Risk Aversion and Portfolio Selection in a Continuous-Time Model*

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Abstract: The comparative statics of the optimal portfolios across individuals is carried out for a continuous-time market model, where the risky assets price process follows a joint geometric Brownian motion with time-dependent and deterministic coefficients. It turns out that the indirect utility functions inherit the order of risk aversion (in the Arrow-Pratt sense) from the von Neumann-Morgenstern utility functions, and therefore, a more risk-averse agent would invest less wealth (in absolute value) in the risky assets.

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1. Introduction

Portfolio selection problem is one of the classical problems in the economics of uncertainty. The optimal portfolios depend on agents' characters (preference and wealth level) and on the market's structure (the risk-free return, the return and risk of the risky assets). Various agents would have different allocations of wealth between the risk-free asset and the risky assets, due to the differences in preference and/or the differences in wealth level. The comparative statics of the optimal portfolios with respect to preference

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and/or wealth level has first been carried out by Arrow (1963) and Pratt (1964), for a static model with a risk-free asset and a risky asset. For this model, if the excess return of the risky asset is positive, then (i) the more risk-averse an agent is, the less wealth is invested in the risky asset; and (ii) if an agent displays decreasing absolute (relative) risk aversion, then the amount (proportion) of wealth invested in the risky asset is increasing in wealth.

Since then, decades have passed. Except for some specific cases such as constant absolute (relative) risk aversion in which the solutions can be explicitly worked out, few works have been reported for dynamic models, as far as we know, until Borell (2007). For a continuous-time complete market model, where the risky assets price process follows a joint geometric Brownian motion, and for an agent who only consumes at the terminal time, Borell (2007) has analyzed the changes of the optimal portfolios across the wealth levels. The similar conclusions that hold for the static models have been obtained there by showing the indirect utility function inherits the decreasing absolute (relative) risk aversion from the von Neumann-Morgenstern utility function. (The indirect utility function also inherits the increasing relative risk aversion from the von Neumann-Morgenstern utility function. The preservation of decreasing (increasing) absolute risk aversion has been presented by Gollier (2001), for static and complete models.)

The purpose of this paper is to investigate how the agents' preference impacts the optimal portfolios for a market model with time-dependent and deterministic coefficients. Here we compare the optimal portfolios across individuals instead of across wealth levels. As a result (see Theorem 5.3 and Section 7), we find that the indirect utility functions inherit the order of risk aversion from the von Neumann-Morgenstern utility functions. Observing that the vector of optimal portfolio proportions is given by the vector of log-optimal portfolio proportions multiplied by the indirect relative risk tolerance, we know it is enough for any agent to replace investments in all assets with investments in the risk-free asset and a single "mutual fund", whose portfolio is log-optimal. Based on these facts, a continuous-time version of comparative statics across individuals can be established: the more risk-averse an agent is, the less wealth is invested in the log-optimal portfolio, and hence, the less wealth in absolute value is invested in the risky assets. Using the result here, almost all conclusions in Borell (2007) on comparisons across wealth levels can be easily recovered, as special cases.

The remainder of this paper is organized as follows: Section 2 describes the market model. Section 3 reviews the optimal solutions of portfolio selection problems, where we present the martingale/duality approach. In particular, it is pointed out that the amount of wealth invested in the log-optimal portfolio equals the indirect absolute risk tolerance. Section 4 gives some representations of the indirect absolute risk tolerance and derives a nonlinear parabolic PDE (partial differential equation) for the indirect absolute risk tolerance function. Section 5 presents the main result of this paper and Section 6 recovers the conclusions of Borell (2007). Section 7 extends the main result to an incomplete market.

We shall make use of the following notation: $M^\top$ stands for transposition of a vector or a matrix M; $|\zeta| = \sqrt{\zeta^\top\zeta}$ is the usual Euclidean norm for a vector ζ; $\mathbb{1}$ is the n-dimensional vector of which each component equals 1; and for a domain $\mathcal{D} \subset [0,T] \times (0,\infty)$, $C^{1,\infty}(\mathcal{D})$ denotes the set of all functions $f : \mathcal{D} \to \mathbb{R}$ such that $f(t,x)$ are continuously differentiable with respect to t and infinitely-many times differentiable with respect to x, for all $(t,x) \in \mathcal{D}$; $C(\mathcal{D})$ denotes the set of all continuous functions $f : \mathcal{D} \to \mathbb{R}$.

