^m = \chi_0 \ln \frac{G_{\max}}{G_{\min}} \exp \left[q \int_0^t \Delta(t') dt' \right],$$

where G_{\min} and G_{\max} are maximal and minimal firm sizes on the market. The solution of this equation has the form

$$\Delta(t) = m/(qt), \quad \chi_0 = Q_0 / \ln(G_{\max}/G_{\min}). \quad (7)$$

First of Eqs. (7) predicts, that the economic grow, see Eq. (5), leads to less actual unemployment, Δ , in qualitative agreement with the famous macroeconomic ‘‘Phillips curve’’. Close quantitative relation between the coalescent theory and the Phillips low is established in Appendix C.

We conclude, that for any monotonically increasing function $Q(t)$ the distribution of firms over their sizes G has Zipf form:

$$f(G, t) = \frac{Q(t)}{\ln(G_{\max}/G_{\min})} \frac{1}{G^2}. \quad (8)$$

This dependence was really observed for extremely wide range of firm sizes, see Fig. 1, where empirically observable distribution

$$F(G, t) \equiv \frac{\int_{G_{\min}}^{G_{\max}} f(G, t) dG}{\int_{G_{\min}}^{G_{\max}} f(G, t) dG} \sim \frac{1}{G} \quad (9)$$

is plotted. The Zipf distribution² (8) is valid for the entire range of US firms¹⁶ (from $G_{\min} = 1$ to $G_{\max} = 10^6$) with Pareto exponent very close to unity.

The same mechanism may be responsible for power distribution function of cities over their population, the amount of assets under management of mutual funds¹⁷, banks¹⁸ and so on. In the analysis of city population in different countries, the exact form of Zipf’s law (9) was confirmed in 20 out of 73 countries¹⁹. Deviations from this low will be studied in next section.

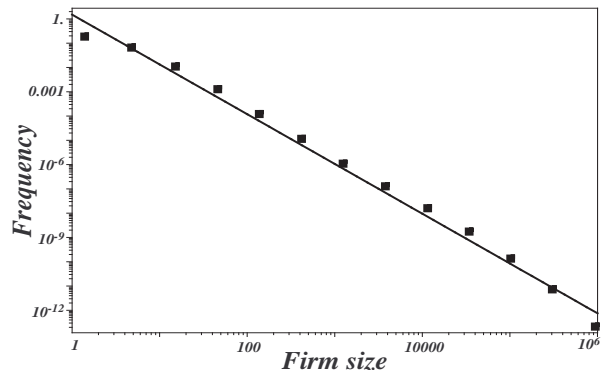


FIG. 1: Size distribution of U.S. business firms in 1997 (Census data)¹⁶. Straight line corresponds to power law distribution $F(G) \sim G^{-\gamma}$ with exponent $\gamma = 1.059$.

2. Stretched exponent

The Pareto exponent (9) can deviate from 1 because of ineffective management, strong influence of industry effects on small firms and so on. With increasing size, these effects gradually trail off, while remaining international, national and regional shocks equally affect all firms. Assuming self-similarity of firm structure, the variation of the firm size can be described by Master equation

$$r \equiv G^{-1} dG/dt = r_G - pG^{-\beta}, \quad (10)$$

with constant p and β .

To derive Eq. (10), consider the firm as the self-similar tree¹⁴ of n generations, each of $G_0 \gg 1$ branches. The size G_0 of each subdivision is described by the same type of equation (10),

$$r_0 = G_0^{-1} dG_0/dt = r_0 - p_0 G_0^{-\beta_0}. \quad (11)$$

Substituting the estimation $G \simeq G_0^n$ for the size of the whole tree in Eq. (10) and comparing with Eq. (11), we find the relation between coefficients of Master equations (10) and (11):

$$\beta = \beta_0/n, \quad r_G = r_0 n, \quad p = p_0 n.$$

In Appendix C we show that while the Gibrat grow rate r_G is fixed by economic factors, the coefficient p of job destruction can experience strong random fluctuations Δp . Neglecting fluctuations of r_G in Eq. (10) we find that fluctuations in size are inversely correlated to the size with an exponent β :

$$\Delta r = -\Delta p G^{-\beta}. \quad (12)$$

In order to estimate the exponent β_0 , consider a hypothetical structureless firm with $n = 1$ of the size $G = G_0 \gg 1$. Fluctuations of its size are characterized by Gaussian exponent $\beta = \beta_0 = 1/2$. The exponent β of

real firms takes small values $\beta = 0.15-0.21^5$, corresponding to the number of tree generations $n = 1/(2\beta) = 3-4$. Using Eq. (12) we find the dependence of the standard deviation of grow rate Δr on the firm size,

$$\langle \Delta r^2 \rangle^{1/2} = \sigma G^{-\beta}, \quad (13)$$

where $\sigma \equiv \langle \Delta p^2 \rangle^{1/2}$ does not depend on firm size G . This relation is in excellent agreement with empirical data^{14,20}.

The condition $r = 0$ (10) determines the critical firm size

$$G_c = (p/r_G)^{1/\beta}. \quad (14)$$

Small firms with $G < G_c$ collapse with time and may leave from the business (or reach a certain fluctuation size), while large firms with $G > G_c$ grow. In Appendix A we find the entropy $S(G)$ of the firm of size G , and show that $G = G_c$ corresponds to its minimum, and also to the minimum point of a “U-shaped” average cost curve in the conventional economic theory (Appendix C). We also derive maximum entropy principle for the market (Appendix A), which is known as the most foundational concepts of Gibbs systems.

In Appendix B we show that the solution of Eqs. (1), (2) with the rate (10) has stretched exponent form:

$$F(G) = \exp \left[- (1/\beta - m) (G/G_c)^\beta \right]. \quad (15)$$

Taking the limit $\beta \rightarrow 0$ we reproduce Eq. (9). It is shown that stretched exponent is the best fitting approximation for many observable distributions (size of cities, population of different countries, popularity of executives, lifetime of different species, strength of earthquakes, indices of quoting, number of coauthors, relative rates of protein synthesis and many others²¹⁻²³), which are determined by the competition of units for common resources. At small but finite $\beta \ll 1$ expanding $(G/G_c)^\beta \simeq 1 + \beta \ln(G/G_c)$ in Eq. (15) we find

$$F(G) \sim G^{-\gamma}, \quad \gamma = 1 - \beta m. \quad (16)$$

We conclude, that the exponent γ of Pareto distribution is, in general, not universal and depends on current rate $m(t)$ of external supply, Eq. (6). This conclusion can be verified by empirical observations: typically, the value of this exponent is in the interval $0.7 < \gamma < 1$. For example, the size distribution of Danish production companies with ten or more employees follows a rank-size distribution with exponent $\gamma = 0.741$ ²⁴.

To confirm the dependence of the exponent γ on the supply rate $m(t)$, consider the distribution of world income across different countries. We assume, that countries could be described by the same Master equation (10) as large firms. Exponential growing of consumable resources leads to linear time dependence of $m \sim t$, see Eq. (6). As the result, the exponent γ linearly decreases

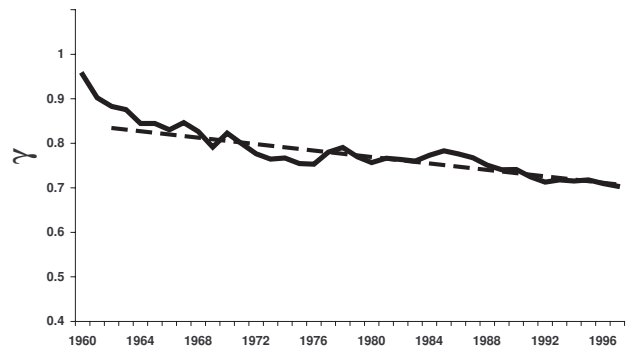


FIG. 2: Temporal path of the exponent γ (continuous line), and its approximation by linear dependence (dotted line)²⁵.

with time, in good agreement with empirical observations, see Fig. 2. Assuming, that $Q(t)$ doubles every 12 years, we estimate $\beta \simeq 0.1$, corresponding to a reasonable number $n \simeq 5$ of hierarchical management ranks in the “typical” country.

3. Income distribution

In order to find the distribution of income between individuals we first introduce the most important economic terms. The total income per state, Q , is shared between all individuals $\{G\}$ and the state expenses, $U(t)$, according to the balance equation (1). There are some minimal expenses of the state, U_* , and the inequality $\Delta = U - U_* > 0$ is usually regulated indirectly, through taxes, which determine the relative income rate, $r_G = q\Delta$, of individuals. Therefore, the income G can be described by a generalization of the Master equation (3),

$$\frac{dG}{dt} = r_G G - p. \quad (17)$$

The last term describes the rate of losses (living-wage), the same for all individuals (linear in G losses renormalize r_G). Since Eq. (17) has the form of Eq. (10) with $\beta = 1$, from Eq. (15) we get exponential distribution of the income:

$$f(G, t) \sim e^{-G/T}, \quad T = p/[r_G(1-m)]. \quad (18)$$

According to Eq. (B3) of Appendix C the average income (the “temperature²⁶”) T linearly grows with time, in good agreement with empirical observations²⁶, and also rises with the supply rate m (6). It is small for countries with low living wage p , producing high inequality in incomes.

Analysis of empirical data shows¹³, that for approximately 95% of the total population, the distribution is exponential, while the income of the top 5% individuals is described by a power-law (16) with time dependent Pareto index γ . This tail is because of speculation in

stocks, when the income is proportional to the volume of sale/buy $G \sim V$. The distribution of large volumes is power tailed, $P(V) \sim V^{-\gamma}$. The exponent γ is not universal, it depends on individual stocks with typical value $\gamma \simeq 3/2$, in good agreement with observable values²⁶ $\gamma = 1.4 - 1.8$ (changing of the most profitable stocks leads to variations in γ).

In general, the income may come from different sources. In the case of n independent sources convolution of n exponential distributions gives the Gamma distribution $P_n(G) \sim G^n e^{-G/T}$, which better describes Russian Rosstat data of salary distribution.

C. Fluctuation theory

1. Cold and hot degrees of freedom

Our approach to the description of fluctuations on the market is related to the main idea of microeconomic theory, based on independent study of “short-time” and “long-time” periods of firm growth. The separation of time scales also has deep analogy with methods of study of complex physical systems with a wide spectrum of relaxation times, as glasses. For given observation time τ degrees of freedoms of such systems can be divided into “hot” and “cold” ones. Hot degrees of freedoms fluctuate in the short-time period ($t < \tau$) given that cold degrees of freedoms are fixed and can only vary in the long-time period ($t > \tau$). Instead of consideration of slow dynamics of one system in the long-time period one usually study statistical properties of an ensemble of such systems at the given time t .

We apply this approach to find PDF of grow rates of firms, which have different dynamics in “short-time” and “long-time” periods. In order to establish general expression for oscillations of the parameter $\Delta p(t)$ (10) it is instructive to consider first single-harmonic case. General expression $\Delta p(t) = \sqrt{2}a \cos(\omega t + \phi)$ can be expanded over two basis functions $\xi'(t) = \cos(\omega t)$ and $\xi''(t) = \sin(\omega t)$:

$$\Delta p(t) = \sqrt{2}a'\xi'(t) + \sqrt{2}a''\xi''(t) \equiv \sqrt{2}(\mathbf{a}, \boldsymbol{\xi}(t)). \quad (19)$$

which are orthogonal:

$$\langle \xi^2 \rangle = \langle (\xi')^2 \rangle + \langle (\xi'')^2 \rangle = 1, \quad \langle \xi'\xi'' \rangle = 0. \quad (20)$$

Here $\langle \dots \rangle$ means time average. Instead of two real basis functions it is convenient to introduce one complex function $\boldsymbol{\xi}(t) = \xi'(t) + i\xi''(t)$ and complex amplitude $\mathbf{a} = a' + ia'' = ae^{i\phi}$, in terms of which the scalar product in Eq. (19) is given by expression $(\mathbf{a}, \boldsymbol{\xi}) = \text{Re}(\mathbf{a}^* \boldsymbol{\xi})$. In the following we use bold notations both for vectors and complex numbers.

In general case, the frequency of quick oscillations $\omega \gtrsim \tau^{-1}$ of $\boldsymbol{\xi}(t)$ (as well as its amplitude) randomly varies with time. Real and imaginary parts of $\boldsymbol{\xi}$ can be considered as random values normalized by condition (20),

where $\langle \dots \rangle$ has the meaning annealed averaging over the noise $\boldsymbol{\xi}(t)$. Complex amplitude \mathbf{a} is fixed in the short-time period, and can be considered as random variable in the long-time period (or for the ensemble of different firms for given time t). The random function $\boldsymbol{\xi}(t)$ and the amplitude \mathbf{a} describe hot and cold degrees of the freedom of the market, respectively.

Notice, that $\Delta p(t)$ (19) is invariant with respect to “gauge” transformation

$$\boldsymbol{\xi} \rightarrow \boldsymbol{\xi}e^{i\varphi}, \quad \mathbf{a} \rightarrow \mathbf{a}e^{i\varphi}, \quad (21)$$

with constant φ , reflecting high degeneracy of market quasi-equilibrium states.

2. Double Gaussian model

We first calculate PDF of fluctuations Δp ,

$$\mathcal{P}(x) \equiv \overline{\langle \delta[x - \Delta p(t)] \rangle}. \quad (22)$$

The bar means ensemble (quenched for the time τ) averaging over amplitudes \mathbf{a} of fluctuations of different firms. The main assumption of “Double Gaussian model” is extremely simple: since tactics of firms at the short-time period is determined by large number of essentially independent factors, we assume Gaussian statistics of random variable $\boldsymbol{\xi}$ at time horizon τ (due to central limit theorem). But two different firms (or the same firm at two different time intervals τ) will have, in general, different amplitude of fluctuations \mathbf{a} at the long-time strategy horizon. Since the strategy of firms is also determined by large number of independent random factors, we assume Gaussian statistics of the random amplitude \mathbf{a} with dispersion $\sigma^2 = \bar{a}^2$.

Due to the gauge invariance (21) the noise and the amplitude PDFs could depend only on moduli $\xi = |\boldsymbol{\xi}|$ and $a = |\mathbf{a}|$. In this section we assume, that hot ($\boldsymbol{\xi}$) and cold (\mathbf{a}) random variables are independent with zero average and Gaussian weights

$$\mathcal{Q}_G(\xi) = \frac{1}{\pi} e^{-(\xi')^2 - (\xi'')^2}, \quad \frac{1}{\pi\sigma^2} e^{-[(a')^2 + (a'')^2]/\sigma^2} \quad (23)$$

respectively.

Fourier transform can be used to calculate the averages:

$$\mathcal{P}(x) = \int G(k) e^{-ikx} \frac{dk}{2\pi}, \quad G(k) = \overline{\langle e^{i\sqrt{2}k(a'\xi' + a''\xi'')} \rangle}$$

We first calculate the average over Gaussian normalized ξ' and ξ'' and get $G(k) = \exp\{-k^2[(a')^2 + (a'')^2]/2\}$. Calculating the average over a' and a'' , we get $G(k) = (1 + \sigma^2 k^2/2)^{-1}$. The last step – is to take the inverse Fourier transform of this $G(k)$:

$$\mathcal{P}(x) = \int_{-\infty}^{\infty} \frac{\cos(kx)}{1 + \sigma^2 k^2/2} \frac{dk}{2\pi} = \frac{1}{2\sigma} e^{-\sqrt{2}|x|/\sigma}. \quad (24)$$

Exponential distribution of firm grow rates (24) was really observed for typical fluctuations $x = \Delta p = -\Delta r G^\beta$, see Eq. (12), with the exponent $\beta = 0.15$. We conclude, that tent-like exponential distribution of firm grow rates is the consequence of Gaussian statistics of all degrees of freedom (hot and cold) of the market.

3. Asymmetry of PDF

The assumption of Double Gaussian model about independence of cold and hot variables is, in general, too strong, and the noise $\xi(t)$ is (anti)correlated with the amplitude \mathbf{a} . Taking such anticorrelations into account, we can write general expression for the noise, satisfying gauge transformation (21):

$$\xi(t) = \tilde{\xi}(t) - \zeta \mathbf{a} / \alpha, \quad \alpha^2 = \overline{a^2}, \quad (25)$$

where $\zeta > 0$ is the dimensionless correlation factor and random variable $\tilde{\xi}(t)$ is not correlated with \mathbf{a} , and has zero average, $\langle \tilde{\xi}(t) \rangle = 0$. In the case $\zeta = 0$ positive and negative fluctuations of firm grow rate, Δr , have equal probability, while in the case of positive $\zeta > 0$ firms will in average grow (because of grow of external resources, see section IIB).

At economic level anticorrelations between firm tactics and strategy (25) reflect the fact that firms prefer to have tactical losses with the hope to get a profit at strategy horizons (say, by pressing out business rivals). And firms (and countries), aimed at the maximum instant profit without significant investments in the short time period will eventually get losses in the long time period.

Repeating our calculations for the model (25), we again find exponential distribution (24)

$$\mathcal{P}_0(x|\sigma) = \frac{1}{\alpha\sqrt{2(1+\zeta^2)}} \begin{cases} e^{-\sqrt{2}x/\sigma_+} & \text{for } x > 0 \\ e^{\sqrt{2}x/\sigma_-} & \text{for } x < 0 \end{cases}, \quad (26)$$

but with different widths σ_\pm ($\sigma_+ < \sigma_-$) of positive and negative PDFs, and the dispersion σ :

$$\sigma_\pm = \alpha \left(\sqrt{1 + \zeta^2} \mp \zeta \right), \quad \sigma^2 = (1 + 2\zeta^2) \alpha^2. \quad (27)$$

The average of this distribution is shifted to negative Δp , corresponding to systematic tendency to grow:

$$\overline{\langle \Delta p \rangle} = -\sqrt{2}\alpha\zeta, \quad \overline{\langle \Delta r \rangle} = -\overline{\langle \Delta p \rangle} G^{-\beta} > 0. \quad (28)$$

Such asymmetrical exponential distribution was really observed in the analysis of empirical data in Ref.²⁷ for large averaging intervals (5 years, see Fig. 3). In Fig. 3 the x-axis is in units of Δr (12), and not Δp . Empirical value $\zeta = 0.23$, and for typical $\langle \Delta r^2 \rangle^{1/2} = 0.5$ we reproduce the observed mean $\overline{\langle \Delta r \rangle} = 0.16$.

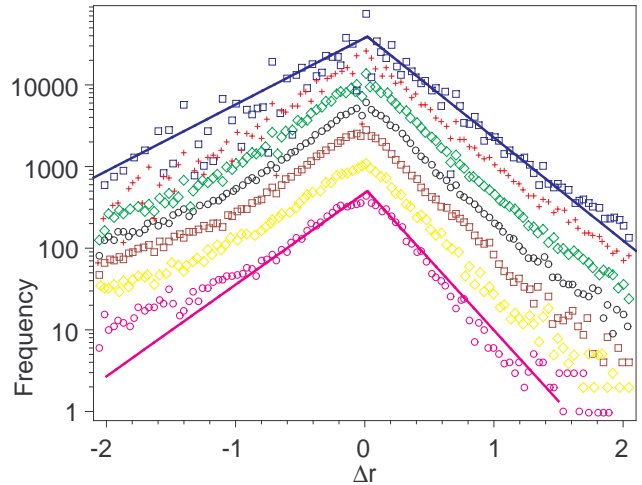


FIG. 3: The distribution of grow rates of US firms in 1998-2003 for seven size groups from $G_{up} = 8-15$ through $G_{down} = 512-1023$ ²⁷. Comparing with the theory we use the same correlation factor $\zeta = 0.23$ and varied only one parameter σ ($\langle \Delta r^2 \rangle^{1/2} = 0.62, 0.45$ and $0.4, 0.3$ respectively for upper and lower curves. Deviations from exponential dependence at large $|\Delta r|$ will be explained in section II C 4.

4. Fat tails

One of the most prominent features of PDF, the fat tail, is usually attributed to large volatility fluctuations (in different stochastic volatility and multifractal models). In this section we show, that the tail originates from large jumps of the noise, and not of the volatility. This new mechanism predicts universal tail exponent $\mu = 3$ for stock jumps, independent on the coarse graining time interval τ .

Fluctuations $\Delta p(t)$ of the Double Gaussian model are characterized by random variable ξ , which is Gaussian at the time interval τ and normalized by the condition $\langle \xi^2 \rangle = 1$ (20). The problem is that even if we normalize Gaussian variable for given time interval τ , this normalization will be broken at next time intervals because of the intermittency effect: relatively rare, but large picks of fluctuations. The only way to normalize $\xi(t)$ for all times is to divide it

$$\xi(t) = \xi_0(t) / \sigma_0(t). \quad (29)$$

by the mean squared average

$$\sigma_0^2(t) = \sum_k w_k \xi_0^2(t - k\tau). \quad (30)$$

$\sigma_0(t)$ slowly varies at time interval τ , and therefore, random variable $\xi(t)$ leaves Gaussian at time scale τ . The division of $\xi_0(t)$ by $\sigma_0(t)$ removes from general Gaussian process $\xi_0(t)$ the long-time (at the time scale τ) trend (long-time variations of the amplitude), leaving only high frequency components.

Standard definition of the mean square σ_0^2 assumes that weights w_k in Eq. (30) do not depend on k , and we reproduce our previous result (24) for PDF. But this definition must be corrected, since there are no any fundamental value of dispersion σ_0^2 , which can only be estimated from the knowledge of past values of ξ_0^2 . As the first step, we have to put $w_k = 0$ at $k \leq 0$ and get

$$\sigma_0^2(t) = w_1 \xi_0^2(t - \tau) + \sum_{k>1} w_k \xi_0^2(t - k\tau). \quad (31)$$

In the case of totally uncorrelated events σ_0^2 is determined only by the “reference” value of $\xi_0^2(t - \tau)$ at previous time interval, and all $w_k \rightarrow 0$ at $k > 1$. Terms with $k > 1$ describe the effect of correlations of events, leading to variations $\Delta p(t)$.

Second, hot variable $\xi(t)$ can vary only on the time scale small with respect to τ . Therefore, all $w_k \rightarrow 0$ at $k > 2$, and random variable $\xi(t)$ has Markovian statistics with correlations only between neighbour time intervals τ . Otherwise it will depend on many time intervals time $k\tau$ in the past, which is prohibited by definition of hot variable $\xi(t)$.

