

The Conformal Constraint in Canonical Quantum Gravity

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Abstract

Perturbative canonical quantum gravity is considered, when coupled to a renormalizable model for matter fields. It is proposed that the functional integral over the dilaton field should be disentangled from the other integrations over the metric fields. This should generate a conformally invariant theory as an intermediate result, where the conformal anomalies must be constrained to cancel out. When the residual metric is treated as a background, and if this background is taken to be flat, this leads to a novel constraint: in combination with the dilaton contributions, the matter lagrangian should have a vanishing beta function. The zeros of this beta function are isolated points in the landscape of quantum field theories, and so we arrive at a denumerable, or perhaps even finite, set of quantum theories for matter, where not only the coupling constants, but also the masses and the cosmological constant are all fixed, and computable, in terms of the Planck units.

1. Introduction

In a previous paper[1], it was argued that the functional integral in canonical quantum gravity,

$$\int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi^{\text{mat}} e^{i(S^{\text{EH}}(g_{\mu\nu}) + S^{\text{mat}}(g_{\mu\nu}, \phi^{\text{mat}}))} , \quad (1.1)$$

where S^{EH} is the Einstein-Hilbert action and S^{mat} is the action of the matter fields, here abbreviated as ϕ^{mat} , should be considered to be taken in two steps:

$$g_{\mu\nu} \equiv \omega^2 \hat{g}_{\mu\nu} ; \quad \det(\hat{g}_{\mu\nu}) = -1 , \quad \int \mathcal{D}g_{\mu\nu} = \int \mathcal{D}\hat{g}_{\mu\nu} \int \mathcal{D}\omega , \quad (1.2)$$

and the integral over the dilaton field ω should be taken together with the integrations over the matter fields ϕ^{mat} .

Rewriting the Einstein-Hilbert action (including a possible cosmological constant) in terms of ω and $\hat{g}_{\mu\nu}$, one finds, in four dimensions,

$$S^{\text{EH}} = \int d^4x \frac{1}{2\kappa^2} \left(\hat{R} \omega^2 + 6 \hat{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega - 2\Lambda \omega^4 \right) , \quad (1.3)$$

where $\kappa^2 = 8\pi G_N$, and \hat{R} is the Ricci scalar associated to $\hat{g}_{\mu\nu}$. It is convenient to split the lagrangian of the matter fields into conformally invariant kinetic parts, mass terms, and interaction terms. In a simplified notation (later we will be more precise), one has

$$\phi^{\text{mat}} = \{A_\mu(x), \psi(x), \bar{\psi}(x), \varphi(x)\} ; \quad (1.4)$$

$$\mathcal{L}^{\text{kin}} = -\frac{1}{4} \hat{g}^{\mu\alpha} \hat{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{12} \hat{R} \varphi^2 - \bar{\psi} \gamma^\mu \hat{D}_\mu \psi ; \quad (1.5)$$

$$\mathcal{L}^{\text{mass}} = -\frac{1}{2} m_s^2 \omega^2 \varphi^2 - \bar{\psi} \omega \mu_d \psi . \quad (1.6)$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; the kinetic term for the Dirac fields is shorthand for the corresponding expression using a vierbein field \hat{e}_μ^a for the metric $\hat{g}_{\mu\nu}$ with its associated connection field. m_s stands short for the scalar masses and m_d for the Dirac masses. The term $\frac{1}{12} \hat{R} \varphi^2$ could be removed by a field redefinition, but is kept here for convenience, making the scalar lagrangian conformally invariant.

Interaction between the matter fields must now be written in the form

$$\mathcal{L}^{\text{int}} = -\frac{1}{4!} \lambda \varphi^4 - \bar{\psi} y_i \varphi_i \psi - \frac{1}{3!} g_3 \varphi^3 \omega , \quad (1.7)$$

where the Yukawa couplings y_i could be matrices in the indices labeling the fermion species, and λ and g_3 could be 4- and 3-index tensors in the scalar field indices.

If now we rescale the ω field:

$$\omega(x) = \frac{\kappa}{\sqrt{6}} \tilde{\omega}(x) , \quad (1.8)$$

we notice two things. First, the action for the $\tilde{\omega}$ field is now nearly identical to the kinetic term for the φ field, both having the same conformal dimension:

$$\mathcal{L}^{\text{EH}} = \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\tilde{\omega}\partial_\nu\tilde{\omega} + \frac{1}{12}\hat{R}\tilde{\omega}^2 - \frac{1}{36}\kappa^2\Lambda\omega^4, \quad (1.9)$$

and secondly, the mass terms turn into *conformally invariant quartic coupling terms* between matter fields and the $\tilde{\omega}$ field:

$$\mathcal{L}^{\text{mass}} = -\frac{1}{2}\tilde{\kappa}^2 m_s^2 \tilde{\omega}^2 \varphi^2 - \tilde{\kappa} m_d \bar{\psi} \tilde{\omega} \psi, \quad (1.10)$$