2. The Financial Market

We consider the typical setup for a continuous-time financial market economy on the finite time span $[0,T]$. The financial market consists of a risk-free asset and n risky assets. The risk-free asset's price process $S^0(t)$ evolves according to the following equation:

$$dS^0(t) = S^0(t)r(t)dt,\ S^0(0) = 1,$$

where $r(t)$ is the interest rate process. The i-th risky asset's price process $S^i(t)$ satisfies the following SDE (stochastic differential equation):

$$dS^i(t) = S^i(t)\left(b^i(t)dt + \sum_{j=1}^{n} \sigma^{ij}(t)dB^j(t)\right),\ S^i(0) > 0,\ 1 \le i \le n.$$

Here $B(t) = (B^1(t), \cdots, B^n(t))^\top$ is an n-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The information structure $(\mathcal{F}_t)$ is the $\mathbb{P}$-augmentation of the filtration generated by $B(t)$ and $\mathcal{F} = \mathcal{F}_T$. Set $b(t) = (b^1(t), \cdots, b^n(t))^\top$ and $\sigma(t) = (\sigma^{ij}(t))_{1\le i,j\le n}$. In this paper, we always assume the coefficients $r(t)$, $b(t)$, and $\sigma(t)$ satisfy the following condition:

Assumption 2.1. *(i) All of $r(t)$, $b^i(t)$ $(1 \le i \le n)$, and $\sigma^{ij}(t)$ $(1 \le i,j \le n)$ are deterministic and continuous functions of t, on $[0,T]$; (ii) the matrix $\sigma(t)$ is non-singular for each t and there exists a constant $c > 0$ such that $\zeta^\top\sigma(t)^{-1}\zeta \ge c|\zeta|^2$ for all $t \in [0,T]$ and $\zeta \in \mathbb{R}^n$.*

In the above setting, the financial market is complete and admits a unique equivalent martingale measure, or risk-neutral measure, denoted by $\mathbb{P}^*$, whose density process is $\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \rho(t)$, where

$$\begin{aligned} \rho(t) &:= \exp\left\{-\int_0^t \theta(\tau)^\top dB(\tau) - \int_0^t \frac{|\theta(\tau)|^2}{2}\,d\tau\right\}, \\ \theta(t) &:= \sigma(t)^{-1}(b(t) - r(t)\mathbb{1}). \end{aligned}$$

By the Girsanov's Theorem,

$$B^*(t) := B(t) + \int_0^t \theta(\tau)\,d\tau$$

is an n-dimensional standard Brownian motion under $\mathbb{P}^*$. Obviously, each risky asset price process satisfies the following equation:

$$dS^i(t) = S^i(t)\left(r(t)dt + \sum_{j=1}^n \sigma^{ij}(t)dB^{*j}(t)\right),\ 1 \le i \le n.$$

The state-price deflator H is defined by

$$H(t) = \exp\left\{-\int_0^t r(\tau)d\tau\right\}\rho(t).$$

It is well known that $S^i(t)H(t)$ is a martingale, for $i = 1, \ldots, n$. For notational simplicity, we set

$$H_t^s = \frac{H(s)}{H(t)},\ 0 \le t \le s \le T.$$

3. Utility Maximization

In this paper, von Neumann-Morgenstern utility functions are defined as follows:

Definition 3.1. *A von Neumann-Morgenstern utility function is a strictly increasing, strictly concave, and twice-continuously-differentiable function* $U : (0, \infty) \to \mathbb{R}$ *that satisfies the Inada condition*

$$U'(0) = \lim_{x\downarrow 0} U'(x) = \infty \quad \textit{and} \quad U'(\infty) = \lim_{x\uparrow\infty} U'(x) = 0.$$

Given a von Neumann-Morgenstern utility function U, the Arrow-Pratt coefficient of absolute risk aversion at x is $-\frac{U''(x)}{U'(x)}$, and the absolute risk tolerance is $-\frac{U'(x)}{U''(x)}$. Accordingly, $-\frac{xU''(x)}{U'(x)}$ is the Arrow-Pratt coefficient of relative risk aversion and $-\frac{U'(x)}{xU''(x)}$ is the relative risk tolerance. Let us introduce an assumption on U that will be used.