And the last: the only information known in future about past fluctuations, is the very increment Δp , which depends only on one component $\xi'_0 = (\xi_0, \mathbf{a})/a$ of ξ_0 along the vector \mathbf{a} . The information about corresponding “perpendicular” component ξ'' do not enter to the increment, and is lost. Therefore, we should drop the contribution of $(\xi''_0)^2$ from correlation terms with $k > 1$ in Eq. (31): $\xi_0^2 = (\xi'_0)^2 + (\xi''_0)^2 \rightarrow (\xi'_0)^2$. After all these corrections we left with expression for the mean square in Eq. (29):

$$\sigma_0^2(t) = w_1 \xi_0^2(t - \tau) + w_2 [\xi'_0(t - 2\tau)]^2 \quad (32)$$

Although we get similar results for any quickly decaying weights w_k , calculations are much simplified in the case of equal weights $w_1 = w_2 = 1/2$ and all $w_k = 0$ at $k > 1$. In order to calculate PDF of the noise $\xi(t)$ (29), we rewrite it in the form

$$\begin{aligned} \mathcal{Q}(\xi) &= \int_0^\infty d\sigma_0 \pi(\sigma_0) \langle \delta[\xi - \xi_0/\sigma_0] \rangle \\ &= \int_0^\infty d\sigma_0 \pi(\sigma_0) \frac{\sigma_0^2}{\pi} e^{-\xi^2 \sigma_0^2}, \end{aligned} \quad (33)$$

where we take the average over Gaussian variable ξ_0 . The probability distribution of the random variable σ_0 (32) is $\pi(\sigma) = 2\sigma \langle \delta(\sigma^2 - \sigma_0^2) \rangle$. Using exponential representation of this δ -function, we get

$$\pi(\sigma) = \frac{\sigma}{\pi} \int \frac{ds e^{is\sigma^2}}{(1 + is/2)^{3/2}} = \sqrt{\frac{2}{\pi}} \sigma^2 e^{-\frac{\sigma^2}{2}}.$$

Substituting this expression into Eq. (33), we come to Student noise distribution:

$$\mathcal{Q}(\xi) = \frac{3}{\pi} (1 + 2\xi^2)^{-5/2}. \quad (34)$$

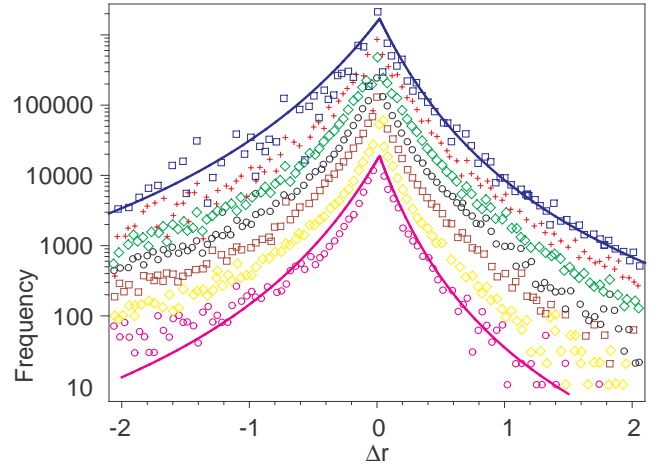


FIG. 4: The distribution of growth rates of US firms in 1998-1999²⁷, the same parameters as in Fig. 3. Tail exponent $\mu = 3$. The varied parameter $\langle \Delta r^2 \rangle^{1/2} = 0.45$ and 0.3 respectively for upper and lower curves.

Using this distribution function, we finally get

$$\mathcal{P}(x|\sigma) = \frac{6}{\sqrt{\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} D_{-4}\left(\sqrt{2}\frac{x}{\sigma}\right), \quad (35)$$

where D is the parabolic cylinder function. The central part of this distribution has exponential shape (24), while its tale has power dependence:

$$\mathcal{P}(x) \sim |x|^{-1-\mu}, \quad |x| \gg \sigma. \quad (36)$$

with the tail exponent $\mu = 3$, well outside the stable Lévy range ($\mu < 2$). One can show, that this exponent does not depend on relation between weights w_1 and $w_2 \sim 1$ in Eq. (32) for Markovian noise. But in the absence of noise correlations, $w_2 \rightarrow 0$, we get the effective exponent $\mu \rightarrow 2$.

If we take into account correlations between the noise and the amplitude (see Eq. (25) and discussion therein), $\langle \xi_0 \rangle = -\zeta \mathbf{a}/\alpha$, and after some calculations we get simple expression for PDF:

$$\begin{aligned} \mathcal{P}(x) &= \int_0^\infty d\sigma_0 \pi(\sigma_0) \mathcal{P}_0(x|\sigma/\sigma_0) = \\ &= \frac{1}{\alpha \sqrt{1 + \zeta^2}} \begin{cases} \sigma_+ \mathcal{P}(x|\sigma_+) & \text{for } x > 0 \\ \sigma_- \mathcal{P}(x|\sigma_-) & \text{for } x < 0 \end{cases}, \end{aligned} \quad (37)$$

where functions $\mathcal{P}_0(x|\sigma)$ and $\mathcal{P}(x|\sigma)$ are defined in Eqs. (26) and (35), and σ_\pm are given in Eq. (27). We show in Fig. 4 that Eq. (37) with $\zeta = 0.23$ allows to explain both the asymmetry and the shape of empirical PDF for different size groups. The size dependence of both Fig. 3 and Fig. 4 follows Eq. (12) with exponent $\beta \simeq 0.1$ and universal Δp .

Now we study the stability of the exponent μ for different time periods τ . The total increment $\Delta p = \sqrt{2}(\mathbf{a}, \xi)$

for two joint intervals τ is the sum of corresponding increments $\Delta p_i = \sqrt{2}(\mathbf{a}_i, \boldsymbol{\xi}_i)$ for each of these intervals. Since the amplitude \mathbf{a} in Eq. (19) slowly varies on the time scale τ , we take it the same for both intervals, $\mathbf{a}_1 = \mathbf{a}_2$, and so the noise $\boldsymbol{\xi}$ is proportional to the sum of noises $\boldsymbol{\xi}_i$ for these intervals. Each of $\boldsymbol{\xi}_i$ can be represented in the form of Eq. (29), and corresponding dispersions σ_1 and σ_2 depend on the same (but shifted over time) series of Gaussian variables $\boldsymbol{\xi}_0$, Eq. (32). Calculating the distribution function of the sum $\boldsymbol{\xi} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$, we find

$$\begin{aligned} \mathcal{Q}(\boldsymbol{\xi}) = & \int_0^\infty d\sigma_1 d\sigma_2 \iint ds_1 ds_2 \frac{\sigma_0^2}{\pi} e^{-\xi^2 \sigma_0^2} \times \\ & \frac{\sigma_1 \sigma_2}{\pi^2} \frac{1}{(1 + i s_1/2)(1 + i s_2/2)} \times \\ & \frac{e^{i s_1 \sigma_1^2 + i s_2 \sigma_2^2}}{\sqrt{1 + i(s_1 + s_2)/2}}, \end{aligned} \quad (38)$$

where

$$\frac{1}{\sigma_0^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2(1 + i s_1/2)}. \quad (39)$$

The tail of the distribution (38) is determined by small $\bar{\sigma}$, corresponding to large $|s_1| \gg |s_2| \sim 1$. As the result we find, that the distribution $\mathcal{Q}(\boldsymbol{\xi}) \sim \xi^{-5}$ for the time interval 2τ is characterized by the same exponent $\mu = 3$, as each of $\boldsymbol{\xi}_i$ for the time interval τ . The only difference is that this asymptotic behavior can be reached at larger ξ , with respect to the distribution function of $\boldsymbol{\xi}_i$.

This observation explains why the fat tail in Fig. 3 for five year period is shifted to higher $|\Delta r|$, with respect to Fig. 4 for one year period data. Experimental observation of the stability of the exponent $\mu = 3$ for widely different economies, as well as for different time periods²⁸ τ , gives strong experimental support of our theory. The stability originates in nonlinear correlations of the noise, see Eq. (29), while linear correlations vanish, $\langle(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)\rangle = 0$. To demonstrate the importance of such correlations, assume, that the noise $\boldsymbol{\xi}_i$ has tail exponent μ , and is uncorrelated at neighboring intervals τ . Then the exponent of $\boldsymbol{\xi} \sim \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2$ for the interval 2τ is equal 2μ , and not μ , as follows from our model.

The systematic study of the distribution of annual growth rates by industry was performed in Ref.²⁹ using Census U.S. data. It is shown, that all sectors but finance can be fitted by exponential distribution (24). We checked the data for finance sector, and show that they can be well fitted by Eq. (35) with exponent $\mu = 3$.

D. Main results

In this section we considered evolution of the market as the result of competition of different firms for external resources, by analogy with coalescent regime in physics of supersaturated solutions. This analogy allows to find informational entropy of the market, and prove the principle of maximum entropy.

We demonstrate that in coalescent regime for Gibrat mechanism of firm growing the distribution of firms over their sizes follows the Pareto power law with the exponent $\gamma = 1$ (Zipf distribution). Taking into account size effects, it turns to stretched exponent distribution, which also describes different processes, related to competition of units for common resources. Coalescent mechanism is also responsible for observable exponential distribution of the income between individuals. The production of real firms can be taken into account by vector models, by analogy with multicomponent solutions.

We propose the theory of market fluctuations, based on separation of all degrees of freedom of the market into cold and hot ones. For Gaussian statistics of all degrees of freedom such separation leads to experimentally observable exponential PDF of firm grow rates. We also prove, that this distribution has power tail with universal stable exponent $\mu = 3$.

We find analytical expression for PDF, and show, that it reproduces observable shape and asymmetry of the distribution of firm grow rates, which is related to existing anticorrelations between tactics of firms at short-time horizon and their strategy at long-time horizon. In next section we apply this approach to study price fluctuations on financial markets.

III. FINANCIAL MARKET

Dynamics of fluctuations is determined by the spectrum of relaxation times of the system. When all times are small with respect to the observation time interval τ , the state of the market at time $t + \tau$ depends only on its state at previous time t , and dynamics is Markovian random process. Short-range correlations of price fluctuations on the market can be studied using stochastic volatility models³⁰, but in order to describe real markets with multi-time dynamics, the model should take infinite-range correlations into account³¹, and has “infinite” number of correction terms. In addition, to take empirically observable excess of volatility into account, one has to go at the boundary of stability of such models.

The real market has enormous number of (quasi-) equilibrium states and extremely wide spectrum of relaxation times, by analogy with turbulence³² and glasses. Multifractal properties of time series can be described by phenomenological Multifractal Random Walk model³³. Although this model well characterizes scaling behavior of price fluctuations, it can not capture correlations at neighboring time intervals, which determine “conditional dynamics of the market” and can be described by the bivariate probability distribution of price increments³⁴.

In previous section IIC we show, that the increment of the random value $P(t)$ of the time series

$$\Delta_\tau P(t) \equiv P(t + \tau) - P(t) \quad (40)$$

has the form of scalar product of two-component random

vectors – the noise $\xi(t)$ and its amplitude $\mathbf{a}(t)$:

$$\Delta_\tau P(t) = \sqrt{2}(\mathbf{a}(t), \xi(t)). \quad (41)$$

Hot variables $\xi(t)$ vary at the scale small with respect to τ , while characteristic times of cold variables $\mathbf{a}(t)$ are large with respect to τ . The time τ plays the role of the effective temperature: at minimal trade-by-trade time, $\tau \simeq \tau_k$, the price is almost frozen, while in the opposite limit $\tau > \tau_0$ it has random walk statistics. In the intermediate time interval $\tau_k < \tau < \tau_0$ (of many decades) the market has “restricted” ergodicity: only hot degrees of freedom are excited, while cold degrees of freedom are frozen and determine the amplitude \mathbf{a} of price fluctuations.

Here we apply this approach to calculate PDF of price increments, as well as various conditional distributions and their moments. The dependence of parameters of these distributions on observation time τ will be studied later, in section IV. In section III A we introduce hot and cold degrees of freedom of the market. Two simplified models are formulated and solved in sections III B and III C. “Markovian” model takes short-time correlations into account and neglects the effect of long-time challenges. “Effective market” model captures such effects, but neglects any short-time correlations because of trader activity. Although both these models capture essential part of observable phenomena of price fluctuations (extremely small linear correlations – the Bachelier’s first law, “dependence-induced volatility smile”, “compass rose” pattern³⁵ and so on), they can not describe all the variety of such “stylized facts”⁴⁰.

In section III D we introduce Double Gaussian model, that takes all correlation effects into account, and show that it allows to explain the behavior of different types of stocks³⁶. Analytical solution of this model is derived in Appendix D. We demonstrate, that this solution reproduces all observable types of “market mill” patterns and gives the mysterious z -shaped response of the market for all kinds of asymmetry of bivariate PDF, as well as other fine characteristics of this distribution. We also show that our theory allows to explain empirically observable Markovian “double dynamics” of signs of returns on the market³⁷.

A. Cold and hot degrees of freedom

The idea of hot and cold degrees of freedom of the market is qualitatively supported by empirical observations: It is shown in Ref.³⁸, that the amplitude of fluctuations for ensemble (quenched) averaging significantly exceeds the amplitude of fluctuations for time (annealed) averaging. This observation can be interpreted as the result of the presence of cold degrees of freedom, which remain “frozen” when considering time fluctuations of hot degrees of freedom. In the case of ensemble averaging such cold degrees of freedom become “unfrozen”, increasing

the amplitude of price fluctuations with respect to its time average value.

Following Ref.³⁹ consider two consecutive price increments, x (push) and y (response) for the time intervals τ :

$$x = \Delta_\tau P(t), \quad y = \Delta_\tau P(t + \tau).$$

According to Eq. 41 price increments can be written in the form of the scalar products:

$$x = \sqrt{2}(\mathbf{a}_1, \xi_1), \quad y = \sqrt{2}(\mathbf{a}_2, \xi_2), \quad (42)$$

of complex noises $\xi_1 = \xi(t)$, $\xi_2 = \xi(t + \tau)$ and complex amplitudes $\mathbf{a}_1 = \mathbf{a}(t)$, $\mathbf{a}_2 = \mathbf{a}(t + \tau)$. Complex random walk $\xi(t)$ in the “tactic” space describes “impatient” agents. Complex random walk $\mathbf{a}(t)$ in the “strategy” space can be thought of as a result of slow variation of composition of the population of such agents on the market, as well as the activity of “patient” agents.

Moduli of complex variables ξ_i and \mathbf{a}_i are normalized as:

$$\langle \xi_i^2 \rangle = 1, \quad \overline{a_i^2} = \sigma^2, \quad (43)$$

σ is the dispersion of price fluctuations

$$\overline{\langle \Delta_\tau P^2(t) \rangle} = \overline{\langle \Delta_\tau P^2(t + \tau) \rangle} = \sigma^2. \quad (44)$$

Eqs. (42) are invariant with respect to “gauge” transformation of noise and amplitude variables, Eq. (21).

We will characterize correlations of price increments by uni- and bivariate PDFs:

$$\mathcal{P}(x) \equiv \overline{\langle \delta[x - \Delta_\tau P(t)] \rangle} = \int dy \mathcal{P}(x, y), \quad (45)$$

$$\mathcal{P}(x, y) \equiv \overline{\langle \delta[x - \Delta_\tau P(t)] \delta[y - \Delta_\tau P(t + \tau)] \rangle}. \quad (46)$$

Using exponential representation of δ -function, these expressions can be rewritten in the form

$$\mathcal{P}(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} G(k, 0), \quad (47)$$

$$\mathcal{P}(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ikx - ipy} G(k, p), \quad (48)$$

where $G(k, p)$ is the Fourier component of PDF

$$G(k, p) \equiv \overline{\langle e^{ik\Delta_\tau P(t) + ip\Delta_\tau P(t + \tau)} \rangle}. \quad (49)$$

The variable y may be interpreted as the response on initial push x , which is characterized by conditional PDF

$$\mathcal{P}(y|x) = \frac{\mathcal{P}(x, y)}{\mathcal{P}(x)}, \quad \mathcal{P}(x) \equiv \int \frac{dk}{2\pi} e^{-ikx} G(k, 0), \quad (50)$$

The average conditional response is

$$\begin{aligned} \langle y \rangle_x &= \int_{-\infty}^{\infty} dy y \mathcal{P}(y|x) \\ &= \frac{i}{\mathcal{P}(x)} \int \frac{dk}{2\pi} e^{-ikx} \left. \frac{\partial G(k, p)}{\partial p} \right|_{p=0}. \end{aligned} \quad (51)$$

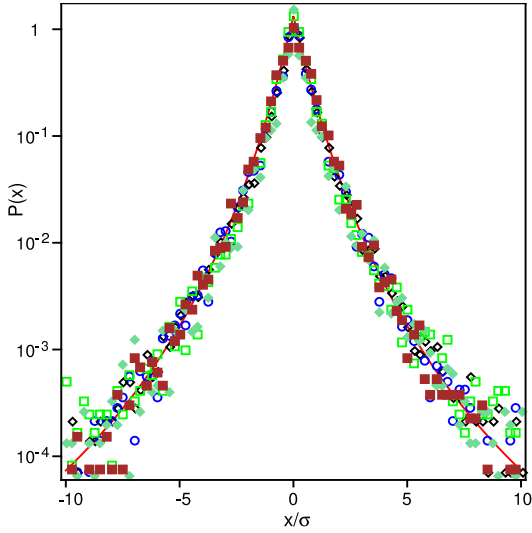


FIG. 5: PDF of Russian financial market (finam.ru, 2006) for $\tau = 5$ min (\diamond - EESR, \circ - LKOH, \square - RTKM, \blacklozenge - SBER, \blacksquare - SNGS), solid line shows theoretical prediction (35) with $\mu = 3$.

The width of the conditional PDF $\mathcal{P}(y|x)$ is characterized by the conditional mean-square deviation

$$\begin{aligned} \sigma_x^2 &\equiv \int dy (y - \langle y \rangle_x)^2 \mathcal{P}(y|x) \\ &= -\frac{1}{\mathcal{P}(x)} \int \frac{dk}{2\pi} e^{-ikx} \left. \frac{\partial^2 G(k, p)}{\partial p^2} \right|_{p=0}. \end{aligned} \quad (52)$$

Large σ_x correspond to a large variety of the behaviors, the “volatility”. The dependence of σ_x on x reflects the volatility clustering: σ_x should not depend of x if there is no volatility clustering.

The conditional response (51) and PDF (46) depend of correlations between noises ξ_i and their amplitudes \mathbf{a}_i in two time intervals. Before formulating general model (see section III D), that takes all such correlations into account, it would be instructive to study some simple limits.

B. Markovian model

We first consider the case when the amplitude $\mathbf{a}(t)$ is not correlated with external challenges at strategy horizons, and $\mathbf{a}_1 = \mathbf{a}_2$ for two neighboring time intervals. We also assume that the noise is not correlated with the amplitude, but take into account short range correlations of the noise, $\langle (\xi_1, \xi_2) \rangle = \varepsilon$. For this Markovian model we find Eq. 35 for the probability distribution, which describes very well Russian financial market for $\tau = 5$ min, see Fig. 5. For Gaussian noise we find exponential PDF (24) of price fluctuations, which is really observed for high frequency fluctuations⁴¹.

Averaging the Fourier component of PDF (49) over fluctuations of Gaussian amplitude $\mathbf{a}_1 = \mathbf{a}_2$ and noise ξ_i we find $G(k, p) = [1 + \sigma^2(k^2/2 + p^2/2 + \varepsilon kp)]^{-1}$. Calculating the Fourier transformation of this function (48), we get the distribution function

$$\mathcal{P}_t(x, y) = \frac{1}{\pi \sigma^2 \sqrt{1 - \varepsilon^2}} K_0 \left[\sqrt{\frac{2(x^2 + y^2 - 2\varepsilon xy)}{\sigma^2(1 - \varepsilon^2)}} \right], \quad (53)$$

where K_0 is the Bessel function. Calculating the integral (51) with function (53), we find the conditional response

$$\langle y \rangle_x = \varepsilon x. \quad (54)$$

Linear dependence (54) with $\varepsilon < 0$ well agrees with data for Russian market, what can be interpreted as indication that Russian investors are oriented only on current benefits, mostly ignoring opening possibilities at strategy horizons. Although linear response (54) is typical for ACOR group of stocks with $\varepsilon < 0$ (according to classification of Ref.³⁶), this model can not describe essentially nonlinear response of other groups of stocks.

C. Effective market model

In general, the amplitude \mathbf{a} is varied in response to unpredictable external challenges. We first study this effect in the model of “Effective market”, neglecting correlations between noise ξ_i^0 in two consecutive time intervals τ , but taking into account random variations of its amplitude \mathbf{a}_i^0 :

$$\langle (\xi_i^0, \xi_j^0) \rangle = \delta_{ij}, \quad \overline{(\mathbf{a}_1^0, \mathbf{a}_2^0)} = \nu \overline{(\mathbf{a}_1^0)^2} = \nu \overline{(\mathbf{a}_2^0)^2}, \quad (55)$$

where ν is dimensionless correlation parameter, $0 < \nu < 1$. As in Markovian model we ignore (anti)correlations between noise and amplitude. Correlations of the noise ξ_i^0 are induced by trader activity, while the change of a stock price in the model of Effective market is determined only by an external information, which may be considered as uncorrelated random process.

PDF $\mathcal{P}(x)$ of this model is proportional to the parabolic cylinder function with power tail exponent $\mu = 2$ (see Eq. (36)). Non Gaussian character of the noise PDF can be ignored when considering the central parts of price distributions, when $\mathcal{P}(x)$ takes exponential form (24). In order to calculate bivariate PDF, we substitute equation (42) in (49) and perform the averaging over fluctuations of Gaussian variables \mathbf{a}_i^0 :

$$G(k, p) = \langle \exp \left[-\sigma^2 k^2 (\xi_1^0)^2 / 2 - \sigma^2 p^2 (\xi_2^0)^2 / 2 - \nu \sigma^2 kp (\xi_1^0, \xi_2^0) \right] \rangle. \quad (56)$$

The averaging over noise ξ_i^0 is performed with Gaussian PDF (23), and the integral over k and p in expression (48)

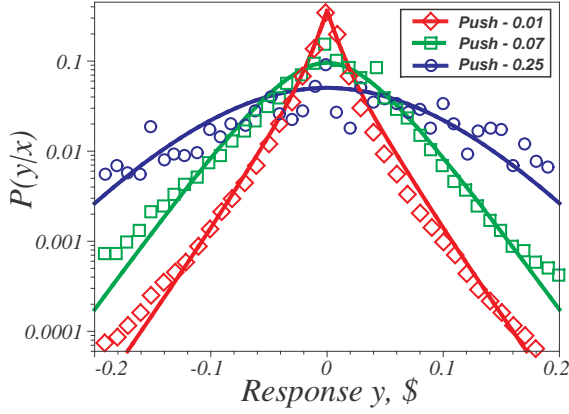


FIG. 6: Profiles of conditional PDF $\mathcal{P}(y|x)$ at different x and $\nu = 0.95$ ($\sigma = \$0.04$) in comparison with empirically observable profiles⁴²

is calculated expanding the function $G(k, p)$ in powers of ν :

$$\mathcal{P}_0(x, y) = \sum_{l=0}^{\infty} \nu^{2l} \mathcal{P}_l(x) \mathcal{P}_l(y). \quad (57)$$

Here $\mathcal{P}_0(x) = \mathcal{P}(x)$ is given by Eq. (24), and functions $\mathcal{P}_l(x)$ are defined by:

$$\mathcal{P}_l(x) = \frac{1}{l!} \frac{d^l}{dz^l} \left[\frac{1}{\sqrt{z}} \mathcal{P} \left(\frac{x}{\sqrt{z}} \right) \right] \Big|_{z=1}. \quad (58)$$

PDF $\mathcal{P}_0(x, y)$ is symmetrical with respect to independent transformations of its variables, $x \rightarrow -x, y \rightarrow -y$, and also with respect to time reversal transformation, which corresponds to push-response interchange, $x \leftrightarrow y$. This function is not analytical in origin, and the geometry of equiprobability levels can be approximated by $|x|^\lambda + |y|^\lambda = \text{const}$, where $\lambda \simeq 1$ near origin and $\lambda \simeq 2$ far away from it.