where $\tilde{\kappa} = \kappa/\sqrt{6} = \sqrt{\frac{4}{3}\pi G_N}$ has the dimension of an inverse mass. Also the scalar 3-field coupling, which originally was not conformally invariant, now turns into a conformally invariant 4 field coupling (see Eq. (1.7)):

$$-\frac{1}{3!}\tilde{\kappa}g_3\varphi^3\tilde{\omega}, \quad (1.11)$$

and a new quartic interaction term for the $\tilde{\omega}$ field is generated by the cosmological constant:

$$-\frac{1}{6}\tilde{\kappa}^2\Lambda\tilde{\omega}^4. \quad (1.12)$$

It is important to observe that the lagrangian (1.9) for the dilaton field $\tilde{\omega}$ has an overall sign opposite to that of ordinary scalar fields φ . For any other field theory, this would be disastrous because it would violate causality. Here however, the unconventional sign is a necessary consequence of the canonical structure of the theory. Since it is an overall sign, it has no net effect on the Feynman rules; this we will exploit by rotating the field in the complex plane:

$$\tilde{\omega}(x) \equiv i\eta(x), \quad (1.13)$$

so that the new field $\eta(x)$ will be indistinguishable from other scalar fields, with one important exception: the conformal interaction terms from the original mass terms, Eq. (1.10), as well as the interaction from the original 3-field interaction, Eq. (1.11), get unconventional factors -1 or i .

Another important thing to observe is that there is no term at all in the lagrangian that could serve as a kinetic term for the $\hat{g}_{\mu\nu}$ field¹. Any such kinetic terms should arise solely from higher order effects due to the interactions with the matter fields. Naturally, since the entire lagrangian now is conformally invariant, we should expect the effective lagrangian for $\hat{g}_{\mu\nu}$ to be conformally invariant as well. It is emphasized in Ref. [1] however that, as is well known [2], this conformal invariance is mutilated by conformal anomalies.

In this paper, we now propose to postpone any attempts to describe the functional integrals over $\hat{g}_{\mu\nu}$. Not only do we have anomalies there, but also there are difficulties

¹Of course, this term reappears if one substitutes $\eta = -i + \mathcal{O}(\kappa\tilde{\eta})$, where $\tilde{\eta}$ is a small oscillating field, by expanding \hat{R} .

with unitarity and Landau ghosts.² As was explained in Ref.[1], the effective interactions with matter and dilaton fields generate an action for $\hat{g}_{\mu\nu}$ that largely coincides with the familiar conformally invariant action obtained from the square of the Weyl curvature, *but with an infinite numerical coefficient*, which would have to be renormalized. The difficulties associated to that are sufficient reason for us now to postpone this sector of the theory entirely.

At first sight, one may well find logical objections to such a procedure: why not also first integrate over $\hat{g}_{\mu\nu}$ before examining whether the amplitudes obtained obey conformal constraints? We argue however that the $\hat{g}_{\mu\nu}$ integration is very different from the rest; $\hat{g}_{\mu\nu}$ determines the location of the local light cones, so that it determines the causal relationships between points in space-time. It may well be that quantum interference of states with light cones at different places will require treatments that differ in essential ways from the standard functional integral.

In any case, it is worth investigating what happens if we follow this procedure. The implications are quite remarkable, as we will show.

2. Renormalization

Let us assume that the matter fields ϕ^{mat} consist of Yang-Mills fields A_μ^a , Dirac fields $\bar{\psi}$, ψ and scalar fields φ , the latter three sets being in some (reducible or irreducible, chiral or non chiral) representation of the local Yang-Mills gauge group. For brevity, we will write complex scalar fields as pairs of real fields, and if Weyl or Majorana fermions occur, the Dirac fields can be replaced by pairs of these.³ Let us rewrite the lagrangian for matter interacting with gravity more precisely than in the previous, introductory section:

$$\begin{aligned} \mathcal{L}(\hat{g}_{\mu\nu}, \eta, \phi^{\text{mat}}) &= -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a - \bar{\psi} \hat{\gamma}^\mu \hat{D}_\mu \psi - \frac{1}{2}\hat{g}^{\mu\nu}(D_\mu\varphi D_\nu\varphi + \partial_\mu\eta \partial_\nu\eta) \\ &\quad - \frac{1}{12}\hat{R}(\varphi^2 + \eta^2) - V_4(\varphi) - iV_3(\varphi)\eta + \frac{1}{2}\tilde{\kappa}^2 m_i^2 \eta^2 \varphi_i^2 - \tilde{\Lambda}\eta^4 \\ &\quad - \bar{\psi}(y_i\varphi_i + iy_i^5\gamma^5\varphi_i + i\tilde{\kappa}m_d\eta)\psi, \end{aligned} \quad (2.1)$$

where $G_{\mu\nu}$ is the (non Abelian) Yang-Mills curvature, and D_μ and \hat{D}_μ are covariant derivatives containing the Yang-Mills fields; $\hat{\gamma}_\mu$ and \hat{D}_μ also contain the vierbein fields and connection fields associated to $\hat{g}_{\mu\nu}$; the Yukawa couplings y_i , y_i^5 and fermion mass terms m_d are matrices in terms of the fermion indices. The scalar self interactions, $V_3(\varphi)$ and $V_4(\varphi)$ must be a third and fourth degree polynomials in the fields φ_i :

$$V_4(\varphi) = \frac{1}{4!}\lambda\varphi^4 = \frac{1}{4!}\lambda^{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l; \quad (2.2)$$

$$V_3(\varphi) = \frac{1}{3!}\tilde{g}_3^{ijk}\varphi_i\varphi_j\varphi_k, \quad \tilde{g}_3 = \tilde{\kappa}g_3. \quad (2.3)$$