Assumption 3.2. *The absolute risk tolerance function of U satisfies the linear growth condition, that is, there is a constant $c > 0$ such that*

$$-\frac{U'(x)}{U''(x)} \leq c(1+x), \text{ for all } x > 0.$$

Following Merton (1971), we assume that (i) there are no transaction costs, taxes, or asset indivisibility; (ii) the agents are price takers; (iii) short sales of all assets, with full use of proceeds, are allowed; and (iv) trading in assets takes place continuously in time.

We consider an agent who consumes only at the terminal time and whose (von Neumann-Morgenstern) utility function for the consumption at the terminal time is U. At any given starting time t, the preference of the agent for the terminal consumption can be represented by the expected utility $\mathbb{E}_t[U(X(T))]$, where $X(T)$ is the value of the terminal wealth and $\mathbb{E}_t$ is the conditional expectation operator at time t. The agent is allowed to allocate the wealth between the risk-free asset and the risky assets to maximize the expected utility. That is, the agent solves the dynamic investment problem

$$\max_{(\phi(s))} \mathbb{E}_t[U(X(T))] \tag{3.1}$$

subject to

$$\begin{cases} dX(s) = \left[X(s)r(s) + \phi(s)^\top(b(s) - r(s)\mathbb{1})\right] ds + \phi(s)^\top \sigma(s) dB(s), \\ X(s) \geq 0, \quad s \in [t, T], \\ X(t) = x, \end{cases} \tag{3.2}$$

where $\phi(s) = (\phi^1(s), \cdots, \phi^n(s))^\top$ is the vector of values of wealth invested in the risky assets at time $s \in [t, T]$, $x > 0$ is the value of wealth at the starting time t. The first constraint in (3.2) is the dynamic budget constraint determining the evolution of the wealth process. The second constraint in (3.2) is the nonnegative wealth constraint ruling out the possibility of create something out of nothing. Here the admissibility of the portfolio process is implicitly assumed to be the nonnegativity of the wealth process.

Following Cox and Huang (1989), see also Karatzas, Lehoczky and Shreve (1987) and Pliska (1986), we can transform the dynamic problem (3.1)-(3.2) into a static one:

$$\begin{aligned} \max_{X(T)\geq 0} \quad & \mathbb{E}_t[U(X(T))] \\ \text{subject to} \quad & \mathbb{E}_t\left[H_t^T X(T)\right] \leq x. \end{aligned} \tag{3.3}$$

In other words, the dynamic budget constraint in (3.2) can be replaced with a static one. The strict concavity of U implies the uniqueness of the solution of problem (3.3).

Since $U'(0) = \infty$, the solution $\hat{X}^{t,x}(T)$ of problem (3.3) is strictly positive and satisfies the first-order condition

$$U'(\hat{X}^{t,x}(T)) = \lambda(t,x)H_t^T, \tag{3.4}$$

where the Lagrangian multiplier $\lambda(t,x) > 0$. Henceforth, we use I to denote the inverse marginal utility function U'^{-1}, that is, $U'(I(y)) = y$, for all $y > 0$. Obviously, I is strictly decreasing and continuously differentiable on $(0,\infty)$, and

$$I(0) = \lim_{y\downarrow 0} I(y) = \infty, \quad I(\infty) = \lim_{y\uparrow\infty} I(y) = 0.$$