Profiles of conditional distribution (50) are shown for different x in Fig. 6. With the rise of the push x the response becomes more flat in origin, in good agreement with empirical data. Slight deviations between the theory and data at large $|y| \gg \sigma$ are related to non-Gaussian character of the noise (leading to power tail), see Eq. 35 for more details. Calculating integral (52) in the case of Gaussian noise, we get the conditional mean-square deviation,

$$\sigma_x^2 = \sigma^2 \left[1 + \frac{1}{2} \nu^2 \left(\sqrt{2} |x| / \sigma - 1 \right) \right]. \quad (59)$$

This function is plotted in Fig. 7. It demonstrates the so called “dependence-induced volatility smile” (“D”-smile), well known from empirical data⁴³. At small $|x| \lesssim \sigma$ the standard deviation of the response (59) is smaller than the unconditional standard deviation σ , while at large $|x| \gtrsim \sigma$ it is larger.

The shape of conditional PDF can also be characterized by the kurtosis, proportional to fourth momentum. One

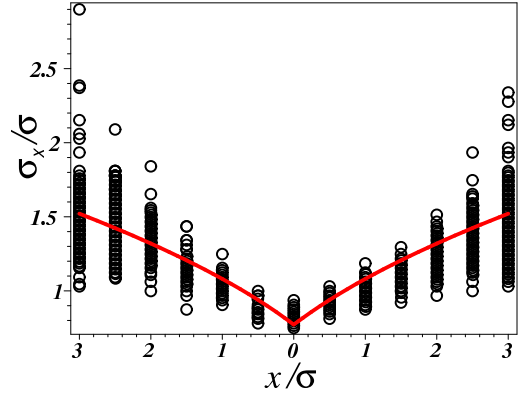


FIG. 7: Conditional mean-squared deviation as function of $x = \Delta p$; the result of Gaussian model with $\nu = 0.95$ and empirically observable D-smile⁴³.

can show that, in agreement with empirical data, the kurtosis of theoretical conditional PDF $\mathcal{P}_0(y|x)$ decreases with the rise of $|x|$. We conclude, that Effective market model captures main features of the market behavior, but it is unable to describe finite response of real stocks.

D. Double Gaussian model

In this section we generalize Effective market model to take into account short-range correlations at strategy horizon because of activity of traders. Such correlations lead to the exponent $\mu = 3$ of the power tail of PDF (section II C 4), and relatively weakly affects the central part of PDF: for time interval τ about several minutes it is estimated as about 5%³⁹. We assume normal distribution of noise fluctuations ξ_i for the central part of PDF, and neglect the effect of noise-amplitude anticorrelations, which is small at short τ , and leads to gain/loss asymmetry (see section III D 4).

For given noise variables ξ_i we introduce random variables ξ_i^0 of Effective market model, which form orthogonal basis in the space of random functions ξ_i , see Eq. (55). Expanding price fluctuations $\Delta P(t)$ and $\Delta P(t + \tau)$ over this basis, we get:

$$\Delta_\tau P(t) = \sqrt{2} (\mathbf{a}_1^0, \xi_1^0) + \sqrt{2} (\varepsilon_1, \xi_2^0), \quad (60)$$

$$\Delta_\tau P(t + \tau) = \sqrt{2} (\mathbf{a}_2^0, \xi_2^0) + \sqrt{2} (\varepsilon_2, \xi_1^0). \quad (61)$$

We consider amplitudes \mathbf{a}_i^0 of Effective market as Gaussian random variables, Eq. (55). Non-diagonal amplitudes ε_i describe the shift of equilibrium on the market because of trader activity. Since there are only two independent amplitudes, \mathbf{a}_1 and \mathbf{a}_2 , for two time intervals, the amplitudes ε_1 and ε_2 , can be expanded over two diagonal amplitudes \mathbf{a}_1^0 and \mathbf{a}_2^0 :

$$\varepsilon_i = \sum_j \tilde{c}_{ij} \mathbf{a}_j^0. \quad (62)$$

In the case $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$ this Double Gaussian model is reduced to Markovian model (section III B), and in the case $\varepsilon_1 = \varepsilon_2 = 0$ – to the model of Effective market (section III C).

PDF of this model is calculated in Appendix D:

$$\mathcal{P}(x, y) = \mathcal{P}_0(x \cos \phi_+ - y \sin \phi_-, y \cos \phi_- + x \sin \phi_+), \quad (63)$$

where \mathcal{P}_0 is PDF of Effective market model, Eq. (57). The distribution (63) depends on only four independent parameters: the dispersion σ , the correlator of the amplitude ν ($0 < \nu < 1$), and two angles ϕ_- and ϕ_+ , depending on starting parameters $\{\tilde{c}_{ij}\}$ of our model. The correlator ν describes the “elasticity” of the market to external challenges at the strategy horizon. The angles ϕ_- and ϕ_+ control the feedback between trader expectations and real price changes at the tactic horizon. Their difference, $\varepsilon = \phi_+ - \phi_-$, is taken as small parameter of our theory, which controls the correlator of neighboring price increments

$$\overline{\langle \Delta_\tau P(t) \Delta_\tau P(t + \tau) \rangle} = \sigma^2 \varepsilon. \quad (64)$$

Eq. (63) turns to corresponding expression (53) for Markovian model in the limit $\nu \rightarrow 1$, and reproduces Eq. (57) of Effective market model for $\phi_- = \phi_+ = 0$.

The sum (57) goes only over even $2l$ because of neglect of noise-amplitude correlations, $\langle \eta_i \rangle = 0$. In general, there are correlations between noise and amplitude, described by a factor ζ (see section II C 3). Such correlations (studied in section III D 4) break the symmetry of the conditional average $\langle y \rangle_x$ with respect to positive and negative x , and are responsible for the so called Leverage effect⁴⁴.

In our theory we have an hierarchy of small parameters, $\zeta \ll |\varepsilon| \ll \phi \ll 1$. PDF of Double Gaussian model with all nonzero $\zeta, \phi, \varepsilon \neq 0$ has no symmetries at all. In the case $\zeta = 0$ but $\phi, \varepsilon \neq 0$ there is a symmetry $\mathcal{P}(x, y) = \mathcal{P}(-x, -y)$, corresponding to rotations on the angle π in the plane (x, y) . In the case $\zeta = \varepsilon = 0$ but $\phi \neq 0$, when there are no linear correlations of price (64) at adjacent time intervals, PDF (57) remains symmetrical only with respect to mixed transformation, $\mathcal{P}(x, y) = \mathcal{P}(-y, x)$, corresponding to rotations by the angle $\pi/2$ in the plane (x, y) . The change of sign of y in the above equation is related to reversion of the time: on reversed time scale one can think about losses in future, $y < 0$, as about gains in the “past”. This approximate push-response invariance was established first time from the analysis of empirical data⁴². And finally, in the case $\zeta = \phi = \varepsilon = 0$ the function $\mathcal{P}(x, y)$ acquires the total symmetry $x \rightarrow -x, y \rightarrow -y$ and $x \leftrightarrow y$ of Effective Market model.

1. Market MILL, ACOR and COR stocks

It is convenient to describe the symmetry of PDF with respect to the axes $y = 0$ by antisymmetric component,

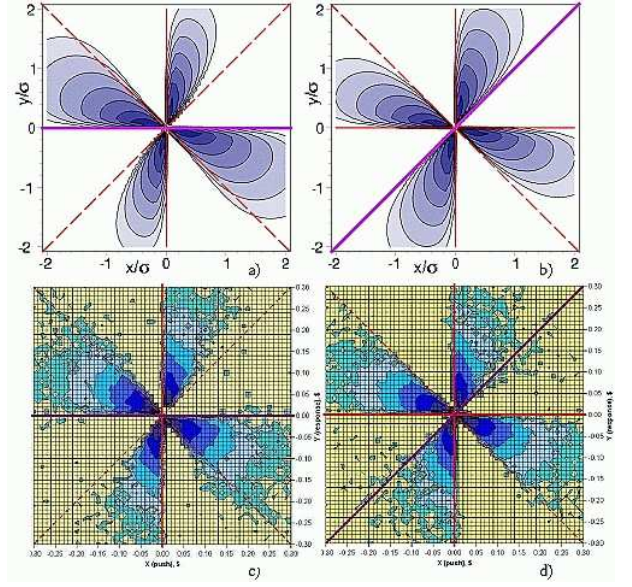


FIG. 8: Two-dimensional projection of $\log_4 \mathcal{P}^a(x, y)$ with respect to $y = 0$ axes a) and $y = x$ axes b) for $\nu = 0.95$, $\phi_- = 8^\circ$ and $\phi_+ = 8.7^\circ$. For comparison sake we show corresponding empirically observed pictures c) and d)³⁶.

$\mathcal{P}^a(x, y) = [\mathcal{P}(x, y) - \mathcal{P}(x, -y)]/2$. In Fig. 8 a) we plot equiprobability levels of positive part of this function, $(\mathcal{P}^a(x, y) + |\mathcal{P}^a(x, y)|)/2$.

For small $|\varepsilon| \ll \phi$ the plot demonstrates four-bladed mill-like pattern (the “market mill” pattern), that was observed first time in Ref.⁴³, see Fig. 8 c). To analyze these pictures it is convenient to divide the push-response plane (x, y) into sectors numbered counterclockwise from I to VIII. In agreement with empirical data at $\varepsilon > 0$ the blades in II and IV quadrants of the (x, y) plane are thinner than their counterparts, which extend out of I and III quadrants. The situation is reversed at $\varepsilon < 0$. With the rise of $|\varepsilon|$ the market mill pattern becomes distorted and only two corresponding blades of the mill pattern left well expressed.

In Figs. 9 a) and b) we show how the market mill pattern is deformed with variation of ε . Varying the angle ϕ_- for fixed σ, ν and ϕ_+ , we get good qualitative agreement with observable patterns, shown in Figs. 9 c) and d). We conclude that the theory allows to explain all the variety of basic patterns for different stocks³⁶, and may be considered as the basis for their quantitative classification: Fig. 8 with $\phi_- \simeq \phi_+ > 0$ ($\varepsilon \simeq 0$) corresponds to the mill pattern (MILL), Fig. 9 a) with $\phi_- > \phi_+$ ($\varepsilon < 0$) corresponds to negative autocorrelation (ACOR), and Fig. 9 b) with $\phi_- < \phi_+$ ($\varepsilon > 0$) corresponds to positive autocorrelation (COR). Anti-mill pattern (AMILL) with $\phi_- \simeq \phi_+ < 0$ was never observed in Ref.³⁶.

Similar patterns are obtained for symmetry properties of the bivariate PDF $\mathcal{P}(x, y)$ with respect to different axes $y = x$, $x = 0$, or $y = -x$. As example we show

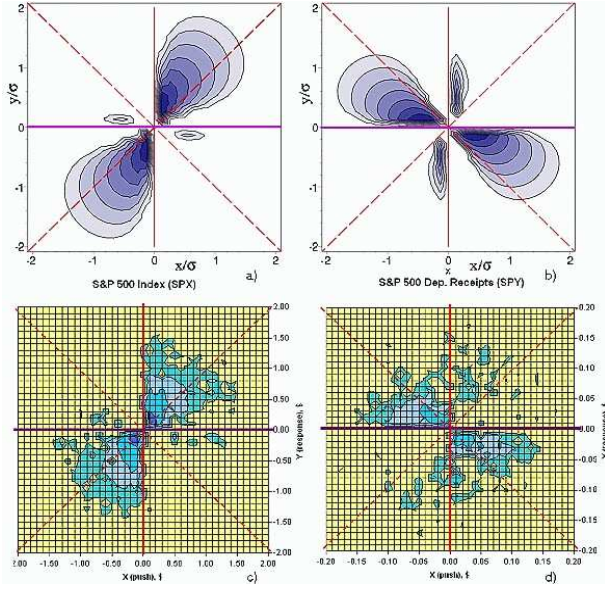


FIG. 9: Changing of asymmetry with respect to $y = 0$ axes with parameter $\varepsilon = \phi_+ - \phi_-$. We use $\nu = 0.95$ and $\phi_+ = 8^\circ$ and vary the angle $\phi_- = 14^\circ$ a), $\phi_- = 6^\circ$ b). For comparison sake we show typical patterns observed for different stocks³⁶ c) and d).

in Fig. 8 b) equiprobability levels of positive part of the function $\mathcal{P}^a(x, y) = [\mathcal{P}(x, y) - \mathcal{P}(y, x)]/2$. The blades of this market mill are more symmetric than those in Fig. 8 a), in agreement with empirical pictures in Figs. 8 c) and d).

An attempt to explain market mill patterns for the asymmetry with respect to the axis $y = 0$ was made in Ref.⁴⁵, where “hand-made” analytical ansatz for conditional PDF was proposed. It was explicitly assumed, that the response y depends only on push x at previous time, and no long-range correlations were taken into account. We do not think, that such Markovian model can give adequate description of real market with extremely wide spectrum of relaxation times.

2. Univariate PDF

Now we calculate one-point PDF, Eq. (47), of Double Gaussian model. Expression (D2) of Appendix D for the Fourier component $G(k, 0)$ can be represented in the form

$$G(k, 0) = (1 + \sigma^2 \alpha_1^2 k^2 / 2)^{-1} (1 + \sigma^2 \alpha_2^2 k^2 / 2)^{-1}, \quad (65)$$

$$\alpha_1 = \cos \theta, \quad \alpha_2 = \sin \theta$$

with the angle θ defined by

$$\sin(2\theta) = \sqrt{1 - \nu^2} \sin(2\phi). \quad (66)$$

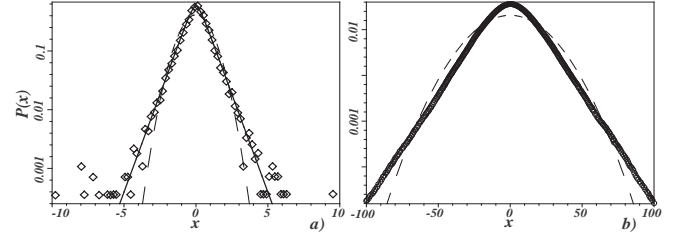


FIG. 10: Probability distribution function for the S&P500. Daily data from 31/1/1950 to 18/7/2003⁴⁶ a) 5 minute increments for 1991-1995⁴⁷ b) We show the best fit by Eq. (67) with $\theta = 0.3$ a) and $\theta = 0.4$ b), for comparison we show by dotted lines the best fit by Gaussian PDF.

Calculating the integral over k in Eq. (50) with this function $G(k, 0)$ we find one-point PDF

$$\mathcal{P}(x) = \frac{\alpha_1 e_1(x) - \alpha_2 e_2(x)}{\sqrt{2\sigma}(\alpha_1^2 - \alpha_2^2)}, \quad e_i(x) = e^{-\sqrt{2}|x|/(\alpha_i \sigma)}. \quad (67)$$

As one can see from Fig. 10 this distribution is in good agreement with observable PDF of the Standard&Poor 500 (S&P500) index, that is one of the most widely used benchmarks for U.S. equity performance.

3. Conditional response

Calculating the integral (51), we find the mean conditional response

$$\langle y \rangle_x = -\text{sign}(x) \sqrt{2\sigma} \frac{2\varepsilon \alpha_1^2 \alpha_2^2 - A}{(\alpha_1^2 - \alpha_2^2)^2} \frac{e_1(x) - e_2(x)}{\alpha_1 e_1(x) - \alpha_2 e_2(x)} + x \frac{(\varepsilon \alpha_1 - A/\alpha_1) e_1(x) + (\varepsilon \alpha_2 - A/\alpha_2) e_2(x)}{(\alpha_1^2 - \alpha_2^2) [\alpha_1 e_1(x) - \alpha_2 e_2(x)]}, \quad (68)$$

where $A = \alpha_1 \alpha_2 \sqrt{(\alpha_1^2 - \alpha_2^2)^2 - \nu^2}$, α_i and $e_i(x)$ are defined in Eqs. (65) and (66). In Fig. 11 a) we show how the dependence (68) of mean conditional response on push x depends on the angle ϕ_- . This dependence has zigzag structure for MILL group ($\varepsilon \simeq 0$), it is almost monotonic for ACOR group ($\varepsilon < 0$), with linear limiting dependence (54)), and is essentially nonlinear for COR group ($\varepsilon > 0$). Similar calculations of conditional mean absolute response $\langle |y| \rangle_x$ (which shows how the response volatility is grow with the amplitude of the given push x) show that in the case $\varepsilon = 0$ to a good accuracy it is linear in the absolute value of the push $|x|$, $\langle |y| \rangle_x \simeq c_0 + c_1 |x|$, in good agreement with empirical data⁴².

To analyze the asymmetry of PDF $\mathcal{P}(x, y)$ with respect to time reversion in the case of MILL group ($\varepsilon = 0$), it is convenient to introduce the total increment of price during the two time intervals, and also the difference of these increments $z = 2^{-1/2}(x + y)$ and

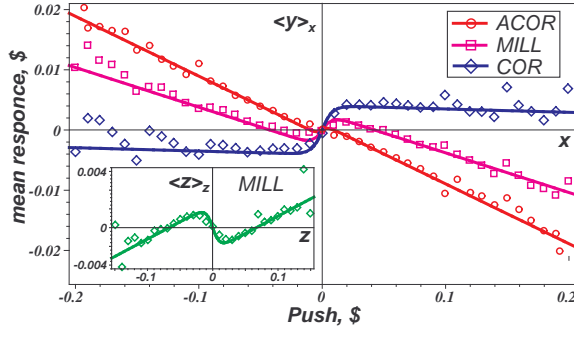


FIG. 11: The dependence of mean conditional response on push x for different angles ϕ_- . We use $\sigma = \$0.04$ and $\phi_+ = 8^\circ$, $\nu = 0.97$ and $\phi_- = 12.5^\circ$ (ACOR), $\nu = 0.9$ and $\phi_- = 9^\circ$ (MILL), and $\nu = 0.8$ and $\phi_- = 4.5^\circ$ (COR). Fitting parameters were taken to reproduce empirical dependences presented for some stocks³⁶. Conditional response $\langle z \rangle_z$ for $\nu = 0.95$, $\phi_+ = 8^\circ$ and $\phi_- = 7.7^\circ$ in line with corresponding empirical data are shown in Insert.

$\bar{z} = 2^{-1/2}(y-x)$. PDF $\mathcal{P}(z, \bar{z})$ of these random variables takes the form of Eq. (63) with the substitution $\phi \rightarrow \varphi = \pi/4 - \phi$, describing the rotation of the push-response plane by the angle $\pi/4$. Therefore, both PDF and all conditional averages are given by above expressions under the substitution $\langle y \rangle_x \rightarrow -\langle \bar{z} \rangle_z$ and $\theta \rightarrow \theta'$, where

$$\sin(2\theta') = \sqrt{1-\nu^2} \sin(2\varphi) = \sqrt{1-\nu^2} \cos(2\phi).$$

Conditional response $\langle \bar{z} \rangle_z$ also has z -shaped structure, and it is shown in Insert in Fig. 11 in comparison with empirical data.

Nonvanishing of average responses $\langle y \rangle_x$ and $\langle \bar{z} \rangle_z$ (Fig. 11) allows one to make some “nonlinear” predictions (68) about future price changes on the market, which can not be obtained from the knowledge of only linear correlations: the response y in the next time interval is correlated with initial increment of the price x at small $|x| \lesssim \sigma$, and is anticorrelated with it at large $|x| \gtrsim \sigma$. The order of price increments is also important for given total increment $\sqrt{2}z$: for small $z < z_0 \sim \sigma$ the average initial variation x is larger than next one, y , and the situation is reverted at large z .

4. Conditional double dynamics

In this section we discuss the hypothesis of Ref.³⁷, that the average return is actually the result of composition of two independent signals with Markovian statistics: one of them positive, and another one negative. It is proposed to characterize this effect by average daily returns $\langle y_- \rangle_{r_c}$ and $\langle y_+ \rangle_{r_c}$ given that the previous day had a re-

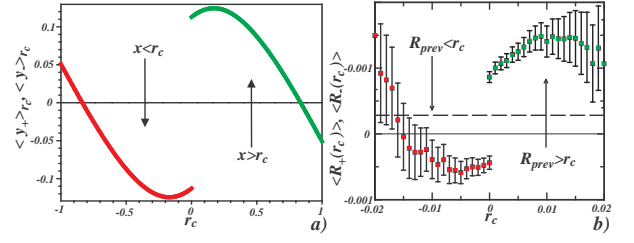


FIG. 12: Average daily return given that the previous day had a return greater than r_c (right) and given that the previous day had a return smaller than r_c (left). Prediction of Gaussian model at $\theta = 0.2$ a) and empirical data³⁷ b).

turn greater than r_c and smaller than r_c :

$$\langle y_- \rangle_{r_c} = \int_{-\infty}^{r_c} dx \langle y \rangle_x \mathcal{P}(x) / \int_{-\infty}^{r_c} dx \mathcal{P}(x), \quad (69)$$

$$\langle y_+ \rangle_{r_c} = \int_{r_c}^{\infty} dx \langle y \rangle_x \mathcal{P}(x) / \int_{r_c}^{\infty} dx \mathcal{P}(x). \quad (70)$$

Calculating integrals (69) and (70) for Double Gaussian model in MILL case $\varepsilon = 0$, we find at $r_c > 0$:

$$\langle y_- \rangle_{r_c} = -\langle y_+ \rangle_{r_c} = \frac{\sigma}{\sqrt{2}(\alpha_1^2 - \alpha_2^2)} \times \frac{A}{\frac{\alpha_1^2 e_1(r_c) - \alpha_2^2 e_2(r_c)}{\alpha_1^2 - \alpha_2^2} - \frac{r_c}{\sqrt{2}\sigma} [e_1(r_c) + e_2(r_c)]}. \quad (71)$$

This function is shown in Fig. 12 in line with empirical data. As one can expect from Fig. 12 a) the average response $\langle y_+ \rangle_{r_c}$ is correlated with r_c at small $r_c \lesssim \sigma$ and is anticorrelated with it at larger r_c . We added horizontal dotted line in Fig. 12 b), shifting the y -axis by unconditional average return³⁷ $\langle y \rangle = 0.00025$. This shift and remaining difference in shape between Figs. 12 a) and b) is related to the buy/sale asymmetry, discussed below.