²In Ref. [3],[4] claims are made that unitarity can be restored. This however requires the integration contours to be rotated in the complex plane such that $\hat{g}_{\mu\nu}$ becomes complex. Such approaches are interesting and may well serve as good starting points for generating promising theories, but they will not be pursued here.

³A single Weyl or Majorana fermion then counts as half a Dirac field.

In Eq. (2.1), like m_d , also $m_i^2 \delta_{ij}$ are mass matrices, in general. Furthermore, $\tilde{\Lambda}$ stands for $\frac{1}{6} \tilde{\kappa}^2 \Lambda$. Of course, all terms in (2.1) must be fully invariant under the Yang-Mills gauge rotations. They must also be free of Adler Bell Jackiw anomalies.[6]

Now that the dilaton field η has been included, the entire lagrangian has been made conformally invariant. It is so by construction, and no violations of conformal invariance should be expected. This invites us to consider the beta functions of the theory. Can we conclude at this point that the beta functions should all vanish?

Let us not be too hasty. In the standard canonical theory, matter fields and their interactions are renormalized. Let us consider dimensional renormalization, and the associated anomalous behavior under scaling. In $4 - \varepsilon$ dimensions, where ε is infinitesimal, the scalar field dimensions are those of a mass raised to the power $1 - \varepsilon/2$, so that the couplings λ have dimension ε . This means that in most of the terms in the lagrangian (2.1) the integral powers of η will receive extra factors of the form $\eta^{\pm\varepsilon}$ or $\eta^{\pm\varepsilon/2}$, which will then restore exact conformal invariance at all values for ε . If we follow standard procedures, we accept that η is close to $-i$, so that singularities at $\eta \rightarrow 0$ or $\eta \rightarrow \infty$ are not considered to be of any significance. Indeed, the limit $\eta \rightarrow 0$ may be seen to be the small-distance limit. This is the limit where gravity goes wrong anyway, so why bother?

However, now one could consider an extra condition on the theory. Let us assume that the causal structure, that is, the location of the light cones, is determined by $\hat{g}_{\mu\nu}$, and that there exist dynamical laws for $\hat{g}_{\mu\nu}$. This was seen to be a very useful starting point for a better understanding of black hole complementarity[5]. The laws determining the scale $\omega(x)$ should be considered to be dynamical laws, and the canonical theory of gravity itself would support this: formally the functional integral over the η fields is exactly the same as that for other scalar fields.

In view of the above, we do think it is worthwhile to pursue the idea that the η field must be handled just as any other scalar component of the matter fields; but then, after renormalization, fractional powers, in the $\varepsilon \rightarrow 0$ limit would lead to $\log(\eta)$ terms, and these must clearly be excluded. Renormalization must be done in such a way that no traces of logarithms are left behind. Certainly then, a scale transformation, which should be identical to a transformation where the fields η are scaled, should not be associated with anomalies. Implicitly, this also means that the region $\eta \rightarrow 0$ is now assumed to be regular. This is the small distance region, so that, indeed, our theory says something non-trivial about small distances. This is why our theory leads to new predictions that eventually should be testable. Predictions follow from the demand that all beta functions of the conformal “theory” (2.1) must vanish.

Note that one set of terms is absent in Eq. (2.1): the terms linear in φ and hence cubic in η . This, of course, follows from the fact that, usually, no terms linear in the scalar fields are needed in the standard matter lagrangians; such terms can easily be removed by translations of the fields: $\varphi_i \rightarrow \varphi_i + a_i$ for some constants a_i . Thus, the classical lagrangian is stationary when the fields vanish: $\varphi = 0$ is a classical solution. In our present notation, this observation is equivalent to the observation that fields may be freely transformed into one another without modifying the physics. One such transformation is

a rotation of one of the scalar fields, say φ_1 , with the η fields:

$$\begin{cases} \varphi_1 & \rightarrow \varphi_1 \cosh \alpha_1 + i\eta \sinh \alpha_1, \\ \eta & \rightarrow \eta \cosh \alpha_1 - i\varphi_1 \sinh \alpha_1, \end{cases} \quad (2.4)$$

where α_1 stands for the original shift of the field φ_1 . The transformation is taken to be a hyperbolic rotation because the “kinetic term” $-\frac{1}{2}(\partial\eta^2 + \partial\varphi^2) = \frac{1}{2}(\partial\tilde{\omega}^2 - \partial\varphi^2)$ in Eq. (2.1) has to be invariant.

In most cases, these transformations need not be considered since terms linear in φ will in general not be gauge invariant.