With this notation, the solution $\hat{X}^{t,x}(T)$ of problem (3.3) is given by

$$\hat{X}^{t,x}(T) = I\left(\lambda(t,x)H_t^T\right). \tag{3.5}$$

Furthermore, since U is increasing, the static budget constraint is binding:

$$\mathbb{E}_t\left[H_t^T \hat{X}^{t,x}(T)\right] = x,$$

that is, the Lagrangian multiplier $\lambda(t,x)$ satisfies the following equation:

$$\mathbb{E}_t\left[H_t^T I\left(\lambda(t,x)H_t^T\right)\right] = x. \tag{3.6}$$

For any $t \in [0,T]$ and $y > 0$, set

$$\mu(t,y) = \mathbb{E}_t\left[H_t^T I\left(yH_t^T\right)\right]. \tag{3.7}$$

Obviously, the independent increments of Brownian motion yield that μ is a deterministic function defined on $[0,T]\times(0,\infty)$, and for any given t, $\mu(t,y)$ is strictly decreasing with respect to y. Particularly, $\mu(T,y) = I(y)$, for $y > 0$.

We can see from (3.6) and the definition of $\mu(t, y)$ that, for all $t \in [0, T]$ and $x > 0$,

$$\mu(t, \lambda(t, x)) = x. \tag{3.8}$$

Hence, $\lambda(t, x)$ is also a deterministic function defined on $[0, T] \times (0, \infty)$, and for any given t, $\lambda(t, x)$ is strictly decreasing with respect to x. Particularly, $\lambda(T, x) = U'(x)$, for $x > 0$. Moreover, we have the following lemma:

Lemma 3.3. *Under Assumptions 2.1 and 3.2, we have the following two assertions:*

(i) *For any given t, $\mu(t, y)$ is strictly decreasing with respect to y, and $\lim_{y \downarrow 0} \mu(t, y) = \infty$, $\lim_{y \uparrow \infty} \mu(t, y) = 0$;*
(ii) *For any given t, $\lambda(t, x)$ is strictly decreasing with respect to x, and $\lim_{x \downarrow 0} \lambda(t, x) = \infty$, $\lim_{x \uparrow \infty} \lambda(t, x) = 0$.*

Moreover, we have

$$\mu \in C^{1,\infty}([0, T) \times (0, \infty)) \cap C([0, T] \times (0, \infty)), \tag{3.9}$$

$$\lambda \in C^{1,\infty}([0, T) \times (0, \infty)) \cap C([0, T] \times (0, \infty)). \tag{3.10}$$

Proof: See Appendix A. □

Let $u(t, x)$ denote the value function of the problem (3.1)-(3.2), or the indirect utility function, given that the value of wealth at the starting time t is x. That is

$$u(t, x) = \mathbb{E}_t \left[U \left(I \left(\lambda(t, x) H_t^T \right) \right) \right], \tag{3.11}$$

for all $(t, x) \in [0, T] \times (0, \infty)$. The independent increments of Brownian motion yield that u is a deterministic function defined on $[0, T] \times (0, \infty)$. Moreover, we have the following lemma:

Lemma 3.4. *Under Assumptions 2.1 and 3.2,*

$$u \in C^{1,\infty}([0, T) \times (0, \infty)) \cap C([0, T] \times (0, \infty)).$$

Proof: See Appendix A. □

We now derive the feedback form of the optimal portfolio policy by the martingale approach, following the argument in Karatzas and Shreve (1998) Section 3.8.

We use $\{\hat{X}^{t,x}(s),\, t \le s \le T\}$ to denote the corresponding optimal wealth process for problem (3.1)-(3.2), that is, the wealth process of the portfolio replicating $\hat{X}^{t,x}(T)$. It is well known that $\{\hat{X}^{t,x}(s)H(s),\, t \le s \le T\}$ is a martingale. Then, for all $s \in [t, T]$,

$$
\begin{aligned}
\hat{X}^{t,x}(s) &= \frac{1}{H(s)}\,\mathbb{E}_s[H(T)\hat{X}^{t,x}(T)] \\
&= \mathbb{E}_s\left[H_s^T I\left(\lambda(t,x)H_t^T\right)\right] \\
&= \mathbb{E}_s\left[H_s^T I\left(\lambda(t,x)H_t^s H_s^T\right)\right] \\
&= \mu\left(s, \lambda(t,x)H_t^s\right), \qquad (3.12)
\end{aligned}
$$

where (3.12) follows from (3.7) and the fact that $\lambda(t,x)H_t^s$ is $\mathcal{F}_s$-measurable. Therefore, by (3.8) and (3.12), we have

$$\lambda\left(s, \hat{X}^{t,x}(s)\right) = \lambda(t,x)H_t^s,\;\; s \in [t,T]. \qquad (3.13)$$