At $r_c = 0$ the average daily return for given sign of the previous day return is finite, reproducing the effect of “double dynamics” of the market, attributed in Ref.³⁷ to “propagation” of two independent signals in “Markovian” market.

In fact, the market is not Markovian, but the sign of price increments is determined only by the noise ξ . Markovian “double dynamics” of signs is direct consequence of Markovian statistics of noise correlations, see section II C 4. Anticorrelations between the noise and the amplitude are responsible for small systematic trend of the price, $\langle y \rangle \simeq \sqrt{2}\alpha\zeta$ (28), reproducing empirical data³⁷ for $\zeta \simeq 0.02$. This positive trend leads to corresponding increase of the probability to have positive price increment, $p_+ = 1/2 + c_1\zeta$ with $c_1 \simeq 1$. Conditional probabilities of the two-state model³⁷ can be expressed through the angle $\phi \simeq \phi_- \simeq \phi_+$, which determines the amplitude of the response $\langle y \rangle_x$ on previous price increment x

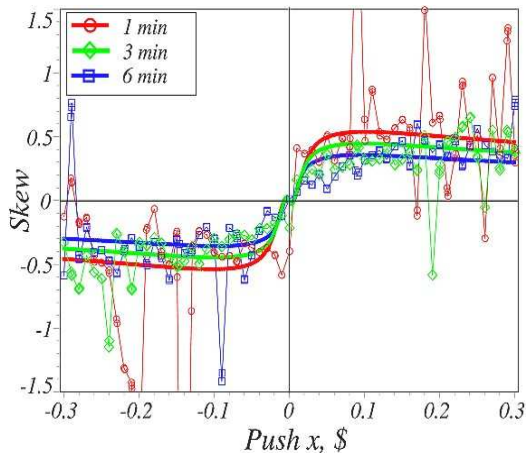


FIG. 13: Theoretical dependence of the skewness ρ_x on x for double Gaussian model at $\phi = 0.2$, $\varepsilon = 0$ and $\nu = 0.9$ and empirically observable dependence³⁹.

($c_2 \sim 1$):

$$\begin{aligned} p_{++} &= 1/2 + c_1\zeta + c_2\phi, \\ p_{--} &= 1/2 - c_1\zeta + c_2\phi. \end{aligned}$$

Empirical observation of “double dynamics” may be considered as direct confirmation of Markovian statistics of noise fluctuations, but not of the whole market, as conjectured in Ref.³⁷. We show in section IV C 7, that multifractal evolution of the amplitude is strongly non-Markovian. But consideration of only signs of returns “erases” information about the amplitude from the time series.

5. Skewness

Asymmetry of the conditional distribution $\mathcal{P}(y|x)$ with respect to the average (51) is characterized by the skewness of the conditional response:

$$\rho_x = \frac{1}{\sigma_x^3} \int_{-\infty}^{\infty} dy (y - \langle y \rangle_x)^3 \mathcal{P}(y|x).$$

The conditional mean-square deviation σ_x is defined in Eq. (52). Positive value of ρ_x indicates that only few agents perform great profits, while many of them have small losses with respect to the mean. A negative ρ_x describes a complementary case. As one can see from Fig. 13, the skewness has the sign of initial push x in accordance with the empirical dependence. Notice that although the skewness is very sensitive characteristic of PDF, our theory reproduces both observed shape and values of ρ_x .

In this section we show, that separation of hot and cold degrees of freedoms allows to reproduce numerous empirical data, known for high-frequency fluctuations on

financial markets. For Gaussian statistics of all degrees of freedom this model captures main features of all groups of stocks, including “market mill” patterns, “dependence-induced volatility smile”, z -shaped response function and so on. Correlations between hot and cold variables are responsible for observable double dynamics of the market, mixing propagating signals of opposite signs and providing systematic positive trend of prices.

E. Results and restrictions

In this section we demonstrate, that the idea of hot and cold variables allows to capture main features of price fluctuations, which can be described by Double Gaussian model – a generalization of the random walk model for the case of multiscale fluctuations. For different sets of parameters the analytical solution of this model reproduces the behavior of all kinds of stocks on financial market, as well as the market as a whole.

Consider some restrictions of this approach. Using coarse-grained description of price fluctuations at times $\tau > \tau_k \lesssim 1$ min we loose information about sale/buy mechanisms^{48,49}. This knowledge (see, for example, Minority and Majority Games⁵⁰) is important to derive parameters of our models for particular markets. We also considered only uni- and bivariate distribution functions, while market dynamics is described by the whole family of n -point correlation functions. In next section we present alternative description of the market at different time scales using ideas of renormalization group approach.

IV. MULTISCALE DYNAMICS OF THE MARKET

Standard thermodynamics can describe only ergodic systems, while the market is the system with “restricted ergodicity”: during the time τ it can explore only small part of the total configuration space near current local equilibrium. Increasing the time t , this equilibrium is shifted because of long-time variations. The resulting multi-time dynamics of fluctuations on the market is not ergodic, and can only be described by continuous set of Langevin or Fokker-Planck-type equations for all time scales.

Note, that this is not exclusive, but standard behavior of complex physical systems. As we will show, different local equilibria on the market are organized into tree cascades (“self-organized criticality”), by analogy with “hot spots” in Quantum Chromodynamics⁵¹, and dynamics of unergodic spin-glasses, which is governed by continuous set of Fokker-Planck equations⁵². Continuous set of equilibria was also predicted for incomplete markets⁵³.

The behavior of the market at the trade by trade level was studied in many details^{54–56}. At larger times col-

lective effects become important, and financial time series display long-time nonlinear correlations^{57–59}, which puzzle many researchers^{60–62}. Different models have been proposed in order to reproduce some “stylized facts”⁴⁰ of empirical time series. Lévy flight processes⁶³ were used to model jumping character of price variations. Volatility (the amplitude of fluctuations) clustering effects have been studied in frameworks of stochastic volatility models⁶⁴ and GARCH-type models⁶⁵. Mixed effect of jumps and stochastic volatility was taken into account in some models⁶⁶. A key to study multiscale properties of price fluctuations is provided by phenomenological multifractal models, see Refs.^{67,68}. Renormgroup approach, describing evolution of multiscale systems, was first proposed for glass systems^{69,70}, and in present work we extend it to the market.

The key question is the source of price fluctuations. Assumption about totally random activity of traders^{71,72} leads to Brownian motion of prices. Although random trading model predicts many qualitative and quantitative properties of the order books^{55,73}, it can not describe existing correlations in price fluctuations. An alternative “efficient market hypothesis” assumes, that the price can be changed only because of unanticipated and totally unpredictable news. This hypothesis lays in the basis of the model of fully rational agents⁷⁴, which also predicts Brownian walk statistics of prices. Observed volatility of the market is too high to be compatible with the idea of fully rational pricing⁷⁵, and can only be reproduced by introducing an artificial random source – “sunspots”⁷⁶. In addition, the analysis of Ref.⁷⁷ shows, that most of large fluctuations in the market are due to trading activity, independently of real news.

The main idea of this paper is that market activity can be described as random trading at all time horizons τ_p from a minute to tenths years. The market tends to reach equilibriums on extremely wide baseband of time scales, and all these equilibriums are continuously changed both because of long-time modes and external events. Multiple local equilibriums can be represented by an hierarchical tree, see Fig. 14. Each generation r of this tree is characterized by its relaxation time τ_r . For any observation time interval $\tau = \tau_r$, all states with times smaller than τ_r are in equilibrium, and fluctuations near this equilibrium are described by hot degrees of freedoms. The states with times larger τ_r are frozen, and are described by cold degrees of freedom.

In this section we derive an analog of renormgroup equations for the market, which relate fluctuations at different coarse grained time scales τ . With decrease of the time interval τ , which can be thought of as effective temperature, the market experiences a cascade of dynamic phase transitions of broken ergodicity, when some hot degrees of freedom become frozen. This cascade can be graphically shown as the hierarchical tree, each branching point of which represents “phase transition” to a state with frozen degrees of freedom with relaxation times $\tau_r > \tau$, see Fig. 14. We show in section IV A 1,

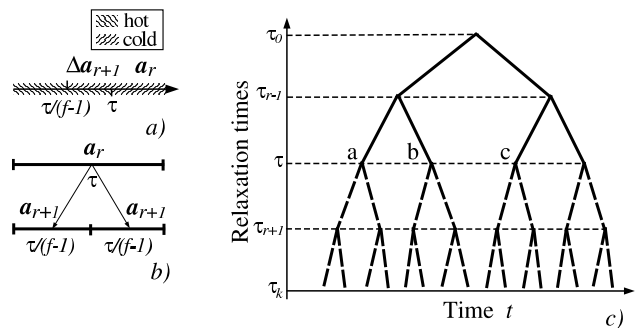


FIG. 14: Hot and cold degrees of freedom have, respectively, times small and large with respect to coarse graining time τ . The scale of relaxation times for two observation times τ and $\tau/(f-1)$, Fig. a). Elementary step of renorm group transformation corresponds to division of “parent” time interval τ into $(f-1)$ “child” time intervals $\tau/(f-1)$, see Fig. b). Hierarchical tree in ultrametric time “space” is shown in Fig. c). For given observation time τ upper part of the tree, shown by solid lines, corresponds to “cold” degrees of freedom, and lower (dotted) part corresponds to “hot” degrees of freedom.

that the topology of this tree reflects ultrametricity of the time “space”.

In section IV A 2 we demonstrate, that fluctuations at given time scale τ are determined by contributions of all “parent” time scales of the hierarchical tree in Fig. 14, what is the reason of non-Markovian dynamics of the market. Cumulative contribution of all time scales allows to explain extremely high volatility of the market (section IV B 1), and is responsible for power law decay of correlation functions (section IV C 1) and their multifractal properties. In section IV C 1 we formulate self-consistency condition, under which the hierarchical tree in Fig. 14 describes coarse-graining dynamics at all levels of the coarse graining time τ , and find the τ -dependence of parameters of our Double Gaussian model, section III D.

In section IV C 3 we derive a set of Langevin equations, describing dynamics of the market with extremely wide range of characteristic times – from minute to tenths years, and calculate the price shift in the response on imbalance of trading volumes (section IV C 4).

We also calculate PDF of volatility (section IV C 5) and show, that it has fat tail with stable exponent $\mu = 3$ for stock jumps and $\mu = 2$ for news jumps. We derive, that coarse grained dynamics of the market can be reduced to the multifractal random walk model^{78,79}, which determines multifractal properties of price fluctuations, related to the ultrametric structure of the tree in Fig. 14. We calculate volatility patterns after news and stock jumps, and find their conditional probabilities. In section IV C 6 we show, that the price $P(t)$ behaves as fractional Brownian motion. We demonstrate in section IV C 7, that Brownian motion, sub- and super-diffusive regimes change each other at the long-time scale. The knowledge of history can be used to estimate the

tendency and risks of future price variations.

Main results of this section are summarized in section IV D. In Appendix E we show details of calculations of the volatility distribution.

A. Renormalization group transformation

Consider statistics of price increments (returns)

$$\Delta_\tau P(t) = P(t + \tau) - P(t), \quad (72)$$

as the function of the coarse-graining time interval τ . Here $P(t)$ is the price or its logarithm at time t . For definiteness sake, we consider only ACOR group of stocks³⁶, when $\Delta_\tau P(t)$ can be represented as scalar product of complex amplitude $\mathbf{a}(t)$ and complex noise $\boldsymbol{\xi}(t)$, Eq. (41). By analogy with renormgroup consideration, cold variable $\mathbf{a}(t)$ slowly varies at time scale τ , while hot variable $\boldsymbol{\xi}(t)$ quickly fluctuates at this scale, see Fig. 14 a).

1. Ultrametricity and restricted ergodicity

In order to establish an analog of renormgroup transformation for the market we first introduce corresponding partitioning of the time “space”. At elementary step of the renormgroup each “parent” interval τ_r of time axis can be divided into $f - 1$ “child” time interval $\tau_{r+1} = \tau_r / (f - 1)$, see Fig. 14 b). Repeating such division, we arrive to the hierarchical tree with functionality f , shown schematically in Fig. 14 c). For this tree the time

$$\tau_r = \tau_0 e^{-\kappa r}, \quad \kappa = \ln(f - 1) \quad (73)$$

depends exponentially on the current rank r , τ_0 is maximal relaxation time. Minimal time $\tau_k = \tau_0 e^{-\kappa k}$ (k is the number of generations of the tree) is about average time between trades. Typically, τ_0 is about several years and τ_k is about minute for liquid markets, and so $\kappa k \gtrsim 13$.

We define the “distance” z between events at times t and t' by the condition $\tau_{r-z} = |t - t'|$:

$$z(t - t') = \frac{1}{\kappa} \ln \frac{|t - t'|}{\tau} \quad \text{at} \quad |t - t'| \gg \tau = \tau_r, \quad (74)$$

which can be identified with the distance (number of generations) along the tree in Fig. 14 c) between these points. One can show, that for three different times t, t', t''

$$z(t - t'') \simeq \max[z(t - t'), z(t' - t'')],$$

and, therefore, the metric (74) generates ultrametric time “space” (with only isosceles and equilateral triangles), which can be really mapped to the tree. For example, in Fig. 14 c) the distance between points a and b is $z_{ab} = 1$, $z_{bc} = 2$, and $z_{ac} = \max(z_{ab}, z_{bc}) = 2$.

Note, that Eq. (73) gives standard relation between time scales of discrete wavelet transformation. The tree in Fig. 14 can be thought of as a skeleton of the wavelet transformation of time series. We turn to wavelet interpretation of our approach in section IV C 3.

Each of horizontal levels at the time $\tau = \tau_r$ on the tree in Fig. 14 c) corresponds to course-grained description of fluctuations at the time scale τ . Hot degrees of freedom are “melted”, and described by complex noise $\boldsymbol{\xi}(t)$ with continuum spectrum of relaxation times extended from τ_k through τ . Cold degrees of freedom are characterized by complex amplitude $\mathbf{a} = \mathbf{a}_r$ of the noise, which is frozen at the time τ , see Eq. (41).

By analogy with glasses, the states of real market are highly degenerated, what is reflected in the presence of gauge transformation (21) of complex noise and amplitude variables, which do not affect price variations $\Delta_\tau P(t)$, Eq. (41). The degeneracy is the reason of high sensibility of the market to external events. Recall, that in spin glasses any observable are not affected by “non-serious” part of disorder, which can be removed by gauge transformation of glass degrees of freedoms.

Following this analogy, the time τ plays the role of the temperature T . With decrease of the temperature $T \sim \tau$ from $\tau = \tau_0$ the market experiences a cascade of dynamic phase transitions of broken ergodicity, when some hot degrees of freedom become frozen (the system is unergodic if its fluctuations can not explore the whole configuration space). This cascade proceeds continuously down to the time τ_k , and can be graphically shown as hierarchical tree, each branching point of which represents phase transition to a state with frozen degrees of freedom with relaxation times $\tau_r > \tau$, see Fig. 14. The parameter $\kappa \ll 1$ determines the probabilities of such transitions, which are relatively rare for real markets.

In this sense at any $\tau < \tau_0$ the market is just at the point of dynamic phase transition of broken ergodicity, and has, therefore, increased amplitude of fluctuations – the volatility. This observation supports the idea that the market is always operating at a critical point as the result of competition between two populations of traders: “liquidity providers”, and “liquidity takers”^{80,81}. Liquidity providers correspond to hot degrees of freedom, creating antipersistence in price changes, whereas liquidity takers correspond to cold degrees of freedom, and they lead to the long range persistence in prices.

Since such separation of the market into hot and cold degrees of freedom takes place at any time scale, $\tau_k \ll \tau \ll \tau_0$, there could not be any unique classification of traders, which can be divided also into “positive feedback” traders and “fundamentalists”⁸², “contrarian” traders and “trend followers”⁸³ and so on. There is, however, important difference between market and spin-glass hierarchical trees: while the states of the glass are not ordered, there is strong time ordering of all “points” of the market tree at any level r of the hierarchy.

2. Recurrence relation

General recurrence relation between amplitudes \mathbf{a}_r and \mathbf{a}_{r+1} at levels r and $r+1$ of the hierarchical tree can be written through the random transition matrix \mathbf{u}_r :

$$\mathbf{a}_{r+1} = \mathbf{u}_{r+1}\mathbf{a}_r + \Delta\mathbf{a}_{r+1}. \quad (75)$$

In general, there could be a term $\sim (\mathbf{a}_r)^*$ in the rhs of this equation, but it is not invariant with respect to gauge transformation, Eq. (21), and should be dropped. The term $\sim \mathbf{u}_{r+1}$ describes the inheritance of the amplitude \mathbf{a}_r of the “parent” levels of the hierarchy, and $\Delta\mathbf{a}_{r+1}$ gives the contribution of “newborn” during the transition $r \rightarrow r+1$ unfrozen degrees of freedom to the “child” amplitude \mathbf{a}_{r+1} .

The recurrence relation (75) can also be rewritten in the multiplicative form, introducing relative increment $\Delta_{r+1} = \Delta\mathbf{a}_{r+1}/\mathbf{a}_r$:

$$\mathbf{a}_{r+1} = e^{\omega_{r+1}}\mathbf{a}_r, \quad e^{\omega_{r+1}} \simeq 1 + \omega_{r+1} = \mathbf{u}_{r+1} + \Delta_{r+1} \quad (76)$$

Random variables \mathbf{u}_{r+1} and $\Delta\mathbf{a}_{r+1}$ (Δ_{r+1}) are determined by degrees of freedom with characteristic times $\tau_r < \tau < \tau_{r+1}$, while \mathbf{a}_r is formed by degrees of freedom with times larger τ_r . Assuming independence of fluctuations of different time scales, \mathbf{a}_r do not depend on \mathbf{u}_{r+1} and $\Delta\mathbf{a}_{r+1}$. We estimate the mean squared amplitudes of fluctuations of $\Delta\mathbf{a}_r$ and \mathbf{u}_r as

$$\overline{(\Delta\mathbf{a}_r)^2} = D_0\tau_r, \quad \overline{\mathbf{u}_r^2} = u^2, \quad (77)$$

There is important difference of Eq. (77) from the case of usual diffusion, when $\langle r^2 \rangle = Dt$ is the consequence of independence of fluctuations at *different times* t . In contrast, diffusion-like dependence (77) with effective coefficient D_0 is the consequence of independence of fluctuations of *different time scales* τ_r . The time t -dependence of price fluctuations is strongly non-diffusive.

B. Amplitude of fluctuations

1. Excess of volatility

In the mean field approximation we neglect fluctuations of $\mathbf{u}_r = u$, and find the solution of Eq. (75) in the form of the sum of independent random signals $\Delta\mathbf{a}_{r-k}$ from time intervals $\tau(f-1)^k$, obtained by multiplicative merging of $(f-1)^k$ previous time intervals τ :

$$\mathbf{a}_r(t) = \sum_{k>0} u^k \Delta\mathbf{a}_{r-k}(t). \quad (78)$$

Weights of these signals exponentially fall with the distance k in time hierarchy from the current rank r . Simulated time series (78) for the amplitude $a(t)$ in the model with random $\Delta\mathbf{a}_r = \pm\sqrt{D_0\tau_r}$ are shown in Insert in Fig. 15. This picture demonstrates multiscale character of resulting price fluctuations.

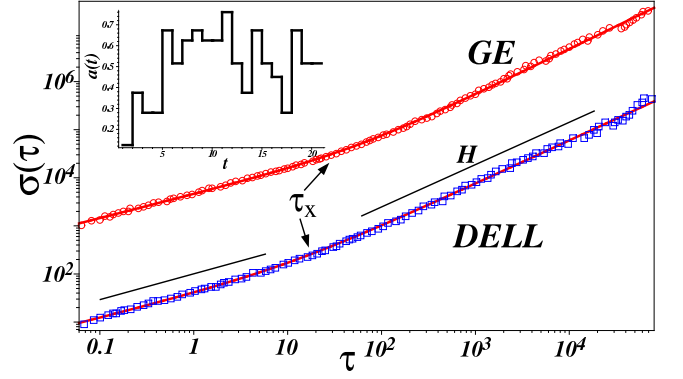


FIG. 15: Empirical dependence⁸⁴ of dispersion $\sigma(\tau)$ on time interval τ , and its fitting by Eq. (82), $\lambda_0^2 = 0.9$ for DELL and $\lambda_0^2 = 0.8$ for General Electric. The effective Hurst exponent H is different at $\tau < \tau_x$ and $\tau > \tau_x$. In Insert – computer simulation of random amplitude, Eq. (78).

Averaging the square of the recurrence relation (75), we find difference equation

$$\sigma^2(\tau_{r+1}) = u^2\sigma^2(\tau_r) + D_0\tau_0 e^{-\kappa(r+1)} \quad (79)$$

for the dispersion $\sigma(\tau_{r+1})$ of the amplitude \mathbf{a}_r , which has the solution

$$\sigma^2(\tau_r) = D\tau_r + Lu^{2r}, \quad (80)$$

where L is the constant of integration and D is the effective diffusion coefficient:

$$D = \frac{D_0}{1 - e^{-\kappa\lambda_0^2}} \gg D_0, \quad e^{-\kappa\lambda_0^2} \equiv u^2 e^\kappa. \quad (81)$$

From Eqs. (80) and (73) we find the dependence of the dispersion of price increments on the coarse-graining time τ :

$$\sigma^2(\tau) = D\tau + L(\tau/\tau_0)^{1+\lambda_0^2}. \quad (82)$$

The dependence (82) for different stocks is in good agreement with empirical data, see Fig. 15. Although at small $\tau \ll \tau_x$,

$$\tau_x = \tau_0 (D\tau_0/L)^{1/\lambda_0^2}, \quad (83)$$

it looks like diffusion with apparent diffusion coefficient D (81), price fluctuations do not really have diffusive behavior. As the sign of it, the amplitude of price fluctuations is anomalously large due to the presence of a big prefactor for $\kappa \ll 1$ in Eq. (81). It was shown by Schiller⁸⁵, that even accounting the volatility of dividends⁸⁶ leaves the empirical volatility at least a factor 5 too large with respect to the random walk model. Such anomalous “excess of volatility”, $D/D_0 \sim 10$, originates from the superposition of signals from all time scales, see Eq. (78). Similar effect (by 10 orders of value) is well

known in spin glasses, where the parameter κ in Eq. (74) is extremely small.