Notice also that the Yang-Mills fields are not directly coupled to the η field. In the “classical limit”, $\eta \rightarrow -i$, we can see why this is so. Since not η , but $\tilde{\omega}$ is real, the invariant quantity is $\varphi^2 - \omega^2$. Rotating φ fields with η fields would therefore be a non-compact transformation, and Yang Mills theories with non-compact Lie groups usually do not work. There is food for thought here, but as yet we will not pursue that.

When constructing a complete theory, the next step should be to consider just any configuration of $\hat{g}_{\mu\nu}(\vec{x}, t)$, formulate the renormalized theory in this background, and finally consider functional integrals over $\hat{g}_{\mu\nu}$. Unfortunately, this is still too difficult. In a non-trivial $\hat{g}_{\mu\nu}$ background, there will be anomalies depending on the derivatives of $\hat{g}_{\mu\nu}$; there are divergences[1] proportional to the Weyl curvature squared, which can be seen to correspond to the field combination $\hat{R}_{\mu\nu}^2 - \frac{1}{3}\hat{R}^2$, and subtracting those leads to new conformal anomalies[2]. It was suggested in Ref. [1] to keep the conformal infinity, which would turn gravitons into “classical” particles, or more precisely, particles that cannot interfere quantum mechanically, but whether this can be held up as a theory remains to be seen. To avoid further complications, we now decided to look at the case when $\hat{g}_{\mu\nu}(\vec{x}, t) = \eta_{\mu\nu}$, or, space-time is basically flat, though we just keep the field $\eta(\vec{x}, t)$.

3. The β functions

Thus, we return to a theory to which all known quantum field theory procedures can be applied, the only new thing being the presence of an extra, gauge neutral, spinless field η , and the perfect local scale invariance of the theory.

We arrived at the lagrangian (2.1), and we wish to impose on it the condition that all its beta functions vanish, since conformal invariance has to be kept. As the theory is renormalizable, the number of beta functions is always exactly equal to the number of freely adjustable parameters. In other words: we have exactly as many equations as there are freely adjustable unknown variables, so that *all coupling constants, all mass terms and also the cosmological constant*, should be completely fixed by the equations $\beta_i = 0$. They are at the stationary points. Masses come in the combination $\tilde{\kappa} m_i$ and the cosmological constant in the combination $\tilde{\kappa}^2 \Lambda$, so all dimensionful parameters of the theory will be fixed in terms of Planck units.

In principle, there is no reason to expect any of these fixed points to be very close yet not on any of the axes, so neither masses nor the cosmological constant can be expected

to be unnaturally small, at this stage of the theory. In other words, as yet no resolution of the hierarchy problem is in sight: why are many of the physical mass terms 40 orders of magnitude smaller than the Planck mass, and the cosmological constant more than 120 orders of magnitude? We have no answer to that in this paper, but we will show that the equations are quite complex, and exotic solutions cannot be excluded.

The existence of infinitely many solutions cannot be excluded. This is because one can still adjust the composition and the rank(s) of the Yang-Mills gauge group, as well as one's choice of the scalar and (chiral) spinor representations.⁴ These form infinite, discrete sets. However, many choices turn out not to have any non trivial, physically acceptable fixed point at all: the interaction potential terms $V(\varphi)$ must be real and properly bounded, for instance. Searches for fixed points then automatically lead to vanishing values of some or more of the coupling parameters, which would mean that the symmetries and representations have not been chosen correctly.

Every advantage has its disadvantage. Since all parameters of the theory will be fixed, we cannot apply perturbation theory. However, we can make judicious choices of the scalar and spinor representations in such a way that the existence of a fixed point for the gauge coupling to these fields can be made virtually certain. The β function for $SU(N)$ gauge theories with N_f fermions and N_s complex scalars in the elementary representation is

$$16\pi^2 \beta(g) = -a g^3 - b g^5 + \mathcal{O}(g^7) , \quad (3.1)$$

$$a = \frac{11}{3}N - \frac{2}{3}N_f - \frac{1}{6}N_s , \quad (3.2)$$

$$b = \mathcal{O}(N^2, N N_f, N N_s) . \quad (3.3)$$

Choosing one scalar extra, or one missing, we can have a as small as $a = \pm\frac{1}{6}$, while a quick inspection in the literature[8][9] reveals that, in that case, b may still have either sign:

$$b = \pm\mathcal{O}(N^2) . \quad (3.4)$$

depending on further details, such as the ratio of fermions and scalars, the presence of other representations, and the values of the Yukawa couplings. Choosing the sign of a opposite to that of b , one then expects that a fixed point can be found at

$$g^2 = -b/a = \mathcal{O}(1/N^2) . \quad (3.5)$$

This, we presume, is close enough to zero that the the following procedure may be assumed to be reliable. Let there be ν physical constants, the ν^{th} one being the gauge coupling g , which is determined by the above equation (3.5). If we take all other coupling parameters to be of order g or g^2 , then the beta function equations are reliably given by the one-loop expressions only, which we will give below. Now these are $\nu - 1$ equations for the $\nu - 1$ remaining coupling parameters, and they are now inhomogeneous equations, since the one coupling, g^2 , is already fixed. All we have to do now, is find physically acceptable solutions. We already saw that non-Abelian Yang-Mills fields are mandatory;

⁴which of course must be free of Adler Bell Jackiw anomalies[6][7].

we will quickly discover that, besides the η fields, both fermions and other scalar matter fields are indispensable to find any non-trivial solutions.