Eq. (80) can be used to estimate the amplitude of the transition matrix in Eq. (76). From Eqs. (77) and (80) with $L = 0$ (at $\tau \ll \tau_x$) we find $\overline{\Delta_{r+1}^2} = e^{-\kappa} \left(1 - e^{-\kappa\lambda_0^2}\right)$, and get

$$\overline{e^{2\omega_r}} = e^{-\kappa}. \quad (84)$$

2. Cross-over time and Hurst exponents

The term $\sim L$ in Eq. (82) appears as a constant of integration of the recurrence equation (79), which is determined by the ‘‘boundary condition’’ at trading time τ_k . Therefore, L is not universal and determined by microstructure of the market.

At small $\tau < \tau_x$ the first term in Eq. (82) gives the main contribution, $\sigma(\tau) \sim \tau^H$, with effective Hurst exponent close to 1/2. At large $\tau > \tau_x$ the second term in Eq. (82) gives dominating contribution to the Hurst exponent:

$$H = (1 + \lambda_0^2)/2. \quad (85)$$

Such behavior with different exponents H at $\tau < \tau_x$ and $\tau > \tau_x$ was really observed for S&P 500 stock index (1984-1996)⁸⁷ with different values of the cross-over time τ_x for individual companies, see Fig. 15.

The removal⁸⁷ of the largest 5 and 10% events kills correlations of the noise $\xi(t)$ at small time scales, reducing the constant L . According to the prediction (83) of our theory, it shifts $\sigma(\tau)$ to lower values, and strongly increases τ_x . Excluding the shift of L from variations of both σ and τ_x , we find linear relation between two these shifts: $\Delta \ln \sigma \simeq -(\lambda_0^2/4) \Delta \ln \tau_x$. Comparison with empirical data⁸⁷ gives the estimation $\lambda_0^2 \simeq 1$, in good agreement with observable exponent $H \simeq 0.93$ for the regime $\tau > \tau_x$, see Eq. (85). Similar behavior is observed for different stocks with typical transition times τ_x about several days.

3. Parameters of Double Gaussian model

Let us show, that in order to represent coarse grained dynamics of price fluctuations for the time interval $\tau = \tau_r$, noise variables ξ_{r+1}^n at different ‘‘child’’ subintervals $n = 1, \dots, f-1$ of the same ‘‘parent’’ interval should be (anti)correlated. According to the idea of the coarse grained description, the price increments for the time interval τ_r is the sum

$$\Delta_{\tau_r} P(t) = \sum_{n=1}^{f-1} \Delta_{\tau_{r+1}} P(t + n\tau_{r+1}) \quad (86)$$

of price increments for all $f-1$ adjacent time intervals τ_{r+1} . Substituting Eq. (41) into Eq. (86), this last equa-

tion can be rewritten in the form

$$\begin{aligned} (\mathbf{a}_r, \xi_r) &= \sum_{n=1}^{f-1} (\mathbf{a}_r, \xi_{r+1}^n) \\ &= \left(\mathbf{u}_r \mathbf{a}_r, \sum_{n=1}^{f-1} \xi_{r+1}^n \right) + \sum_{n=1}^{f-1} (\Delta \mathbf{a}_{r+1}^n, \xi_{r+1}^n), \end{aligned} \quad (87)$$

where we substituted Eq. (75) for the amplitude \mathbf{a}_{r+1}^n to the right hand side of Eq. (87). Calculating the average (both quenched and annealed) of the square of this equation, we find the self-consistency equation

$$\sigma^2(\tau_r) = u^2 \sigma^2(\tau_r) (f-1)(1+\varepsilon) + D_0 \tau_r (f-1),$$

where ε is the noise correlator at neighboring time intervals,

$$\begin{aligned} \varepsilon &\equiv \overline{\langle \Delta_{\tau_{r+1}} P(t) \Delta_{\tau_{r+1}} P(t + \tau_{r+1}) \rangle} / \sigma^2(\tau_{r+1}) \\ &= \left\langle \left(\xi_{\alpha}^{r+1}, \xi_{\beta}^{r+1} \right) \right\rangle, \end{aligned}$$

Substituting here $u^2 = e^{-\kappa(1+\lambda_0^2)}$ (80) with $e^{\kappa} = f-1$ we find in the case $L=0$

$$\varepsilon \simeq -(e^{\kappa} - 1) \kappa \lambda_0^2 \simeq -\kappa^2 \lambda_0^2.$$

This value is always negative and small at $\kappa \ll 1$.

4. Time and size dependence of fluctuations

The effect of noise/amplitude anticorrelations, studied in section IIC4, is small in the parameter $\zeta \ll 1$, and leads to the asymmetry of the tails of probability distributions (see Fig. 4), observed for PDF of individual companies in Ref.⁸⁸. It is also responsible for different apparent exponents for positive ($\mu_+ > 3$) and negative ($\mu_- < 3$) power tails. This effect is illustrated in Insert in Fig. 16, where we show, that the function $(x - \bar{x})^{-4}$ at $\bar{x} = 1$ is indistinguishable at about two decades in x from power tails $|x|^{-1-\mu_{\pm}}$ with exponents $\mu_+ = 3.25$ and $\mu_- = 2.8$. Increasing $\bar{x} \sim \zeta$ leads to larger deviations of these exponents from the universal value 3, see Insert.

Simple analytical expression for the whole distribution $\mathcal{P}(x)$ for general τ can be easily constructed by consecutive matching Gaussian, exponential and power distributions, $\mathcal{P}(x) \sim x^{-4}$, at some points x_{\pm} and y_{\pm} , respectively. The resulting expression well reproduces most of empirical data for PDF $\mathcal{P}(x)$. Below we use this expression to explain main features of PDF at $\tau \ll \tau_x$ and $\tau \gg \tau_x$:

We can estimate the effective exponents μ_{\pm} re-expanding $x^{-4} \sim (x - \bar{x})^{-1-\mu_{\pm}}$ about correlation induced systematic shift $\bar{x} \simeq \sqrt{2}\zeta\alpha \simeq \sqrt{2}\zeta\sigma$ (see Eq. 28), and find $\mu_{\pm} \simeq 3 + 4\bar{x}/x_{\pm}$ near cross-over points $x = x_{\pm}$. At small $\tau \ll \tau_x$ the central part of the distribution has exponential shape (26) with maximum at \bar{x} . Matching it

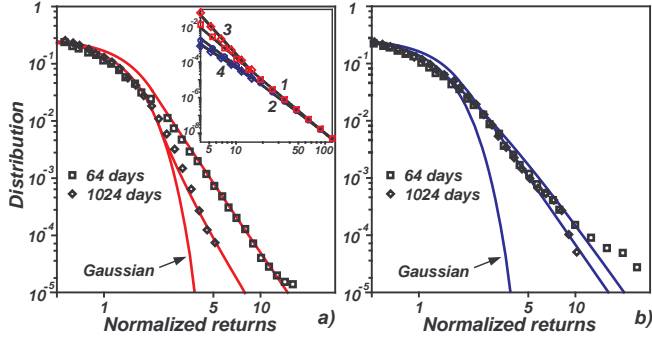


FIG. 16: Empirically observed positive a) and negative b) tails⁸⁸ of the probability distribution, and their matching by shifted Gaussian and power tail $(x - \bar{x})^{-4}$, $\bar{x} = 0.1\sigma$ for time interval $\tau = 64$ days and $\bar{x} = 1.1\sigma$ for $\tau = 1024$ days. In Insert we show matching of x^{-4} by $(x - \bar{x})^{-1-\mu_{\pm}}$ with $\mu_{+} = 3.25$ (1), $\mu_{-} = 2.8$ (2) at $\bar{x} = 1$, and $\mu_{+} = 4.5$ (3), $\mu_{-} = 2.3$ (4) at $\bar{x} = 2$.

at $x = x_{\pm}$ with power tails $A_{\pm}x^{-4}$ we find $x_{\pm} = \pm 2\sqrt{2}\sigma_{\pm}$ and $\mu_{\pm} \simeq 3 \pm 2\zeta$, in good agreement with Ref.⁸⁸.

At large $\tau \gg \tau_x$ the central part has Gaussian shape. Matching it at $x = y_{\pm}$ with power tails, we get $y_{\pm} = \bar{x}/2 \pm \sqrt{\bar{x}^2/4 + 4\sigma^2}$. With increase of ζ (at $\bar{x} \simeq \sqrt{2}\zeta\sigma \gtrsim \sigma$) Gaussian region of the positive distribution is progressively extended, while negative distribution remains fat-tailed, explaining corresponding mysterious behavior of empirical data⁸⁸, see Fig. 16.

The size dependence of the volatility was studied for individual companies in Ref.⁸⁸. It is shown, that Eq. (13), $\sigma \sim G^{-\beta}$, well describes the dependence of dispersion of returns on market capitalization. For $\tau = 1$ day $\beta \simeq 0.2$, while it gradually decreases with the rise of τ , approaching the value $\beta \simeq 0.09$ for $\tau = 1000$ days. This effect supports our self-similar model of companies (see section II B 2), when the index $\beta = 1/(2n)$ is determined by the number n of generations of the hierarchical tree. The effective number n of tree generations logarithmically depends on relaxation time τ , Eq. (73), and for $\tau_k \ll \tau \ll \tau_0$ the dependence β on τ can be approximated by

$$\beta \simeq \beta_0 - \beta_1 \ln \tau. \quad (88)$$

From equation $\sigma \sim G^{-\beta} \sim \tau^H$ we find that the Hurst exponent (85) should grow logarithmically,

$$H = H_0 + \beta_1 \ln G,$$

with market capitalization G , in good agreement with empirical data⁸⁴.

C. Nonlinear dynamics of fluctuations

1. Correlation functions: multifractality

From Eqs. (75) and (80) we find simple analytical expression for the correlation function of amplitudes:

$$\overline{(\mathbf{a}(t), \mathbf{a}(t'))} = D\tau_0 e^{-\kappa r - \kappa \lambda_0^2 z(t, t')} + Lu^{2r}, \quad (89)$$

where z is the logarithmic distance (74) in the ultrametric space, see Fig. 14. Therefore, observed power autocorrelations in the time series are the consequence of the self-similarity of the hierarchical tree in Fig. 14. Neglecting the term L at large enough r (small coarse-graining time $\tau < \tau_x$) in Eq. (89) we find that amplitude correlation function decays as the power of the time

$$\overline{(\mathbf{a}(t), \mathbf{a}(t'))} \sim \exp(-\lambda_0^2 \ln |t - t'| / \tau), \quad |t - t'| \gg \tau.$$

Now consider fluctuations of the modulus $a(t)$ of the amplitude $\mathbf{a}(t)$. The solution of the multiplicative recurrent relation (76) for the coarse graining time $\tau = \tau_r$ is

$$a(t) \simeq \sigma_0 e^{\omega(t)}, \quad \omega(t) = \sum_{p \leq r} \omega_p(t). \quad (90)$$

From expansion (90) we find

$$\overline{a^q(t) a^q(t')} \sim \left(\frac{\tau}{|t - t'|} \right)^{\tau(q)}, \quad \tau(q) = \frac{g(2q) - 2g(q)}{\kappa} \quad (91)$$

with

$$g(q) \equiv \ln \overline{e^{q\omega_r}} = -q\kappa/2 + \kappa\lambda^2 (q^2/2 - q) + \dots,$$

where we expanded $g(q)$ over irreducible correlators of $\omega_r = \bar{\omega}_r + \Delta\omega_r$, $\kappa\lambda^2 = \overline{\Delta\omega_r^2}$, and used Eq. (84) to find the linear in q term. For Gaussian ω_r there are only two first terms in this expansion, and we get $\tau(q) = \lambda^2 q^2$. We also find

$$\overline{a^q(t)} \sim \tau^{q\tilde{H}(q)}, \quad q\tilde{H}(q) = -g(q)/\kappa.$$

In the case of Gaussian ω_p this gives us the generalized Hurst exponent $\tilde{H}(q) = 1/2 + \lambda^2 - \lambda^2 q/2$, see also Ref.⁸⁹. The intermittence parameter λ^2 characterizes the uncertainty on the market, and we expect, that λ^2 is relatively large for emerging markets with large uncertainty, and small for well-developed markets (see section IV C 3).

We conclude, that hierarchical structure of market times, see Fig. 14, generates multifractal time series with q -dependent generalized Hurst exponent. The amplitude a of fluctuations is randomly renewed with time t : with the probability $p_0 \sim \tau_0^{-1}$ for the root of the hierarchical tree in Fig. 14, ..., and with the probability $p_k \sim \tau_k^{-1} \gg \gamma_r$ for maximum rank $i = k$ of the tree. This random process generalizes the Markov-Switching Multi-Fractal process⁶⁷ with $a^2 = \sigma^2 \prod_{r=1}^k M^{(r)}$. The multiplier $M^{(r)}$ is renewed with probability p_r exponentially depending on its rank r within the hierarchy of multipliers.

2. Volume statistics

In this section we introduce an analog of canonical action-angle variables, in which coarse-grained market dynamics can be described by a set of linear Langeven equations for all time scales τ_p in the market. The “thermodynamic force” of price variations is the imbalance of trading volumes, $V(t)$, which can be considered as random function of time (volume time series). The increment of the volume at the time interval $\tau = \tau_r$ can be written by analogy with price increment (Eq. 90) in the form:

$$\Delta_\tau V(t) \simeq \sigma_0 e^{v(t)} \eta(t), \quad v(t) = \sum_{p \leq r} v_p(t). \quad (92)$$

Normalized random noise $\eta(t)$ is proportional to the sign of the increment $\Delta_\tau V(t)$. Gaussian random variable $v(t)$ slowly varies at the time scale τ , and can be expanded over modes p covering the frequency band from τ_p^{-1} to τ_{p+1}^{-1} . Explicit expression for $v_p(t)$ can be obtained expanding its variation $\Delta v(t) = v_p(t) - \bar{v}_p$ over normalized wavelet functions ψ with expansion coefficients $\hat{v}_p(t')$ (\bar{v}_p describes regular trend):

$$\begin{aligned} \Delta v_p(t) &= \int \tau_p^{-1/2} \psi[(t-t')/\tau_p] \hat{v}_p(t') dt', \\ \hat{v}_p(t') &= \int \tau_p^{-3/2} \psi^*[(t-t')/\tau_p] \Delta v(t) dt. \end{aligned} \quad (93)$$

Similar equations relate modes $\omega_p(t)$ (90) with the volatility $\omega(t)$. In the case of random activity of traders $\hat{v}_p(t')$ at all time horizons τ_p should be considered as uncorrelated random values:

$$\overline{\hat{v}_p(t) \hat{v}_{p'}(t')} \simeq \lambda^2 \kappa \delta_{pp'} \delta(t-t'). \quad (94)$$

The noise function $\eta(t)$ in Eq. (92) can be presented in the form $\eta(t) \simeq \cos \phi(t)$, where $\phi(t)$ is the phase of corresponding complex amplitude (see Eq. 41). Random function $\phi(t)$ can be expanded over wavelet modes (Eq. 93). Assuming that corresponding expansion coefficients $\hat{\phi}_p(t')$ are independent random values at all time horizons,

$$\overline{\hat{\phi}_p(t) \hat{\phi}_{p'}(t')} \simeq \gamma \kappa \delta_{pp'} \delta(t-t'), \quad (95)$$

we find:

$$\begin{aligned} \overline{\eta(t) \eta(t')} - \overline{\eta(t)}^2 &\simeq \exp \left[- \sum_{\tau_k < \tau_p < |t-t'|} \overline{\hat{\phi}_p(t) \hat{\phi}_p(t')} \right] \\ &\simeq \exp \left[- \int_{\tau_k}^{|t-t'|} \frac{d\tau_p}{\kappa \tau_p} (\gamma \kappa) \right]. \end{aligned}$$

Calculating the integral over τ_p we arrive to

$$\overline{\eta(t) \eta(t')} - \overline{\eta(t)}^2 \sim |t-t'|^{-\gamma}. \quad (96)$$

From above equations we find that the amplitude of volume increments $|\Delta_\tau V|$ is log-normally distributed and

uncorrelated with the sign of $\Delta_\tau V(t)$. These predictions are supported by empirical data⁹⁰, which also show, that signs $\eta(t)$ of trade volumes (and, therefore, the very $\eta(t)$) have long-range power correlations, Eq. (96), with stock dependent exponent $\gamma < 1$. We conclude, that observed power-low correlations in signs of volume are the consequence of the self-similarity of price fluctuations at different time scales, which lead to scale invariant intermittence parameter λ^2 (Eq. 94) and the amplitude of fluctuations γ (Eq. 95). Such long range correlations are usually considered as the result of cutting of large trades into small chunks (see Ref.⁹⁰).

3. Langeven equations and market entropy

The key idea of Langeven formulation of multi-time market dynamics is that fluctuations at different time scales τ_p are statistically independent. Therefore, the logarithmic volatility $\omega_p(t)$ of the mode with relaxation time τ_p is induced only by corresponding volume mode $v_p(t)$ (93). Since both $\omega_p(t)$ and $v_p(t)$ have Gaussian statistics, general Langeven equations are linear (different choice of fluctuation modes makes these equations highly nonlinear):

$$\tau_p \frac{\partial \omega_p(t)}{\partial t} + \omega_p(t) = v_p(t) \quad (97)$$

with δ -correlated noise:

$$\overline{\Delta v_p(t) \Delta v_{p'}(t')} = 2 \delta_{pp'} \kappa \lambda^2 \tau_p \delta(t-t'). \quad (98)$$

At time scale τ_p this equation is in agreement with Eq. (94). After standard calculations we find correlation function of volatility modes

$$\overline{\Delta \omega_p(t) \Delta \omega_{p'}(t')} = \delta_{pp'} \kappa \lambda^2 e^{-|t-t'|/\tau_p}, \quad (99)$$

and fluctuations of logarithmic volatility for the coarse graining time $\tau = \tau_r$:

$$\begin{aligned} G(t-t') &= \overline{\Delta \omega(t) \Delta \omega(t')} = \sum_{p \leq r} \overline{\Delta \omega_p(t) \Delta \omega_p(t')} \\ &\simeq \lambda^2 \ln(\tau_0/|t-t'|), \quad \tau \ll |t-t'| \ll \tau_0. \end{aligned} \quad (100)$$

This expression lays in the basis of multifractal random walk model^{78,79}, and reproduces Eq. (91):

$$\overline{a^q(t) a^q(t')} \sim \overline{\exp q [\omega(t) + \omega(t')]} \sim (\tau/|t-t'|)^{\tau(q)}.$$

Eqs. (90), (97) and (98) present Langeven formulation of multifractal market dynamics. Using standard transformations, they can be rewritten in the form of Smoluchowski equations for probability function $\Psi\{\omega_p\}$. The importance of this function is that it defines rigor entropy of the market

$$S[\Psi] = \int D\omega_p \Psi\{\omega_p\} \ln \Psi\{\omega_p\},$$

which can only increase with time. The entropy $S[\Psi]$ characterizes informational content of the market.

From Eq. (97) we find the relation between averages: $\bar{v}_p = \bar{\omega}_p \sim -\kappa$ (see Eq. 84), which allows one to express parameters κ and λ^2 of our theory through corresponding moments of the trade volume V :

$$\kappa \sim \frac{2}{k} \overline{\frac{|V|}{V_k}}, \quad \lambda^2 \sim \frac{(\Delta \ln V)^2}{k\kappa} \sim \frac{(\Delta \ln V)^2}{\ln |V/V_k|}, \quad (101)$$

where $V_k^2 \simeq \sigma_0^2 \tau_k / \tau_0$ is about squared bid-ask spread.

4. Response functions

From Langeven equation (97) we find the response of the mode p on volume imbalance $v_p(t)$:

$$\begin{aligned} \omega_p(t) &= \int_{-\infty}^t \chi_p(t-t') v_p(t') dt', \\ \chi_p(t-t') &= \frac{\kappa}{\tau_p} e^{-(t-t')/\tau_p} \theta(t-t'). \end{aligned} \quad (102)$$

Using Eq. (93) one can check, that the response function χ of volatility on volume imbalance at the coarse graining time $\tau_r = \tau$ is determined by the sum of contributions of modes $p > r$:

$$\begin{aligned} \omega(t) &\simeq \int_{-\infty}^t \chi(t-t'|\tau) v(t') dt', \\ \chi(t-t'|\tau) &= \sum_{p=0}^r \chi_p(t-t') \simeq \frac{\theta(t-t')}{t-t'}. \end{aligned} \quad (103)$$

The average price shift because of a single trade at the time $t = 0$ of the volume

$$|\Delta V| = V_k (e^{\Delta v} - 1)$$

(-1 is only important at small $|\Delta V| \sim \Delta v$) is determined by all modes with times from τ_k through τ_0 (the noise $\xi_k \simeq 1$ at τ_k):

$$\begin{aligned} \Delta P(t) &\simeq \sigma_k \left[e^{\Delta \omega(t)} - 1 \right] \simeq \sigma_k \Delta \omega(t) \\ &\simeq \sigma_k \chi(t|\tau_k) \tau_k \Delta v. \end{aligned} \quad (104)$$

The dispersion σ_k at time interval $\tau = \tau_k$ is obtained by averaging over fluctuations of random variables (98), describing variations of the liquidity of the market. In general, the liquidity (at physical language, susceptibility) strongly depends on the history: small volumes can initiate large jumps or make almost no effect. Similar ‘‘aging’’ effect exists for spin glasses, where the susceptibility strongly depends on the history of temperature and magnetic field changes.

Using Eq. (92) we get

$$\begin{aligned} \Delta P(t) &= G_0(t) \text{sign}(\Delta V) \ln(1 + |\Delta V|/V_k), \\ G_0(t) &\simeq \sigma_k \theta(t) \tau_k (t + \tau_k)^{-1}. \end{aligned} \quad (105)$$

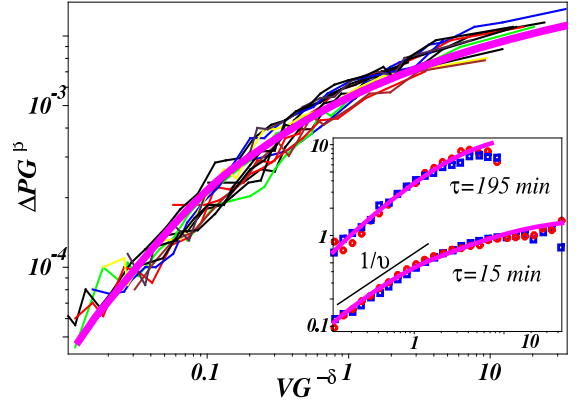


FIG. 17: The price shift ΔP per trade vs. transaction size V , for buy orders in 1996, renormalized by powers of market capitalization G .⁹¹ Theoretical prediction, Eq. (105), is shown by solid line. Results for 1995, 1997 and 1998 are very similar. In Insert: normalized $\Delta_\tau P$ for different τ as function of V , $V > 0$ (\circ) and $V < 0$ (\square), data from Ref.⁹². Solid lines show fitting by Eq. (106).

In general, 1 under logarithm is out of accuracy of our calculations, and we hold it to reproduce expected linear response $\Delta P \sim \Delta V$ at extremely small ΔV . The result $G_0(\tau_k) \simeq \sigma_k = \sigma(\tau_k)$ extremely well supported by data⁹⁰. Weak logarithmic dependence of average price shift ΔP on the trade volume ΔV is related to multi-time character of volume fluctuations, described by Eq. (92).

The dispersion, $\sigma_k \sim G^{-\beta}$, is inversely correlated to the capitalization G of the market, see Eq. (13). The exponent $\beta \simeq 0.3$ (Eq. 88 at $\tau = \tau_k$) is smaller than the Gaussian value $1/2$, because of hierarchical structure of financial markets, see Ref.⁹³. It is shown in Fig. 17 that price impact curves for 1000 stocks are collapsed very well by Eq. (105) with $V_k \sim G^\delta$ and $\delta \simeq 0.3 - 0.4$.

The average price shift during the time interval $\tau > \tau_k$ can be estimated considering several trades as one large trade of summary volume ΔV , and renormalizing minimal relaxation time $\tau_k \rightarrow \tau$ in Eq. (105):

$$\Delta_\tau P \simeq \sigma \text{sign}(\Delta V) \ln(1 + |\Delta V|/V_\tau), \quad (106)$$

with $V_\tau^2 = V_k^2 \tau / \tau_k$. We show in Insert in Fig. 17, that the above dependence $\Delta_\tau P$ well agrees with empirical data at different τ . Eq. (106) can be approximated by power dependence $\Delta P \simeq h \Delta V^{1/v}$, with time, volume and capitalization dependent effective exponent (see Insert in Fig. 17):

$$v \equiv \frac{d \ln V}{d \ln \Delta_\tau P} = \left(1 + \frac{V_\tau}{|\Delta V|} \right) \ln \left(1 + \frac{|\Delta V|}{V_\tau} \right).$$

At small ΔV the apparent exponent v is large at small τ , and $v \simeq 1$ at large τ ($v \simeq 3$ for $\tau = 5$ min and $v \simeq 1$ for $\tau = 195$ min, see Ref.⁹² and Insert in Fig. 17). At large ΔV typical value $v \simeq 2$, see Ref.⁹⁴, while $1/v$ slowly decreases⁹⁵ with ΔV , and⁹⁶ $1/v \rightarrow 0$ for very large ΔV .

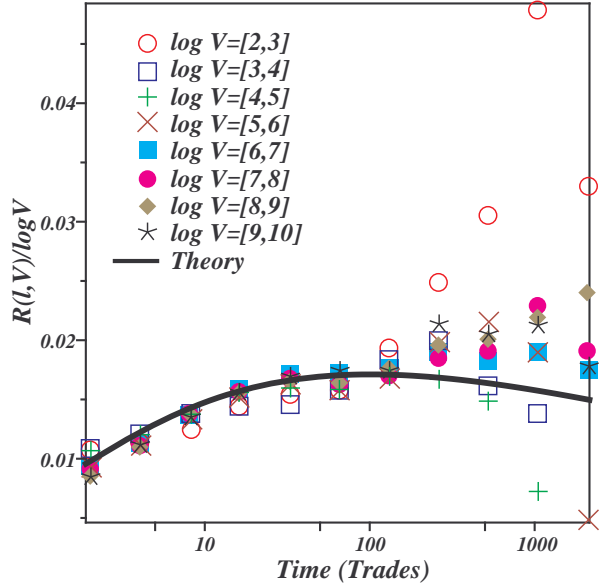


FIG. 18: Response function $R(l, V)$, conditioned to a certain volume V , as a function of dimensionless time l . Data for France-Telecom⁹⁰. The thick line is the prediction of Eq. (107).

In the end of this section we estimate introduced in Ref.⁹⁰ response function conditioned to a volume V :

$$\begin{aligned} \mathcal{R}(l, V) &\simeq \overline{\sum_{n < l} \Delta P[(l-n)\tau_k] \eta(0)} \Big|_{\Delta V=V} \\ &\simeq \int_0^l |\Delta P[(l-n)\tau_k]| \overline{\eta(n\tau_k) \eta(0)} dn. \end{aligned}$$

Here $|\Delta P[(l-n)\tau_k]|$ is the value of average price shift at time $l\tau_k$ because of a trade at time $n\tau_k$. It is important, that $\ln|V|$ at time $l\tau_k$ for given value of $\ln|V|$ at time $n\tau_k$ logarithmically weakly depends on time interval $(l-n)\tau_k$ because of multi-time relaxation of this value (see Eq. 112 below as an example of calculation of such conditional average). Substituting Eqs. (105) and (96) we find with logarithmic accuracy

$$\begin{aligned} \mathcal{R}(l, V) &\simeq \mathcal{R}(l) \ln|V/V_k|, \\ \mathcal{R}(l) &\sim \int_0^l \frac{1}{l-n+1} \frac{dn}{n^\gamma} \simeq \frac{\ln(1+l)}{l^\gamma}, \end{aligned} \quad (107)$$

Plotted in Fig. 18 function $\mathcal{R}(l)$ is in good agreement with empirical data for France-Telecom⁹⁰ with the sole parameter $\gamma = 1/5$. In general, $\mathcal{R}(l)$ initially grows, reaching maximum at certain $l^* \simeq e^{1/\gamma}$, and than decreasing back with dimensionless time $l > l^*$.

Notice, that Eqs. (105), (106) and (107) were obtained by pre-averaging over fluctuations of the liquidity of market, and can not be applied to find higher moments of price increments, like dispersion $\sigma(\tau)$. Multifractality changes power dependences of these values: in contrast to the first moment, Eq. (107), when the intermittency

effect is not important, it gives a major contribution to higher moments. We show in section IV C 6, that the liquidity fluctuations lead to strong variations of the virtual trading time, the rate of which is determined by local time between trades. Therefore correlation function $\mathcal{R}(l, V)$ with pre-averaged time intervals τ_k carries no information about dispersion $\sigma(\tau)$. The relation between $\mathcal{R}(l, V)$ and $\sigma^2(\tau) \simeq D\tau$ was used in Ref.⁹⁰ to demonstrate a very delicate balance between liquidity takers and liquidity providers to put the market at the border between sub- and super-diffusive behavior. In section IV B 1 we show, that apparent diffusive behavior $\sigma^2(\tau) \simeq D\tau$ is really a result of random trader activity at all time scales.

5. Stock and news jumps

In this section we study volatility patterns of large price jumps in the market. The volatility variable $\omega(t)$ (90) can be measured empirically as the average over $n \gg 1$ time intervals τ of the logarithmic modulus of price increments:

$$\omega(t) = \frac{1}{n} \sum_{k=1}^n \ln |\Delta P_k(t)|, \quad \Delta P_k(t) = \Delta_\tau P(t - k\tau).$$

In the limit $n \rightarrow \infty$ $\omega(t)$ can be considered as asymptotically Gaussian random variable. To prove this it is instructive to define generalized volatility

$$V_q(t) = \frac{1}{n} \sum_{k=1}^n |\Delta P_k(t)|^q, \quad (108)$$

which turns to standard definition of volatility at $q = 1$, while in the limit $q \rightarrow 0$ we have

$$\omega(t) = \left. \frac{dV_q(t)}{dq} \right|_{q=0}. \quad (109)$$

In Appendix E we show, that at large $n \gg 1$ PDF of volatility converges to universal function, which depends only on q , exponent $\mu = 3$ of the fat tail of PDF, $\mathcal{P}(\Delta P) \sim |\Delta P|^{-1-\mu}$, and non-universal constant $c > 0$ (will be calculated later):

$$P(V_q) = \frac{x^{-1-\mu/q}}{c\Gamma(c\mu/q)V_m} e^{-x^{-1/c}}, \quad x = \frac{V_q}{V_m}. \quad (110)$$

The maximum of this distribution is reached at the point $V_{\max} = [c(1 + \mu/q)]^{-c} V_m$.

In Fig. 19 we show that expression (E2) of Appendix E for $q = 1$ with $\mu = 3$ and $c_1 = 2/3$ is in excellent agreement with known empirical data. Usually this dependence is fitted by log-normal, Eq. (E3) of Appendix E, or inverse gamma distributions⁹⁷ (Eq. (110) with $c = 1$) with extremely high exponent $\mu = 5 - 7$. Our calculations show, that the distribution $P(V_q)$ has the same tail exponent $\mu = 3$ as PDF of price increments. This result clearly demonstrates the absence of the self-averaging of

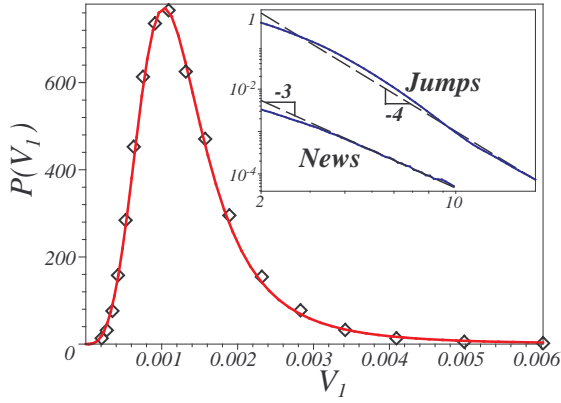


FIG. 19: Comparison of theoretical (with $\mu = 3$ and $c = 2/3$) and empirical⁸⁷ volatility distributions for $\tau = 30$ min, $n\tau = 120$ min. In insert we show the tail distribution of volatility of jumps and news⁷⁷.

price fluctuations: large variations of the coarse-grained “volatility” variable $V_q(t)$ (108) are induced by large short time jumps, the contribution of which is dominated even after averaging over $n\tau \rightarrow \infty$ time interval.

In Insert in Fig. 19 we show the probability of large volatility fluctuations⁷⁷. As one can see, the probability of large “stock jumps” has power tail, $P(V) \sim V^{-1-\mu}$ with $\mu = 3$, while $\mu = 2$ for “news jumps”, induced by independent macroeconomic events. We show in section II C 4 that μ equals to the number of essential degrees of freedom of the noise: complex uncorrelated noise of “news jumps” has $\mu = 2$ components, while there is additional component of price at previous time interval for Markovian noise of stock jumps. The prediction $\mu = 3$ is strongly supported by the analysis of distinct databases with extremely large number of records⁹⁸ for the interval τ from a minute through several months.

Both predictions for the tail exponent of PDF, $\mu = 2$ and $\mu = 3$, are quite general. For example, they describe two major universal classes of city grow (discovered from empirical data in Ref.⁹⁹), because of adding new street lines. The PDF describes the distribution of lengths of these lines. New lines are created randomly for cities with $\mu = 2$, while there are strong local correlations in line creation for cities, characterized by the exponent $\mu = 3$. Similar Gutenberg-Richter power law describes earthquakes of a given strength.

Since $\mu/q \rightarrow \infty$ at $q \rightarrow 0$, the distribution of V_q (110) in this limit becomes asymptotically Gaussian, and random variable $\omega(t)$ at large $n \gg 1$ has Gaussian statistics with the probability

$$P\{\omega\} \sim e^{-H[\omega]}, \quad (111)$$

$$H\{\omega\} = \frac{1}{2} \iint dt dt' \omega(t) \omega(t') G^{-1}(t-t'),$$

where G^{-1} is an inverse to the kernel G , Eq. (100) (explicit expression for G^{-1} is given in section IV D). We

checked Gaussian character of $\omega(t)$ by numerical simulations in the model of Eq. (78). Eqs. (90) and (100) lay in the basis of the famous Multifractal Random Walk model^{78,79}.

Minimizing $H\{\omega\}$ (111) under the condition of fixed $\omega(t_0)$, we find deterministic component of $\omega(t)$ at $t-t_0 > \tau$:

$$\omega(t) = \Lambda G(t) = \omega(t_0) h(t-t_0), \quad (112)$$

$$h(t) \equiv \epsilon \ln(\tau_0/|t|), \quad \epsilon = 1/\ln(\tau_0/\tau), \quad (113)$$

where we expressed the Lagrange multiplier $\Lambda = \epsilon \omega(t_0)/(2\lambda^2)$ through $\omega(t_0)$.

Eq. (112) describes the result of trading activity, while news coming at time $t = 0$ induce additional volatility, see Eq. (103):

$$\omega_n(t) = \omega_0 \tau / t, \quad t > \tau, \quad (114)$$

ω_0 is the amplitude of the news jump. Combining both contributions, Eq. (112) and (114), we find differential equation for the most probable $\omega(t)$ after a news jump $d\omega = -\epsilon \omega d \ln(t/\tau) + d\omega_n$, with the solution

$$\omega(t) = \frac{\omega_0}{1-\epsilon} \left[\frac{\tau}{t} - \epsilon \left(\frac{\tau}{t} \right)^\epsilon \right], \quad t > \tau. \quad (115)$$

The resulting volatility pattern, $a(t) = a_0 e^{\omega(t)}$, can be measured by averaging over all significant news.

Last term in Eq. (115) appears because of long memory effect, known as “aging” effect in spin glasses. It can also be interpreted⁷⁷ as the reduction of measure of uncertainty after news, since some previously unknown information becomes available. In Fig. 20 we show the prediction of Eq. (115) in comparison with empirical data⁷⁷. Initial increase of $\omega(t)$ at $t < 0$ is related to finite waiting time τ_w , see Eq. (115).

Although stock jumps have Markovian statistics, correlations quickly decay at the coarse graining time τ , which is slightly longer than the time of returns, and at larger times these events can be described by uncorrelated random variable $\eta(t)$. The volatility, Eq. (100), can be rewritten through $\eta(t)$ as¹⁰⁰

$$\Delta\omega(t) = \int_{-\infty}^t \frac{dt' \eta(t')}{\sqrt{t-t'}}, \quad \overline{\eta(t)\eta(t')} = \lambda^2 \delta(t-t'). \quad (116)$$

Substituting $\eta(t) = \omega_0 \delta(t)$ in Eq. (116), we find, that volatility after stock jump at $t = 0$ is relaxing as $\omega(t) = \omega_0 t^{-1/2}$, in good agreement with empirical observations⁷⁷. Stock jumps can be interpreted as stochastic resonance of different fluctuation modes p in Eq. (90) because of their random concurrence. Such resonance is usually happened because of delaying the jump until it will be anchored by jumps of larger time scales. The amplitude of the resulting stock jump is significantly increased, and slowly decays with time. The model predicts diffusion dependence $\omega_0^2 \sim \lambda^2 t_w$ of the amplitude of a jump on the waiting time t_w between neighboring stock jumps.

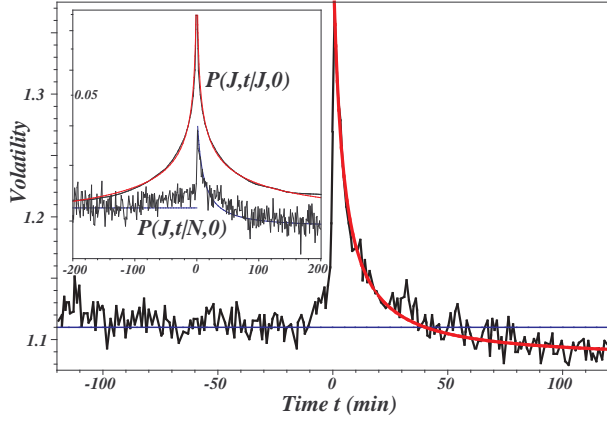


FIG. 20: The volatility pattern before and after news jump, given by Eq. (115) with $\tau = 3, \tau_0 = 10$ years, in comparison with empirical data⁷⁷. The amplitude ω_0 is chosen from the condition of best fitting. In Insert we show, that the probabilities of stock jump after jump, $P(J, t|J, 0)$, decays to jump probability as $|t|^{-1/2}$, as predicted by our theory. We also show the probability to observe a jump after news, $P(J, t|N, 0)$, Eq. (118), which is increased at small times and decreased at intermediate times.

The central part of PDF of the volatility fluctuation $V_1(t_0) = a_0 e^{\omega(t_0)}$, averaged over all fluctuations, can be found by substituting Eq. (112) into (111):

$$P(V_1) \sim \exp \left[-G(0) \frac{\Lambda^2}{2} \right] \simeq \exp \left[-\frac{\epsilon}{4\lambda^2} \left(\ln \frac{V_1}{a_0} \right)^2 \right]. \quad (117)$$

The log-normal of this distribution is supported by numerous empirical data⁹⁷. Comparing this expression with Eq. (E3) of Appendix E, we find the value of constant $c \simeq \lambda^2 (\mu + 1) / \epsilon$ in Eq. (110) for $q = 1$.

The volatility pattern of a news jump at time $t = 0$ followed by a stock jump at time t can be presented as the sum of corresponding volatility patterns, Eq. (115) and (112). Substituting it into Eq. (111) we find the probability of this pattern:

$$P(J, t|N, 0) = P_n(\omega_0) P(V_1) \exp \left[-\frac{\epsilon\omega(t)}{2\lambda^2} \ln \frac{V_1}{a_0} \right]. \quad (118)$$

Here P_n and P are probabilities of news and stock jumps, and function $\omega(t)$ is calculated in Eq. (115). We conclude, that at small time interval t there is asymmetric increase of the probability to see a jump induced by a news, following by the decrease of this probability at intermediate times. Similar expression (118) with $\omega(t) = \omega_0 t^{-1/2}$ can be found for the probability $P(J, t|J, 0)$ to find a jump at time t after a jump at time $t = 0$. These predictions are in good agreement with empirical data, see Insert in Fig. 20.

6. Virtual trading time

In this section we show, that fluctuations of the liquidity of the market lead to corresponding variations of the virtual trading time, $\Theta(t)$, which is proportional to the number of trades per given time interval. Logarithmic volatility $\omega(t)$, Eq. (112), gives the deterministic part of time dependence of the amplitude at $t - t_0 > \tau$:

$$a(t) \simeq a_0 e^{\omega(t)} \simeq a(t_0) [(t - t_0) / \tau]^\alpha, \quad (119)$$

with the “feedback parameter”

$$\alpha = -\epsilon\omega(t_0). \quad (120)$$

In Multifractal models¹⁰¹ the (logarithmic) price is assumed to follow

$$P(t) = B[\Theta(t)], \quad (121)$$

where $B(t)$ is Brownian motion and $\Theta(t)$ is the random trading time, which is an increasing function of t . Differentiating Eq. (121) over t , we can represent the increment of price in the form $\Delta_\tau P(t) = a(t) \dot{\xi}(t)$ with $a(t) \sim \Theta'(t)$. Substituting “classical trajectory” $a(t)$ from Eq. (119), we find

$$\Theta(t) - \Theta(t_0) \sim (t - t_0)^{1+\alpha}, \quad (122)$$

and the mean square increment of the price is

$$\overline{\langle [P(t) - P(t_0)]^2 \rangle} \sim \Theta(t) - \Theta(t_0) \sim (t - t_0)^{2H} \quad (123)$$

with the local Hurst exponent

$$H = (1 + \alpha) / 2. \quad (124)$$

Expression (123) is valid for any t_0 with current $H(t_0)$.

The price $P(t)$ experiences different types of fractional Brownian motion in time intervals Δt_i with different feedback index α , which randomly change each other, see Fig. 21. The case $H \simeq 1/2$ ($\alpha \simeq 0$) describes usual Brownian motion. A Hurst exponent value $0 < H < 1/2$ ($\alpha < 0$) will exist for a time series with sub-diffusive (anti-persistent) behavior. A Hurst exponent value from the interval $1/2 < H < 1$ ($\alpha > 0$) indicates super-diffusive (persistent) behavior. $H \neq 1/2$ can be interpreted as the result of local unbalance between the competing liquidity providers and liquidity takers⁹⁰.

The fractal dimension of the fractional Brownian motion is $D_{ext} = 2 - H$. At large H the motion becomes more regular ($D_{ext} \rightarrow 1$), with large up- and downturns, while at small H it quickly fluctuates, trying to covers the whole plane ($D_{ext} \rightarrow 2$). Therefore, establishing of a super- and sub-diffusive behavior leads to significant suppression/creation of short-time fluctuations, see Fig. 21. This effect was really observed¹⁰², and may be used as an indicator of establishment of large volatility at long time scales, which is hard to detect at short time intervals.

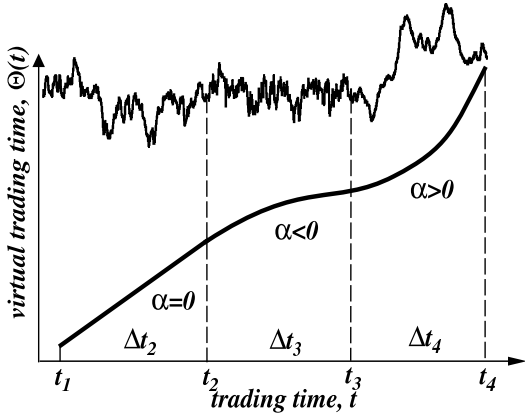


FIG. 21: Price fluctuations and corresponding virtual trading time $\Theta(t)$ as functions of real time t for Brownian motion ($\alpha = 0$), sub- ($\alpha < 0$) and super-diffusive behavior ($\alpha > 0$).

7. Brownian motion, sub- and super-diffusion

Switching of fluctuation regimes between Brownian motion ($\alpha \simeq 0$), sub- ($\alpha < 0$) and super-diffusion ($\alpha > 0$) is happened randomly at “frustration times” (with equal probability of different choices) with the probability (see Eq. (117))

$$p(\alpha) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{\alpha^2}{2\sigma_0^2}\right), \quad \sigma_0^2 = 2\epsilon\lambda^2. \quad (125)$$

Because of multifractality, such change of fluctuation regimes is happened at all time scales τ .

We also define a multivariate PDF

$$p(\alpha_0, \dots, \alpha_k) \equiv \overline{\prod_{l=0}^k \delta[\alpha_l + \epsilon\omega(t_l)]}, \quad (126)$$

which determines information entropy conditional to the set of indexes $\alpha_0, \dots, \alpha_k$:

$$S(\alpha_0, \dots, \alpha_k) \simeq \ln p(\alpha_0, \dots, \alpha_k). \quad (127)$$

The entropy, Eqs. (127) and (125), is maximal for Brownian motion, $\alpha = 0$, sub- and super-diffusion lower the entropy production, since the market behavior is more predictable in these regimes.