One trivial, yet interesting solution must be mentioned: $\mathcal{N} = 4$ super Yang-Mills. We take its lagrangian, and add to that the η field while postulating that this η field does not couple to the $\mathcal{N} = 4$ matter fields at all. Then indeed all β functions vanish[10]. However, since the η field is not allowed to couple, the physical masses are all strictly zero, which disqualifies the theory physically. Note, however, that also the cosmological constant is rigorously zero. Perhaps the procedure described above can be applied by modifying slightly the representations in this theory, so that a solution with masses close to zero, and in particular a cosmological constant close to, but not exactly zero, emerges.

The one loop β functions are generated by an algebra[11], in which one simply has to plug the Casimir operators of the Yang Mills Lie group, the types of the representations, the quartic scalar couplings and the fermionic couplings. If we take the scalar fields φ_i and η together as σ_i , the generic lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{2}(D_\mu \sigma_i)^2 - V(\sigma) - \bar{\psi}(\gamma D - (S_i + i\gamma^5 P_i)\sigma_i)\psi , \quad (3.6)$$

where σ_i and $\bar{\psi}$, ψ are in general in reducible representations of the gauge group, D_μ is the gauge covariant derivative, $V(\sigma)$ is a gauge-invariant quartic scalar potential, and S_i and P_i are matrices in terms of the fermion flavor indices. Everything must be gauge invariant and the theory must be anomaly free[6][7].

The covariant derivatives contain the hermitean representation matrices T_{ij}^a , $U_{\alpha\beta}^{La}$ and $U_{\alpha\beta}^{Ra}$:

$$D_\mu \sigma_i \equiv \partial_\mu \sigma_i + iT_{ij}^a A_\mu^a \sigma_j ; \quad (3.7)$$

$$D_\mu \psi_\alpha \equiv \partial_\mu \psi_\alpha + i(U_{\alpha\beta}^{La} P^L + U_{\alpha\beta}^{Ra} P^R) A_\mu^a \psi_\beta ; \quad P^{L,R} \equiv \frac{1}{2}(1 \pm \gamma^5) . \quad (3.8)$$

The gauge coupling constant(s) g are assumed to be included in these matrices T and U . The operators P^L and P^R are projection operators for the left- and right handed chiral fermions.

The group structure constants f^{abc} are also assumed to include a factor g , and they are defined by

$$[T^a, T^b] = if^{abc}T^c ; \quad [U^{La}, U^{Lb}] = if^{abc}U^{Lc} ; \quad [U^{Ra}, U^{Rb}] = if^{abc}U^{Rc} . \quad (3.9)$$

Casimir operators C_g , C_s and C_f will be defined as

$$f^{apq} f^{bpq} = C_g^{ab} , \quad \text{Tr}(T^a T^b) = C_s^{ab} , \quad \text{Tr}(U^{La} U^{Lb} + U^{Ra} U^{Rb}) = C_f^{ab} . \quad (3.10)$$

All beta functions are given by writing down how the entire lagrangian (3.6) runs as a function of the scale μ [11]:

$$\frac{\mu \partial}{\partial \mu} \mathcal{L} = \beta(\mathcal{L}) = \frac{1}{8\pi^2} \Delta \mathcal{L} , \quad (3.11)$$

$$\Delta \mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^b \left(\frac{11}{3}C_g^{ab} - \frac{1}{6}C_s^{ab} - \frac{2}{3}C_f^{ab} \right) - \Delta V - \bar{\psi}(\Delta S_i + i\gamma^5 \Delta P_i)\sigma_i \psi . \quad (3.12)$$

Here,

$$\begin{aligned} \Delta V = & \frac{1}{4}V_{ij}^2 - \frac{3}{2}V_i(T^2\sigma)_i + \frac{3}{4}(\sigma T^a T^b \sigma)^2 + \\ & \sigma_i V_j \text{Tr}(S_i S_j + P_i P_j) - \text{Tr}(S^2 + P^2)^2 + \text{Tr}[S, P]^2, \end{aligned} \quad (3.13)$$

where $V_i = \partial V(\sigma)/\partial \sigma^i$, $V_{ij} = \partial^2 V(\sigma)/\partial \sigma_i \partial \sigma_j$.

It is convenient to define the complex matrices W_i as

$$W_i = S_i + iP_i, \quad (3.14)$$

Then,

$$\begin{aligned} \Delta W_i = & \frac{1}{4}W_k W_k^* W_i + \frac{1}{4}W_i W_k^* K_k + W_k W_i^* W_k - \\ & - \frac{3}{2}(U^R)^2 W_i - \frac{3}{2}W_i (U^L)^2 + W_k \text{Tr}(S_k S_i + P_k P_i). \end{aligned} \quad (3.15)$$

If now we write the collection of scalars as $\{\sigma_i = \varphi_i, \sigma_0 = \eta\}$, taking due notice of the factors i in all terms odd in η , we can apply this algebra to compute all β functions of the lagrangian (2.1).