Conditional dynamics of mode switching can be described by the probability $p(\alpha_1|\alpha_0) = p(\alpha_0, \alpha_1)/p(\alpha_0)$ to find given value of the index α_1 at time $t_1 = t_0 + \Delta t_1$ under the condition that it was α_0 at previous time t_0 . Calculating the average (126) at $k = 1$ with Gaussian distribution function, Eq. (111), we find:

$$p(\alpha_1|\alpha_0) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(\alpha_1 - \bar{\alpha}_1)^2}{2\sigma_1^2}\right], \quad (128)$$

$$\bar{\alpha}_1 = \alpha_0 h(\Delta t_1), \quad \sigma_1^2 = \sigma_0^2 [1 - h^2(\Delta t_1)] \quad (129)$$

Conditional average $\bar{\alpha}_1$ decreases with time Δt_1 , while the conditional dispersion σ_1 grows with this time. The

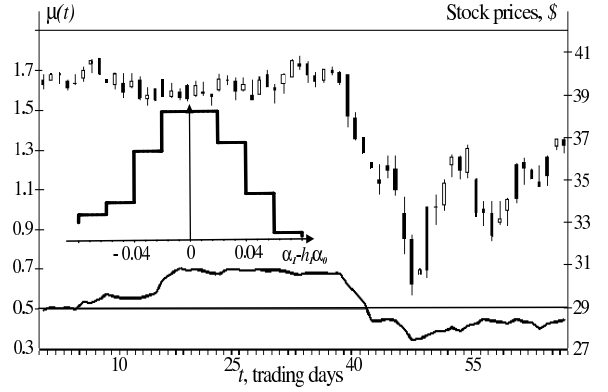


FIG. 22: Daily series of Exxon Mobil Corporation and corresponding $\mu(t)$ ¹⁰². Probability distribution of $\alpha_1 - h_1\alpha_0$ with $h_1 = 0.9$ at $\Delta t_1 = 16$ days is shown in Insert.

transition to a new state is happened in average when two these amplitudes become of the same order:

$$\Delta t_1 \simeq \tau_0 \left(\frac{\tau}{\tau_0}\right)^{1/\sqrt{1+z}}, \quad z = \frac{\alpha_0^2}{4\epsilon^2\lambda^2}. \quad (130)$$

It is surprising, that the length of the time interval Δt_1 grows with the rise of $|\alpha_0|$ (although the probability of large initial $|\alpha_0|$, Eq. (125), is small). Such counter-intuitive behavior is related to the absence of any “restoring force” to $\alpha = 0$.

The characteristic time of sub- and super-diffusion behavior can be roughly estimated substituting in the above expression the most probable value $\alpha_0^2 \simeq \sigma_0^2$ from Eq. (125), giving $z \simeq \frac{1}{2} \ln(\tau_0/\tau)$. For $\tau = 1$ day, $\tau_0 \simeq 10^3$ days and $\lambda^2 \simeq 0.1$ this gives Δt_1 about a month, in agreement with empirical observations, see Fig 22 and Ref.¹⁰². The effect of sub- and super-diffusion switching can be hardly detectable at small time intervals τ , because of small α (see Eq. (120)), but it is well pronounced for daily time intervals τ .

The feedback index α is related to “variation index” $\mu = (1 - \alpha)/2$, introduced in Ref.¹⁰² from the fractal analysis of empirical data (local extension of the R/S analysis¹⁰³ of the Hurst exponent H). In Insert in Fig. 22 we plot estimated probability distribution of $\alpha_1 - h_1\alpha_0$ for $\Delta t_1 = 16$ days, which is proportional to the conditional probability (128). The value h_1 is chosen to get a maximum at the beginning, and it decreases with the rise of the time interval Δt_1 from 0.99 at $\Delta t_1 = 1$ to 0.9 at $\Delta t_1 = 16$ days. The dispersion of this distribution $\sigma_0 \simeq 0.2$, and from Eq. (125) we get reasonable estimation $\lambda^2 \simeq 0.08$. From second Eq. (129) we find the dispersion $\sigma_1 = \sigma_0\sqrt{1 - h_1^2} \simeq 0.07$, close to observed value.

We can also calculate the probability to find the feedback index α_k at time t_k under the condition that it was

α_{k-1} at time t_{k-1} , α_{k-2} at time t_{k-2} , and so on:

$$\begin{aligned} & p(\alpha_k | \alpha_0, \dots, \alpha_{k-1}) \\ & \equiv p(\alpha_0, \dots, \alpha_k) / p(\alpha_0, \dots, \alpha_{k-1}) \quad (131) \\ & = \frac{1}{\sqrt{2\pi}\sigma_k} \exp \left[-\frac{(\alpha_k - \bar{\alpha}_k)^2}{2\sigma_k^2} \right], \end{aligned}$$

The logarithm of this probability determines the entropy lowering because of the knowledge about previous events $\alpha_0, \dots, \alpha_k$. The conditional average $\bar{\alpha}_k$ corresponds to the maximum of entropy production, Eq. (127). It has the meaning of average response of α on previous values $\alpha_0, \dots, \alpha_{k-1}$:

$$\bar{\alpha}_k = \sum_{i=0}^{k-1} K_{ki} \alpha_i, \quad K_{ki} = -a_{ki}/a_{kk}, \quad (132)$$

a_{kj} are adjoints of the matrix \mathbf{h} with elements $h(t_i - t_j)$. Conditional dispersion

$$\sigma_k^2 = \sigma_0^2 \det \mathbf{h} / a_{kk} \quad (133)$$

estimates the accuracy of the prediction (132). Probabilities (131) depend not only on index α_{k-1} at previous time t_{k-1} , but also on all $\alpha_0, \dots, \alpha_{k-1}$ – the random process is not Markovian. As the consequence, the probability to have the same value of all three indexes $\alpha_2 = \alpha_1 = \alpha_0$ (continuation of a sub- and super-diffusive regimes) grows with increase of the initial time interval Δt_1 .

8. Fluctuation corrections

The ‘‘classical trajectory’’ (119) predicts that the feedback index α (120) can be changed only because of fluctuations. In this section we demonstrate, that fluctuations lead to an additive shift of α in the super-diffusion direction. Calculating the average of $V_q(t)$ with Gaussian weight (111) under the condition $\omega(0) = \omega_0$, we find:

$$\begin{aligned} \langle V_q(t) \rangle_{\omega_0} & \simeq \langle a^q(t) \rangle_{\omega_0} \simeq \langle V_q(0) \rangle_{\omega_0} (t/\tau)^{q\alpha + q^2\beta(t)}, \\ \beta(t) & = \lambda^2 [1 + h(t)] / 2, \quad (134) \end{aligned}$$

where $h(t)$ is defined in Eq. (113). The β -term describes deviation from fractional Brownian motion because of multifractal behavior. At $q = 2$ this expression can be interpreted as the response on ‘‘endogenous event’’ $\omega(0)$ ¹⁰⁰, while at $q = 1$ it gives fluctuation correction to the feedback index α .

Empirical study of year correlations shows¹⁰⁴, that financial market is really ‘‘locked’’ in sub- and super-diffusive or Brownian motion states at extremely long periods (conventions can persist up to 30 years). The change in convention can be rather smooth, like during the second part of the century, or occur suddenly, triggered by an extreme event, like it did after 1929. From the data, it was also observed a systematic bias towards the persistent following convention.

D. Universality of fluctuations

Non-universal properties of market at trading times $\lesssim \tau_k$ can only be described by models of agent-based strategies. In this section we consider only universal properties of price fluctuations at time scales $\tau > \tau_k \simeq 1$ min, see Ref.¹⁰⁵. At qualitative level the presence of universality is known for a long time as ‘‘stylized facts’’⁴⁰. We show, that our approach captures such stylized facts and gives explanation for many others empirical observations.

The universality is related to self-similarity of price fluctuations at different time scales^{79,106,107}: the change of time interval τ corresponds to the change of characteristic scale along the hierarchical tree in Fig. 14. We demonstrate, that resulting time series have complex non-periodic behavior with chaotic changes of usual Brownian motion, sub- and super-diffusion, reflecting cyclic dynamics of the market. We show, that the impact function of the market logarithmically depends on volume imbalance.

Fluctuations on financial market have unexpected physical interpretation, reflecting the unified nature of physics. The effective Hamiltonian (111) can be rewritten as

$$H\{\omega\} = \frac{\eta}{2\pi} \iint \left(\frac{\omega(t) - \omega(t')}{t - t'} \right)^2 dt dt'.$$

This expression describes diffusion of quantum Brownian particle with the coordinate $\omega(t)$ and the coefficient of linear friction $\eta = 1/(2\pi\lambda^2)$.

A microscopic model of quantum diffusion is based on coupling to a thermostat – the reservoir of harmonic oscillators¹⁰⁸, presenting the ‘‘army’’ of traders in the case of the market. The resulting dynamics is intrinsically non-Markovian in that the evolution depends on history rather than just on present state¹⁰⁹. Brownian particle can respond to a very wide range of reservoir frequencies, and this is the origin of time-irreversible behavior and slow relaxation after fluctuation cast, see Eq. (112). The production of information entropy (see Eq. (127) and Appendix A) is related to environment-induced decoherence of the quantum particle¹¹⁰, and it is at the peak of many recent studies.

V. CONCLUSION

Many concepts of equilibrium macroeconomic (resources, unemployed, different firm dynamics at small- and long- time horizons, taxes and so on) naturally enter into proposed coalescent theory, which integrates both physical and economic concepts of essentially non-equilibrium market in one unique approach. We also developed new approach to study fluctuations on the market, well describing empirical data of both firm grow rates and price increments on financial markets. We propose the set of Langeven equations, describing multi-time dynamics of price and volume fluctuations at different time

scales on the market. Using these equations, we derived analytically equations of multifractal random walk model.

In the end we discuss physical meaning of our theory:

a) What is physics of extreme events on the market (problem of fat tails)?

There are two sources of fluctuations: macroeconomic events – news, and traders activity because of uncertainty of equilibrium prices at different time scales, which generate two types of large price jumps: News jumps are created by the inflow of news, while stock jumps are generated during random concurrence of different fluctuation modes. We show, that the decay of volatility observed after news jumps is related to the effect, similar to “aging” effect in spin glasses.

Fluctuations on the market are characterized by the normalized noise ξ and its amplitude \mathbf{a} (volatility). New key idea of our approach is that ξ and \mathbf{a} are independent complex random variables, separated on time scale: the noise is generated by hot degrees of freedom on times small with respect to observation time interval τ , while evolution of the amplitude is determined by cold degrees of freedom on times large with respect to τ .

In stochastic volatility and multifractal models jumps are predicted as the result of volatility fluctuations, and characterized by large non-universal tail exponent $\mu \gg 1$, while the noise is assumed to be Gaussian uncorrelated random variable. In fact, the noise is strongly correlated, and can experience large non Gaussian jumps. We calculate the contribution of such jumps to PDF and show, that the distribution of stock jumps is characterized by the tail exponent $\mu = 3$, while the distribution of news jumps has tail exponent $\mu = 2$. The exponent μ remains stable with the rise of τ (recall, that Levy distribution with $\mu > 2$ is unstable).

b) Why market dynamics is so complex: can it be described by simple Markovian or Gaussian processes?

We show, that local equilibriums on the market are self-organized in the hierarchical tree, according to their relaxation times. The amplitude \mathbf{a} of the noise at given time scale is determined by cumulative signal from all “parent” time scales, and its dynamics is complex multifractal process. But the information about the amplitude can be “erased” from time series considering only signs of price fluctuations. The resulting Markovian process describes propagation of positive and negative signals, and

determines conditional double dynamics of the market.

Typical noise ξ and amplitude \mathbf{a} are determined by signals from large number of, respectively, short and long (with respect to τ) time scales, and they have asymptotically Gaussian statistics. We propose and solve Double Gaussian model of market fluctuations, and show good agreement with empirical data for different groups of stocks.

c) What physics stands behind “random trading time” in Multifractal models¹⁰¹?

At large time intervals the price randomly cycles between Brownian motion, sub- and super-diffusive regimes, which change each other because of liquidity fluctuations. The virtual trading time is proportional to the real time for Brownian motion and experiences time shifts in sub- and super-diffusive regimes. The theory predicts systematic bias to persistent behavior, observed for many markets and exchange rates.

d) And finally, can price behavior be described by universal physical laws or it is dictated only by the zoo of microstructures of markets (see Refs.^{111,112})?

The universality of price fluctuations on financial markets was demonstrated at time scales from a minute to tenths years in many studies, see for example, Refs.^{28,36,37}. We show, that it is related to the self-similarity of the underlying hierarchical tree of amplitudes, see Fig. 14 (we do not give here lists of all stocks, used for comparison with our theory, since they are shown in corresponding references). In contrast, statistics of trades and volumes is not universal, and strongly depends on details of market microstructure.

Our theory can also be used to study other time series, such as variations of cloudiness, temperature, earthquake frequencies, rate of traffic flow and so on. It looks attractive to apply analytical approach of this paper for the description of social processes, which are driven by frustrations at turning points of the mankind history. Events between these points support the social activity, but do not change the state of the society.

Acknowledgement *I would like to thank Andrey Leonidov for attracting my attention to the problem of market fluctuations and helpful notes, M. Dubovikov for discussion of some results, and J.-P. Bouchaud for critical comments.*

¹ R. Gibrat, Lesin’ egalit’es’ economiques, Sirey, Paris (1931).

² R. Perline, *Phys. Rev. E* **54**, 220 (1996).

³ M. Levy, S. Solomon, *Int. J. Mod. Phys. C* **7**, 595 (1996); D. Sornette, R. Cont, *J. Phys. I* (France) **7**, 431 (1997).

⁴ L.A.N. Amaral, S.V. Buldyrev, S. Havlin, M.A. Salinger, and H.E. Stanley, *Phys. Rev. Lett.* **80**, 1385 (1998).

⁵ Y. Lee, L.A.N. Amaral, D. Canning, M. Meyer, and H.E. Stanley, *Phys. Rev. Lett.* **81**, 3275 (1998).

⁶ G. Sutton, *Physica A*, **312**, 577 (2002).

⁷ P.A. Samuelson, *Economics*. Mc-Graw Hill Int., Auckland (1980).

⁸ B.K. Chakrabarti, S. Marjit, *Indian J. Phys. B* **69**, 681 (1995); S. Ispolatov, P.L. Krapivsky, S. Redner, *Eur. Phys. J. B* **2**, 267 (1998).

⁹ A.A. Drăgulescu, V.M. Yakovenko, *Eur. Phys. J.*, **B17**, 723 (2000).

¹⁰ A. Chatterjee, B.K. Chakrabarti, S.S. Manna, *Physica A*,

- 335**, 155 (2004).
- ¹¹ F. Slanina, *Phys. Rev. E*, **69**, 046102 (2004).
- ¹² J.J. Ramsden and Gy. Kiss-Hayp'al, *Physica A*, **277**, 220 (2000).
- ¹³ A. Drăgulescu, V.M. Yakovenko, *Physica A* **299**, 213 (2001); V.M. Yakovenko, [arXiv:physics/0709.3662v3](https://arxiv.org/abs/physics/0709.3662v3) [physics.soc-ph].
- ¹⁴ L.A.N. Amaral, S.V. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, M. A. Salinger, H. E. Stanley, and M. H. R. Stanley, *J. Phys. I France*, **7**, 621 (1997); S. V. Buldyrev, L.A.N. Amaral, S. Havlin, H. Leschhorn, P. Maass, M.A. Salinger, H.E. Stanley, and M.H.R. Stanley, *J. Phys. I France*, **7**, 635 (1997).
- ¹⁵ M. Friedman, *Amer. Econ. Rev.*, **58**, 8 (1968).
- ¹⁶ R. Axtell, *Science*, **293**, 1818 (2001).
- ¹⁷ X. Gabaix, P. Gopikrishnan, V. Plerou and H.E. Stanley, *Nature*, **423**, 267 (2003).
- ¹⁸ D. Pushkin and A. Hassan, *Physica A*, **336**, 571 (2004).
- ¹⁹ Soo, K.T., Zipf's law for cities: a cross country investigation. London School of Economics, preprint (2002).
- ²⁰ M.H.R. Stanley, L.A.N. Amaral, S. Buldyrev, S. Havlin, H. Leschhorn, P. Maass, M.A. Salinger, H. E. Stanley, *Nature*, **319**, 804 (1996).
- ²¹ J. Laberrère, D. Sornette, *Eur. Phys. J. B* **2**, 525 (1998).
- ²² J.A. Davies, *Eur. Phys. J. B* **27**, 445 (2002).
- ²³ M.E.J. Newman, *Phys. Rev. E* **64**, 016131 (2001).
- ²⁴ T. Knudsen, *Amer. J. of Economics and Sociology* **60**, 123, (2001).
- ²⁵ C.Di Guilmi, E. Gaffeo, and M. Gallegati, *Econom. Bull.*, **15**, No. 6, 1 (2003).
- ²⁶ A.C. Silva and V.M. Yakovenko, *Europhys. Lett.*, **69**, 304 (2005).
- ²⁷ R. Perline, R. Axtell and D. Teitelbaum, *Small Business Research Summary*, N285 (2006).
- ²⁸ H.E. Stanley, L.A.N. Amaral, P. Gopikrishnan, V. Plerou. and M.A. Salinger, *J. Phys.: Condens. Matter*, **14**, 2121 (2002).
- ²⁹ D. Teitelbaum and R. Axtell, *Small Business Research Summary*, N247 (2005).
- ³⁰ T. Bollerslev, R.F. Engle, D.B. Nelson, ARCH models, Handbook of Econometrics, vol 4, ch. 49, R.F Engle and D.I. McFadden Edts, North-Holland, Amsterdam, 1994.
- ³¹ L. Borland, J.-Ph. Bouchaud, [arXiv:physics/0507073v1](https://arxiv.org/abs/physics/0507073v1) [physics.soc-ph].
- ³² R.N. Mantegna and E Stanley, *Physica A* **239**, 255 (1997).
- ³³ E. Bacry, J. Delour and J.F. Muzy, *Phys. Rev. E*, **64** 26103 (2001).
- ³⁴ B. Mandelbrot, R.L. Hudson, "The (Mis)behavior of Prices: A Fractal View of Risk, Ruin, and Reward". New York: Basic Books; London: Profile Books, 2004.
- ³⁵ T.F. Crack, O. Ledoit, *Journal of Finance*, **51**, 751 (1996).
- ³⁶ A. Leonidov, V. Trainin, S. Zaitsev, A. Zaitsev, [arXiv:physics/0605138](https://arxiv.org/abs/physics/0605138).
- ³⁷ M. Boguñá and J. Masoliver, *Eur. Phys. J. B* **40**, 347 (2004).
- ³⁸ F. Lillo and R.N. Mantegna, *Physica A*, **299**, 161, (2001).
- ³⁹ A. Leonidov, V. Trainin, S. Zaitsev, A. Zaitsev, [arXiv:physics/0601098](https://arxiv.org/abs/physics/0601098).
- ⁴⁰ R. Cont, *Quant. Finance*, **1**, 223 (2001).
- ⁴¹ A.C. Silva, R.E. Prange, V.M. Yakovenko, *Physica A* **344**, 227 (2004).
- ⁴² A. Leonidov, V. Trainin, S. Zaitsev, A. Zaitsev, [arXiv:physics/0603103](https://arxiv.org/abs/physics/0603103).
- ⁴³ A. Leonidov, V. Trainin, A. Zaitsev, [arXiv:physics/0506072](https://arxiv.org/abs/physics/0506072).
- ⁴⁴ J. Perello, J. Masoliver, *Phys. Rev. E*, **67**, 037102 (2003).
- ⁴⁵ A. Leonidov, V. Trainin, S. Zaitsev, A. Zaitsev, [arXiv:physics/0701158](https://arxiv.org/abs/physics/0701158); *Physica A*, **386**, 240 (2007).
- ⁴⁶ M. Bartolozzi and A.W. Thomas, *Physical Review E* **69**, 046112 (2004).
- ⁴⁷ R. Cont, [arXiv:cond-mat/9705075v3](https://arxiv.org/abs/cond-mat/9705075v3).
- ⁴⁸ B. Rosenow, *Int. J. Mod. Phys. C*, **13**, 419 (2002).
- ⁴⁹ P. Weber and B. Rosenow, *Quant. Finance*, **5**, 357 (2005).
- ⁵⁰ See web page: <http://www.unifr.ch/econophysics/minority/>
- ⁵¹ R. Peschanski, [arXiv:hep-ph/0610019v1](https://arxiv.org/abs/hep-ph/0610019v1).
- ⁵² K. Binder and A.P. Young, *Rev. Mod. Phys.* **58**, 801 (1986).
- ⁵³ Y. Balasko and D. Cass, *Econometrica*, **57**, 135 (1989).
- ⁵⁴ B. Biais, P. Hilton, C. Spatt, *Journal of Finance*, **50**, 1655 (1995).
- ⁵⁵ J.-P. Bouchaud, M. Mezard, M. Potters, *Quantitative Finance*, **2**, 251 (2002).
- ⁵⁶ S. Maslov, M. Millis, *Physica A*, **299**, 234 (2001).
- ⁵⁷ R. Mantegna, H. E. Stanley, An Introduction to Econophysics, Cambridge University Press, Cambridge, 1999.
- ⁵⁸ M.M. Dacorogna, R. Gencay, U.A. Muller, R.B. Olsen, O.V. Pictet, An Introduction to High-Frequency Finance, Academic Press, San Diego, 2001.
- ⁵⁹ R. Cont, *Quantitative Finance*, **1**, 223 (2001).
- ⁶⁰ J.-P. Bouchaud and R. Cont, *Eur. Phys. J.*, **B6**, 543 (1998).
- ⁶¹ I. Giardina and J-P. Bouchaud, *Eur. Phys. J. B*, **31**, 421 (2003).
- ⁶² A. Krawiecki, J.A. Holyst, and D. Helbing, *Phys. Rev. Lett.*, **89**, 158701 (2002).
- ⁶³ R.N. Mantegna, H.E. Stanley, *Nature* **376**, 46 (1995).
- ⁶⁴ S.J. Taylor. Modelling Financial Time Series. Chichester, Wiley (1986).
- ⁶⁵ R. Engle, *Econometrica*, **50**, 987 (1982).
- ⁶⁶ P. Carr, H. Geman, D. Madan, M. Yor, *Math. Finance*, **13**, 345 (2003).
- ⁶⁷ L. Calvet and A. Fisher, *J. Financ. Econometrics*, **2**, 49 (2004).
- ⁶⁸ B. Mandelbrot, A. Fisher, L. Calvet, Cowles Foundation Disc. Paper, 1164 (1997).
- ⁶⁹ V.S. Dotsenko, *J. Phys. C* **20**, 5473 (1987); *J. Phys.: Condens. Matter.*, **2**, 2721 (1990).
- ⁷⁰ V.S. Dotsenko, *J. Phys. C* **18**, 6023 (1985).
- ⁷¹ P. Bak, M. Paczuski, and M. Shubik, *Physica A*, **246**, 430 (1997).
- ⁷² M.G. Daniels, J. D. Farmer, G. Iori, E. Smith, *Phys. Rev. Lett.* **90**, 108102 (2003).
- ⁷³ E. Smith, J. D. Farmer, L. Gillemot, S. Krishnamurthy, *Quant. Finance*, **3**, 481, (2003).
- ⁷⁴ E.F. Fama, *J. of Finance* **25**, 383 (1970).
- ⁷⁵ R.J. Schiller, *Am. Econ. Rev.*, **71**, 421 (1981).
- ⁷⁶ D. Cass and K. Shell, *J. Polit. Econ.*, **92**, 193 (1983).
- ⁷⁷ A. Joulin, A. Lefevre, D. Grunberg, J-P. Bouchaud, [arXiv:0803.1769v1](https://arxiv.org/abs/0803.1769v1) [physics.soc-ph].
- ⁷⁸ J.F. Muzy, J. Delour and E. Bacry, *Eur. Phys. J. B* **17**, 537 (2000).
- ⁷⁹ E. Bacry, J. Delour and J.F. Muzy, *Phys. Rev. E* **64**, 026103 (2001).
- ⁸⁰ J.-P. Bouchaud, *Chaos*, **15**, 026104 (2005).
- ⁸¹ P. Bak, How Nature Works: The Science of Self-Organized Criticality, Copernicus Springer, New York,

- (1996).
- ⁸² M. Youssefmir, B.A. Huberman and T. Hogg, *Comp. Econ.* **12**, 97 (1998).
- ⁸³ M. Marsili, *Physica A*, **299**, 93 (2001).
- ⁸⁴ Z. Eisler, Ph.D. thesis, “Fluctuation Phenomena on the Stock Market”, Budapest, (2007).
- ⁸⁵ A. Shleifer, *Inefficient Markets, An Introduction to Behavioral Finance*, Oxford University Press (2000).
- ⁸⁶ R. J. Schiller, *Amer. Econ. Rev.*, **71**, 421 (1981).
- ⁸⁷ Y.I. Liu, P. Gopikrishnan, P. Cizeau, M. Meyer, C-K. Peng, and H.E. Stanley, *Phys. Rev. E*, **60**, 1390 (1999).
- ⁸⁸ V. Plerou, P. Gopikrishnan, L.A.N. Amaral, M. Meyer and H.E. Stanley, *Phys. Rev. E* **60**, 6519 (1999).
- ⁸⁹ F. Schmitt, *Eur. J. Phys. B*, **34**, 85 (2003).
- ⁹⁰ J.-P. Bouchaud, Y. Gefen, M. Potters, M. Wyart, *Quant. Finance*, **4**, 176 (2004).
- ⁹¹ F. Lillo, J.D. Farmer, R.N. Mantegna, *Nature*, **421**, 123 (2003).
- ⁹² V. Plerou, P. Gopikrishnan, X. Gabaix and H.E. Stanley, *Phys. Rev. E* **66**, 027104 (2002).
- ⁹³ R.N. Mantegna, *Europ. Phys. J. B* **11**, 193 (1999).
- ⁹⁴ X. Gabaix, P. Gopikrishnan, V. Plerou and H.E. Stanley, *Physica A* **324**, 1 (2003).
- ⁹⁵ F. Lillo, J.D. Farmer and R.N. Mantegna, *Nature* **421**, 129 (2003).
- ⁹⁶ M. Potters and J.-P. Bouchaud, *Physica A* **324**, 133 (2003).
- ⁹⁷ P. Cizeau, Y. Liu, M. Meyer, Ci-K. Peng, H.E. Stanley, *Physica A*, **245**, 441 (1997).
- ⁹⁸ P. Gopikrishnan, V. Plerou, L.A.N. Amaral, M. Meyer, and H.E. Stanley, *Phys. Rev. E* **60**, 5305 (1999).
- ⁹⁹ R. Carvalho and A. Penn, *Physica A*, **332**, 539 (2003).
- ¹⁰⁰ D. Sornette, Y. Malevergne and J.-F. Muzy, [arXiv:cond-mat/0204626v1](https://arxiv.org/abs/cond-mat/0204626v1).
- ¹⁰¹ B. Mandelbrot, *J. of Business*, **36**, 307 (1963).
- ¹⁰² M.M. Dubovikov, N.V. Starchenko, M.S. Dubovikov, *Physica A*, **339**, 591 (2004).
- ¹⁰³ Jens Feder, *Fractals*. New York, NY: Plenum Press, 1988.
- ¹⁰⁴ M. Wyart and J.-P. Bouchaud, [arXiv:cond-mat/0303584v2](https://arxiv.org/abs/cond-mat/0303584v2).
- ¹⁰⁵ A.P. Nawroth and J. Peinke, *Eur. Phys. J. B*, **50**, 147 (2006).
- ¹⁰⁶ M.M. Dacorogna, R. Gençay, U.A. Müller, R.B. Olsen, O.V. Pictet, *An Introduction to High Frequency Finance* Academic Press, San Diego, (2001).
- ¹⁰⁷ T. Di Matteo, T. Aste, and M. Dacorogna, *Physica A*, **324**, 183 (2003).
- ¹⁰⁸ A.O. Caldeira and A.J. Leggett, *Physica A*, **121**, 587 (1983).
- ¹⁰⁹ B.L. Hu, J.P. Paz and Y. Zhang, *Phys. Rev. D*, **45**, 2843 (1992).
- ¹¹⁰ S.V. Panyukov, A.D. Zaikin, *JETP*, **91**, 1677 (1986).
- ¹¹¹ W. Brock, C. Hommes, *Econometrica* **65**, 1059 (1997); C. Hommes, *Quantit. Finance*, **1**, 149 (2001).
- ¹¹² I. Giardina, J.P. Bouchaud, *Eur. J. of Phys. B* **31**, 421 (2003).
- ¹¹³ E.M. Lifshitz and L.P. Pitaevsky, *Physical Kinetics*, (Pergamon Press, 1981).
- ¹¹⁴ A. Phillips, *Economica*, **25**, (1958).
- ¹¹⁵ S.J. Davis, J. Haltiwanger and S. Schuh, *Job Creation and Destruction*, MIT Press: Cambridge, Mass.

APPENDIX A: ENTROPY FORMULATION

In order to reveal the economic meaning of Master equation (10), it is convenient to rewrite it in the form

$$\frac{dG}{dt} = p_c(G) - p_d(G), \quad (\text{A1})$$

where $p_c(G)$ and $p_d(G)$ are probabilities of job creation and destruction per unit time in the firm of G people. In the absence of any external supply, $U = 0$, the probability of job creation is zero. In the main order in “concentration” U (we use physical term to emphasize the analogy with coalescence) $p_c(G)$ is proportional to U , while the probability of job destruction $p_d(G)$ is determined mainly by internal firm structure, and do not depend of U . Comparing Eqs. (A1) and (10) we find explicit expressions for these probabilities for our model

$$p_c(G) = qUG, \quad p_d(G) = qU_*G + pG^{1-\beta}. \quad (\text{A2})$$

We define the “entropy” $S(G)$ of the firm of size G as logarithm of equilibrium distribution function of firms over their sizes, $f_{eq}(G)$. Since equilibrium values do not depend of a way how the system is assembling, consider the process when the size G is varying by one. In this case $f_{eq}(G)$ is determined by the detailed balance condition, $p_c(G)f_{eq}(G) = p_d(G+1)f_{eq}(G+1)$. The solution of this equation at $G \gg 1$ relates firm entropy with probabilities of job creation and destruction:

$$f_{eq}(G) = e^{S(G)}, \quad S(G) = \int dG \ln \frac{p_c(G)}{p_d(G)}. \quad (\text{A3})$$

As one can naively expect, the entropy of the firm increases with the rise of the probability to get a job and decreases with the rise of the probability to loose it. In the case of overheated market $U < U_*$ the entropy $S(G)$ monotonically grows with the firm size G , while in the “supersaturated” case $U > U_*$ it initially decreases with G , reaching its minimum for firms of critical size, $G = G_c$, Eq. (14).

Calculating this integral (A3) with functions (A2), we find

$$S(G) \simeq \mu G - G \ln \left[U_*/U_0 + (G/e)^{-\beta} \right] + const,$$

where $U_0 = p/g$ and $\mu = \ln(U/U_0)$ has the meaning of chemical potential. The entropy of the whole market

$$S = -U \ln \frac{U}{eU_0} + \int dGS(G) f(G, t) - \mu(Q - U) \quad (\text{A4})$$

is the sum of the entropy of the “ideal gas” of unemployments, the entropy of all firms and the term $\sim \mu$, which takes into account the supply of external resources (1). By analogy with thermodynamics, there is maximum principle for the entropy: maximizing it with respect to U we reproduce the balance condition (1). Using Eqs. (A3) – (A4) one can check that in the case when $Q(t)$ is not

(quickly) decreasing function, the market entropy always increases with time

$$dS/dt > 0. \quad (\text{A5})$$

Since the variation of entropy ΔS is opposite to the variation of information, $\Delta I = -\Delta S$, Eq. (A5) means, that the activity of the market leads to “erasing” of initial information – the effect, well known for some “laundering” schemes.

APPENDIX B: SOLUTION OF COALESCENCE EQUATIONS

To find PDF of coalescent model (10) we use the method of Ref.¹¹³. We define dimensionless time τ and introduce the function $u(\tau)$:

$$\tau = \ln[G_c(t)/G_c(t_0)], \quad u(\tau) = G(t)/G_c(t),$$

where t_0 is coalescent time. In new variables the Master equation (10) takes the form

$$du/d\tau = v(u) = \gamma(\tau)(u - u^{1-\beta}) - u, \quad (\text{B1})$$

where

$$\gamma(\tau) = p \frac{dt}{G_c^{\beta-1} dG_c}. \quad (\text{B2})$$

The balance equation (5) can only be satisfied if the plot of function $v(u)$ lays below the axis u , and touches it at one point $u = u_0$. Such locking point $u = u_0$, $\gamma = \gamma_0$, for Eq. (B1) is determined by equations

$$v(u_0) = 0, \quad dv(u_0)/du_0 = 0, \quad d^2v(u_0)/du_0^2 < 0,$$

and we find that $u_0 \rightarrow \infty$ and $\gamma_0 = 1$.

From Eq. (B2) we get that the critical size $G_c(t)$ grows and “supersaturation” $\Delta(t)$ (we use physical terms here) decreases with time as

$$G_c(t) = \left(\frac{\beta qt}{\gamma_0}\right)^{1/\beta}, \quad \Delta(t) = \frac{\gamma_0}{q\beta t}. \quad (\text{B3})$$

PDF of firms can be rewritten through PDF of variables u and τ : $f(G, t) = \varphi(u, \tau)/G_c(t)$, and neglecting the diffusion inflow of new firms we find

$$\frac{\partial \varphi}{\partial \tau} + \frac{\partial}{\partial u} [v_0(u) \varphi] = 0, \quad (\text{B4})$$

where the velocity $v_0(u) = du/d\tau = -u^{1-\beta}$ is given by Eq. (B1) with $\gamma = \gamma_0$.

General solution of Eq. (B4) is

$$\varphi(u, \tau) = u^{\beta-1} \chi[\tau - \tau(u)], \quad \tau(u) = -u^\beta/\beta \quad (\text{B5})$$

with arbitrary function $\chi(\tau)$. To find $\chi(\tau)$ substitute this expression into the balance equation (1) and (5) with $Q(t) \gg U(t)$:

$$Q_0 \left(\frac{\gamma_0}{\beta h}\right)^m G_c^{m-1}(t_0) e^{\tau(\beta m-1)} = \int_0^\infty du u \varphi(u, \tau). \quad (\text{B6})$$

This condition can be satisfied only if the function χ has the form $\chi[\tau - \tau(u)] = A e^{(\beta m-1)[\tau - \tau(u)]}$. Substituting this function into Eq. (B5), we find

$$\varphi(u, \tau) = A (1 - \beta m)^{-1} e^{(\beta m-1)\tau} dF(u)/du, \quad (\text{B7})$$

with

$$F(u) = e^{-(1/\beta-m)u^\beta}, \quad A \simeq Q_0 \left(\frac{\gamma_0}{\beta p}\right)^m G_c^{m-1}(t_0). \quad (\text{B8})$$

APPENDIX C: MACROECONOMIC INTERPRETATION

To get better understanding of coalescent model, consider its microeconomic interpretation. Optimal firm size, $G = G_c$, is determined from the maximum of the profit function

$$\pi(G) = Py(G) - wG, \quad (\text{C1})$$

where $y(G)$ is the number of units produced by firm of G peoples, P is the price of one unit, and w is the average wage per one man. The technology is usually characterized by the standard Cobb-Douglas function $y(G) = K^\eta G^{1-\eta}$, where K is the firm capital and the exponent $\eta > 0$. Both the price P and the capital K are reduced to initial time. Maximizing the profit function (C1) we get $G_c \sim w^{-1/\eta}$.

Variation of wages $w(t)$ with the time is determined by the Fillips low¹¹⁴:

$$\frac{1}{w(t)} \frac{dw(t)}{dt} \simeq a[U_* - U(t)] = -a\Delta(t), \quad (\text{C2})$$

with positive constant $a > 0$. Substituting expression (B3) for $\Delta(t)$ in Eq. (C2), we find its solution $w(t) = \text{const} \times t^{-\zeta}$ with $\zeta = a/(\beta q)$. Substituting this function into $G_c \sim w^{-1/\eta}$, we get the optimal firm size $G_c \sim t^{\zeta/\eta}$. Comparing this dependence with Eqs. (14) and (B3), we find the Fillips parameter in Eq. (C2): $a = \eta q$. We conclude, that coalescent approach is consistent with the maximum profit principle and the Fillips low.

Notice that while the parameter $q = a/\eta$ of Master equation (A1), (A2) is determined by technology (η) and market structure (a), economic analysis do not impose any restrictions on the second parameter p of Master equation. Therefore, p can depend on management ability of firm head, relations between firm staff, industry shocks and so on, and can experience strong random fluctuations Δp . This observation explains the empirical fact, that there is much more variance in job destruction than in job creation time series¹¹⁵ (as was noted by Lev Tolstoy: all fortunate families are happy alike – each unfortunate family is unhappy in own way).

APPENDIX D: PDF OF DOUBLE GAUSSIAN MODEL

It is convenient in Eq. (62) to use instead of $\{\mathbf{a}_i^0\}$ Gaussian random variables $\{\boldsymbol{\alpha}_i\}$:

$$\mathbf{a}_i^0 = c_i \boldsymbol{\alpha}_i, \quad \varepsilon_i = \sum_j c_{ij} \boldsymbol{\alpha}_j,$$

such as $\overline{\alpha_i^2} = 1$, $\overline{(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)} = \nu$. Averaging over fluctuating Gaussian variables $\boldsymbol{\xi}_i^0$ and $\boldsymbol{\alpha}_i$, we get general expression for inverse Fourier component of PDF:

$$G^{-1}(k, p) = 1 + k^2 \sigma_{11}/2 + p^2 \sigma_{22}/2 + kp \sigma_{12} + (1 - \nu^2) (\varkappa_{11} k^2 + \varkappa_{22} p^2 + \varkappa_{12} kp)^2$$

with

$$\begin{aligned} \sigma_{11} &= c_1^2 + c_{12}^2 + c_{11}^2 + 2\nu c_{11} c_{12}, \\ \sigma_{22} &= c_2^2 + c_{21}^2 + c_{22}^2 + 2\nu c_{22} c_{21}, \\ \sigma_{12} &= c_1 c_{21} + c_2 c_{12} + \nu (c_1 c_{22} + c_2 c_{11}), \\ \varkappa_{11} &= c_1 c_{12}, \quad \varkappa_{22} = c_2 c_{21}, \\ \varkappa_{12} &= c_1 c_2 - c_{11} c_{22} + c_{12} c_{21}. \end{aligned}$$

An important relation between elements of matrixes $\boldsymbol{\sigma}$ and \varkappa follows from the condition of stationarity of PDF, which leads to physical constraint $G(k, 0) = G(0, k)$ on Fourier components of univariate PDFs of $\Delta_\tau P(t)$ and $\Delta_\tau P(t + \tau)$, and we get $\sigma_{11} = \sigma_{22} = \sigma^2$, $\varkappa_{11}^2 = \varkappa_{22}^2$. Here σ is the dispersion of price fluctuations,

$$\begin{aligned} \overline{\langle \Delta_\tau P^2(t) \rangle} &= \overline{\langle \Delta_\tau P^2(t + \tau) \rangle} = \sigma^2, \\ \overline{\langle \Delta_\tau P(t) \Delta_\tau P(t + \tau) \rangle} &= \sigma_{12} = \varepsilon \sigma^2. \end{aligned} \quad (\text{D1})$$

We conclude, that in addition to σ and ε , our model is characterized by dimensionless constant ν and the angle ϕ :

$$G^{-1}(k, p) = 1 + [(k^2 + p^2)/2 + \varepsilon kp] \sigma^2 + (1 - \nu^2) \times \frac{\sigma^4}{4} \left[\frac{k^2 - p^2}{2} \sin(2\phi)/2 - kp \cos(2\phi) \right]^2. \quad (\text{D2})$$

The quadratic part of this expression can be diagonalized by changing variables, $K = k \cos \psi - p \sin \psi'$, $P = k \sin \psi + p \cos \psi'$, with $\psi' = \phi - \frac{1}{2} \arcsin \varepsilon$, $\psi = \phi + \frac{1}{2} \arcsin \varepsilon$. In new variables Eq. (D2) takes the form

$$G^{-1}(k, p) = 1 + \frac{\sigma^2}{2} (K^2 + P^2) + \frac{1 - \nu^2}{(1 - \varepsilon^2)^2} \frac{\sigma^4}{4} [KP - \tau(K, P)]^2,$$

with

$$\begin{aligned} \tau(K, P) &= \frac{\varepsilon}{2} \cos(2\phi) (K^2 + P^2) + \frac{1}{2} (1 - \sqrt{1 - \varepsilon^2}) \\ &\times \sin(2\phi) [2KP \sin(2\phi) + (K^2 - P^2) \cos(2\phi)]. \end{aligned}$$

Eq. (D2) can be simplified if we note, that correlations of price increments, Eq. (D1), are always very small, $|\varepsilon| \ll 1$. In polar coordinates $K = |\mathbf{K}| \cos \varphi$, $P = |\mathbf{K}| \sin \varphi$ we have in the main order in ε

$$\begin{aligned} [KP - \tau(K, P)]^2 &\simeq \frac{1}{4} |\mathbf{K}|^4 [\sin(2\varphi) - \varepsilon \cos(2\varphi)]^2 \\ &\simeq \frac{1}{4} |\mathbf{K}|^4 \sin^2 \left[2 \left(\varphi - \frac{\varepsilon}{2} \cos(2\varphi) \right) \right] = (K' P')^2, \end{aligned}$$

where we introduced new orthogonal rotated coordinate system $K \simeq K' - (\varepsilon/2) P' \cos(2\phi)$, $P \simeq P' + (\varepsilon/2) K' \cos(2\phi)$. Changing integration variables $(k, p) \rightarrow (K', P')$ in the integral (48) in the main order in the small parameter ε we get Eq. (63), where angles ϕ_+ and ϕ_- are defined by $\phi_- = \phi - \varepsilon \cos^2 \phi$, $\phi_+ = \phi + \varepsilon \sin^2 \phi$, and functions $\mathcal{P}_l(x)$ are determined by Eqs. (58) and (67).

APPENDIX E: PDF OF VOLATILITY FLUCTUATIONS

PDF of the volatility variable $V_1(t)$ (108) at $q = 1$ can be expressed through the n -point PDF $\mathcal{P}\{\Delta P_k\}$ of variables ΔP_k ,

$$P_n(V_1) \equiv \prod_{k=1}^n \int_0^\infty d\Delta P_k \delta \left(V_1 - \frac{1}{n} \sum_{k=1}^n |\Delta P_k| \right) \mathcal{P}\{\Delta P_k\} \quad (\text{E1})$$

Asymptotes of $P_n(V_1)$ can be found both for $V_1 \ll \sigma$ and for $V_1 \gg \sigma$. At $V_1 \ll \sigma$ only small $|\Delta P_k| \ll \sigma$ contribute to the integral (E1) and we have $P_n(V_1) \sim V_1^{n-1}$. In the opposite case of large $V_1 \gg \sigma$ the integral is dominated by the power tail of PDF $\mathcal{P}\{\Delta P_k\}$ with typical $|\Delta P_k| \sim V_1 \gg \sigma$. From dimension consideration we find for such ΔP_k that $\mathcal{P}\{\Delta P_k\} \sim V_1^{-n-\mu}$, where μ is the exponent of one-point PDF, Eq. (36), and the integral (E1) is estimated as $P_n(V_1) \sim V_1^{-1-\mu}$.

Both these limits are matched by the function

$$P_n(V_1) = V_0^{-1} N_0^{-1} f(V_1/V_0), \quad (\text{E2})$$

$$f(z) = z^{-1} \left(z^{-n/s} + z^{\mu/s} \right)^{-s}$$

with $V_0 \sim \sigma$. The dependence of a new parameter $s > 0$ on n will be found later from the condition that at large n the distribution $P_n(V_1)$ should not depend of n . Momentums of this distribution $\langle V_1^k \rangle = V_0^k N_k / N_0$ are determined by normalization integrals,

$$N_k = \int_0^\infty z^k f(z) dz = mB[m(n+k), m(\mu-k)],$$

$$m = s/(n + \mu),$$

where B is the Beta-function.

The function $f(z)$ (E2) reaches its maximum at $z_{\max} = (n-1)^m / (\mu+1)^m$. The central part of the distribution is obtained by expanding the probability $P_n(V_1)$ over

$\ln(V_1/V_{\max})$ near its maximum at $V_{\max} = z_{\max}V_0$, and it has log-normal form:

$$\ln P_n(V_1) = \text{const} - \frac{1}{2} \frac{n-1}{s} (\mu+1) \left(\ln \frac{V_1}{V_{\max}} \right)^2. \quad (\text{E3})$$

Since the distribution $P_n(V_1)$ (E3) should not depend on

n at large n , we find $s = c(n-1)$ with certain constant c . In the limit $n \rightarrow \infty$ $P_n(V_1)$ becomes universal function of V_1/V_{\max} , Eq. (110) with $q = 1$. Repeating our calculations for general $q > 0$, we find that it is given by the substitution $\mu \rightarrow \mu/q$ and $n \rightarrow n/q$ in the above expressions.