The values of the various β functions depend strongly on the choice of the gauge group, the representations, the scalar potential function and the algebra for the Yukawa terms, and there are very many possible choices to make. However, the signs of most terms are fixed by the algebra (3.12)–(3.15). By observing these signs, we can determine which are the most essential algebraic constraints they impose on possible solutions. As we will see, they are severely restrictive.

4. Adding the dilaton field to the algebra for the β functions.

Consider the dilaton field η added to the lagrangian, as in Eq. (2.1). This requires extending the indices i, j, \dots in the lagrangian (3.6) to include a value 0 referring to the η field. The unusual thing is now that the terms odd in η are purely imaginary, while all terms in Eq. (2.1) are of dimension 4. Let us split off this special component. In Eq. (3.6), we then write

$$V(\sigma) = \tilde{\Lambda}\eta^4 - \frac{1}{2}m_i^2\eta^2\varphi_i^2 + iV_3(\varphi)\eta + V_4(\varphi); \quad (4.1)$$

$$S_0 = im_d; \quad P_0 = 0; \quad S_i = y_i; \quad P_i = y_i^5. \quad (4.2)$$

Here, m_d is the fermionic mass matrix, y_i are matrices representing the scalar Yukawa couplings, and y_i^5 are the pseudoscalar Yukawa coupling matrices. $m_i^2\delta_{ij}$ is the scalar mass matrix, which is allowed to have negative eigenvalues, so we allow the Higgs mechanism to take place. We henceforth choose modified Planck units by setting $\tilde{\kappa}^2 = \frac{4}{3}\pi G_N = 1$.

Note, that in the Standard Model, there is no gauge-invariant Dirac mass matrix and no gauge-invariant cubic scalar interaction, so there m_d and V_3 are zero, but we will need to be more general.

We write $W = S + iP$, and now $\tilde{W} = S - iP$. The algebra (3.12) — (3.15) is then found to become

$$\Delta V(\sigma) = \Delta V^0(\varphi) \quad (a)$$

$$-\frac{1}{2}V_{3i}^2 - V_3\varphi_i\text{Tr}(m_d y_i) + \frac{1}{4}(m_i^2\varphi_i^2)^2 \quad (b)$$

$$+i\eta \left(-\frac{3}{2}V_{3i}(T^2\varphi)_i + \frac{1}{2}V_{3ij}V_{4ij} + \varphi_i V_{3j}\text{Tr}(y_i y_j + y_i^5 y_j^5) + V_{4i}\text{Tr}(m_d y_i) \right) \quad (4.3)$$

$$-4\text{Tr}(m_d(y_i\varphi_i)^3) + 2\text{Tr}([m_d, y_i^5\varphi_i][y_j\varphi_j, y_k^5\varphi_k]) - 2\text{Tr}((y_i^5\varphi_i)^2(m_d y_j + y_j m_d)\varphi_j) \\ -2V_{3i}m_i^2\varphi_i - V_3\text{Tr}m_d^2 - m_j^2\varphi_j^2\varphi_i\text{Tr}(m_d y_i) \quad (c)$$

$$+\eta^2 \left(-\frac{1}{4}V_{3ij}^2 - V_{3i}\text{Tr}(m_d y_i) - \frac{1}{2}m_i^2 V_{4ii}^2 + 2\text{Tr}((m_d y_i\varphi_i)^2) + 4\text{Tr}(m_d^2(y_i\varphi_i)^2) \right. \\ \left. +2\text{Tr}(m_d^2(y_i^5\varphi_i)^2 - \text{Tr}([m_d, y_i^5\varphi_i]^2) - m_j^2\varphi_i\varphi_j\text{Tr}(y_i y_j + y_i^5 y_j^5) + \frac{3}{2}m_i^2\varphi_i(T^2\varphi)_i \right. \\ \left. +2m_i^4\varphi_i^2 + \text{Tr}(m_d^2)m_i^2\varphi_i^2 - 6\tilde{\Lambda}m_i^2\varphi_i^2 \right) \quad (d)$$

$$+i\eta^3 \left(-\frac{1}{2}m_i^2 V_{3ii} - m_i^2\varphi_i\text{Tr}(m_d y_i) + 4\text{Tr}(m_d^3 y_i\varphi_i) + 4\tilde{\Lambda}\varphi_i\text{Tr}(m_d y_i) \right) \quad (e)$$

$$+\eta^4 \left(36\tilde{\Lambda}^2 - 4\tilde{\Lambda}\text{Tr}(m_d^2) + \frac{1}{4}\sum_i m_i^4 - \text{Tr}(m_d^4) \right), \quad (f)$$

where $V_{3i} = \partial V_3/\partial\varphi_i$, etc, and we apply summation convention: double indices are summed over starting with 1, except the index in the scalar mass matrix m_i^2 , which is only summed over if it occurs twice elsewhere as well, or if this is explicitly indicated. ΔV^0 is the expression that we already had, in Eq. (3.13).

The beta coefficients for the Yukawa couplings follow from adding the index 0 to Eq. (3.15):

$$\Delta W_i = \Delta W_i^0 - \frac{1}{4}(m_d^2 W_i + W_i m_d^2) + m_d \tilde{W}_i m_d - m_d \text{Tr}(m_d S_i), \quad (4.4)$$

$$-i\Delta W_0 = \Delta m_d = -\frac{3}{2}m_d^3 - m_d \text{Tr}(m_d^2) + \frac{1}{4}(y_k^2 + y_k^5)m_d + \frac{1}{4}m_d(y_k^2 + y_k^5) \\ + y_k m_d y_k - y_k^5 m_d y_k^5 + y_k \text{Tr}(y_k m_d) + i\gamma^5 y_k^5 \text{Tr}(y_k m_d), \quad (4.5)$$

where ΔW_i^0 stands for the standard β function for the corresponding dimension 4 interaction terms.

Before demanding that the β functions all vanish, we observe that we can still allow for an infinitesimal field transformation of the form (2.4) in the original lagrangian. This adds to the counter terms:

$$\delta V(\sigma) = i\alpha_i \left(\eta \frac{\partial V(\sigma)}{\partial\varphi_i} - \varphi_i \frac{\partial V(\sigma)}{\partial\eta} \right)$$

$$\begin{aligned}
&= \alpha_i V_3 \varphi_i + i\eta \alpha_i (V_{4i} + m_j^2 \varphi_j^2 \varphi_i) \\
&+ \eta^2 \alpha_i (-V_{3i}) + i\eta^3 \alpha_i (-m_i^2 \varphi_i - 4\tilde{\Lambda} \varphi_i), \tag{4.6}
\end{aligned}$$

and a similar rotation in the Yukawa couplings. This can be used to eliminate the term (4.3 e) by adjusting α_i ; it corresponds to a field shift in the non-gravitational case. In most cases, however, such as in the Standard Model, the terms in (4.3 e) are forced to vanish anyhow due to gauge invariance. In a similar way, an infinitesimal chiral rotation among the fermions can be used to eliminate the last term in Eq. (4.5).

Thus, after term (e) has been made to vanish by hand, the demand that all β functions vanish, for all values of η , applies in particular to the terms in Eq. (4.3 a — d) and (f), and to Eqs. (4.4) and (4.5).

We have already assumed that the non-Abelian Yang Mills field coupling(s) g have a small but non-vanishing fixed point. Through the effects of the group matrices T^a , U^{La} and U^{Ra} , the coupling(s) g determine the values of the other parameters, by as many coupled non-linear equations as there are unknowns. It follows that, in this theory, we *must* have non-Abelian Yang-Mills fields. In contrast, Abelian $U(1)$ components are not allowed since those do not have fixed points close to the origin.

Next, let us consider the requirement that the term (f) in Eq. (4.3) vanishes:

$$36\tilde{\Lambda}^2 = \text{Tr}(m_d^4) + 4\tilde{\Lambda} \text{Tr}(m_d^2) - \frac{1}{4} \sum_i m_i^4. \tag{4.7}$$

The r.h.s. of this equation resembles a supertrace. Since its sign must be positive, we read off right away that there *must* be fermions. If furthermore we like to have a very small or vanishing cosmological constant Λ , we clearly need that the sum of the fourth power of the Dirac fields (approximately) equals the sum of the fourth powers of the masses of the real scalar particles divided by 4.

Can we do without the scalar fields φ_i ? Eq. (4.7) would have a solution, although the cosmological constant would come out fairly large. However, now there is only one more equation to consider: Eq. (4.5), with all Yukawa couplings y_i and y_i^5 replaced by 0. That gives:

$$\frac{3}{2}m_d^3 + m_d \text{Tr}(m_d^2) \stackrel{?}{=} 0. \tag{4.8}$$

Whenever m_d has a real, non vanishing eigenvalue, this would imply that the trace of m_d^2 is negative, an impossible demand. Therefore, our theory also *must* have scalars φ_i , besides the dilaton field η .

It appears that in today's particle models not only the cosmological constant $\tilde{\Lambda}$ but also the mass terms are quite small, in the units chosen, which are our modified Planck units. Also, if there is a triple scalar coupling, $V_3(\varphi)$, it appears to be small as well. This is the hierarchy problem, for which we cannot offer any solution other than suggesting that we may have to choose a very complex group structure — as in the landscape scenarios often proposed in superstring theories. Perhaps, the small numbers in our present theory are all related.

If the masses are indeed all small, then the only large terms in our equations are the ones that say how the coupling constants and masses run with scale. Our theory suggests that they might stop running at some scale; in any case, a light Higgs particle indeed follows from the demand that the Higgs self coupling is near an UV fixed point.

The author did not (yet) succeed in finding a physically interesting prototype model with a non-trivial fixed point; this is a very complex, but interesting technical problem. Let us briefly set out a strategy.

We search for a solution where all mass terms, and of course also the cosmological constant, are small. Start with a theory that has nearly, but not quite, a set of β functions that vanish at one-loop. Assume that it has a fixed point near the origin. It is known that Eqs. (3.12)—(3.15) allow this. For simplicity, let us assume that there are no triple scalar couplings, $V_3 = 0$, and no gauge invariant scalars, so that the terms (4.3 c) and (e) are forbidden by gauge invariance. If we deviate slightly from the fixed point, term (b) dictates that we must be at a point where the β function for the scalar self couplings is negative. This gives us a first guess for m_i^4 , the absolute values of the scalar masses, but not the signs of m_i^2 .

Knowing the approximate values of the Yukawa couplings y_k and y_k^5 allows us to fix the Dirac mass matrix m_d using Eq. (4.5). Since we want these masses to be small, we must assume that the pseudoscalar Yukawa couplings largely cancel against the scalar ones in this equation. Then, Eq. (4.4) can be obeyed by moving slightly away from the original fixed point in that direction.

The only remaining equation is then the vanishing of term (d), the running of the scalar mass-squared terms. Knowing V_{4ii}^2 and $\text{Tr}(m_d^2)$, and assuming that m_i^4 is very small, then gives us an equation for m_i^2 . Notice that its sign was still free, so that there is some freedom here. Various further attempts to refine this procedure may well lead to interesting models with fixed points. We do note that, apparently, relatively large pseudoscalar Yukawa couplings y_i^5 are wanted. Also, we are talking about primary, gauge invariant Dirac masses, which have to be there due to the term (4.3 f), while they do not occur in the presently known Standard Model.

5. Discussion.

Our theory derives constraints from the fact that matter fields interact with gravity. The basic assumption could be called a new version of relativity: the scalar matter fields should not be fundamentally different from the dilaton field $\eta(\vec{x}, t)$. Since there are no singular interactions when a scalar field tends to zero, there is no reason to expect any singularity when $\eta(\vec{x}, t)$ tends to zero at some point in space-time. Standard gravity theory does have singularities there: this domain refers to the short distance behavior of gravity, which is usually considered to be “not understood”. What if the short distance behavior of gravity and matter fields is determined by simply demanding the absence of a singularity? Matter and dilaton then join smoothly together in a perfectly conformally invariant theory. This, however, only works if all β functions of this theory vanish: its

coupling parameters must be at a fixed point. There are only discrete sets of such fixed points. Many theories have no fixed point at all in the domain where physical constants are real and positive — that is, stable. Searching for non trivial fixed points will be an interesting and important exercise.

Indeed, *all* physical parameters, including the cosmological constant, will be fixed and calculable in terms of the Planck units. This may be a blessing and a curse at the same time. It is a blessing because this removes all dimensionless freely adjustable real numbers from our theory; everything is calculable, using techniques known today; there is a strictly discrete set of models, where the only freedom we have is the choice of gauge groups and representations. It is difficult to tell how many solutions there are; the number is probably infinite.

This result is also a curse, because the values these numbers have in the real world is strange mix indeed: the range of the absolute values cover some 122 orders of magnitude:

$$\tilde{\Lambda} = \mathcal{O}(10^{-122}) ; \quad \mu_{\text{Higgs}}^2 \approx 3.10^{-36} . \quad (5.1)$$

The question where these various hierarchies of very large, or small, numbers come from is a great mystery called the “hierarchy problem”. In our theory these hierarchies will be difficult to explain, but we do emphasize that the equations are highly complex, and possibly theories with large gauge groups and representations have the potential to generate such numbers.

Our theory is a “top-down” theory, meaning that it explains masses and couplings at or near the Planck domain. It will be difficult to formulate any firm predictions about physics at energies as low as the TeV domain. Perhaps we should expect large regions on a logarithmic scale with an apparently unnatural scaling behavior. There is in principle no supersymmetry, although the mathematics of supersymmetry will be very helpful for constructing the first non-trivial models.

What is missing furthermore is an acceptable description of the dynamics of the remaining parts $\hat{g}_{\mu\nu}$ of the metric field. In Ref. [1], it was suggested that this dynamics may be non quantum mechanical, although this does raise the question how $\hat{g}_{\mu\nu}$ can back react on the presence of quantum matter. Standard quantum mechanics possibly does not apply to $\hat{g}_{\mu\nu}$ because the notion of energy is absent in a conformal theory, and consequently the use of a hamiltonian may become problematic. A hamiltonian can only be defined after coordinates and conformal factor have been chosen, while this is something one might prefer not to do. The author believes that quantum mechanics itself will have to be carefully reformulated before we can really address this problem.

Our theory indeed is complex. We found that the presence of non-Abelian Yang-Mills fields, scalar fields and spinor fields is required, while $U(1)$ gauge fields are forbidden (at least at weak coupling, since the β function for the charges here is known to be positive). Because of this, one “prediction” stands out: there will be magnetic monopoles, although presumably their masses will be of the order of the Planck mass.

Finally, there is one other firm prediction: the constants of nature will indeed be truly constant. Attempts to experimentally observe variations in constants such as the

finestructure constant or the proton electron mass ratio, with time, or position in distant galaxies, are predicted to yield negative results.

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