Moreover, applying Itô's formula to (3.12) leads to

$$
\begin{aligned}
& d\hat{X}^{t,x}(s) \\
= \;& \mu_y\left(s, \lambda(t,x)H_t^s\right)\lambda(t,x)\, d_s H_t^s + \text{ a } ds\text{-term} \\
= \;& -\mu_y\left(s, \lambda(t,x)H_t^s\right)\lambda(t,x)\, H_t^s\, \theta(s)^\top dB(s) + \text{drift part} \\
= \;& -\mu_y\left(s, \lambda\left(s, \hat{X}^{t,x}(s)\right)\right)\lambda\left(s, \hat{X}^{t,x}(s)\right)\, \theta(s)^\top dB(s) + \text{drift part} \\
= \;& -\frac{\lambda\left(s, \hat{X}^{t,x}(s)\right)}{\lambda_x\left(s, \hat{X}^{t,x}(s)\right)}\,\theta(s)^\top dB(s) + \text{drift part},
\end{aligned}
$$

where the last two equalities follow from (3.13) and (3.8), respectively. Comparing the preceding equation and the first constraint in (3.2), we know the optimal portfolio $\hat{\phi}$ satisfies

$$\hat{\phi}(s) = -\frac{\lambda\left(s, \hat{X}^{t,x}(s)\right)}{\lambda_x\left(s, \hat{X}^{t,x}(s)\right)}(\sigma(s)\sigma(s)^\top)^{-1}(b(s) - r(s)\mathbb{1}).$$

The Lagrangian multiplier $\lambda(t,x)$ gives the marginal, or shadow, value of relaxing the static budget constraint in (3.3). It therefore equals the agent's marginal utility of wealth at the optimum, that is

$$\lambda(t,x) = u_x(t,x). \qquad (3.14)$$

Therefore, the optimal portfolio $\hat{\phi}$ satisfies

$$\hat{\phi}(s) = -\frac{u_x\left(s, \hat{X}^{t,x}(s)\right)}{u_{xx}\left(s, \hat{X}^{t,x}(s)\right)}(\sigma(s)\sigma(s)^\top)^{-1}(b(s) - r(s)\mathbb{1}),$$

which implies that the optimal portfolio policy $\hat{\phi}$ is in the following feedback form:

$$\hat{\phi}(s, x) = -\frac{u_x(s, x)}{u_{xx}(s, x)}\,(\sigma(s)\sigma(s)^\top)^{-1}(b(s) - r(s)\mathbb{1}).$$

That is,

$$\hat{\phi}(s, x) = f(s, x)(\sigma(s)\sigma(s)^\top)^{-1}(b(s) - r(s)\mathbb{1}), \tag{3.15}$$

where

$$f(s, x) = -\frac{u_x(s, x)}{u_{xx}(s, x)}, \text{ for all } (s, x) \in [t, T] \times (0, \infty).$$

Remark 3.5. *For every time $s \in [t, T]$, we have represented the optimal portfolio $\hat{\phi}(s)$ in feedback form on the level of wealth at time s, in terms of the indirect utility function $u(s, \cdot)$ and the instantaneous market coefficients $r(s)$, $b(s)$, and $\sigma(s)$. The optimal portfolio policy depends on neither the starting time nor the starting wealth level. Thus* (3.15) *always gives the optimal portfolio policy in feedback form, regardless of the specification of starting time and the starting wealth level.*

Remark 3.6. *Usually, the feedback form of the optimal portfolio policy is derived from HJB (Hamilton-Jacobi-Bellman) equation. Actually, following Merton (1971), the principle of dynamic programming leads to the following HJB equation for u:*

$$\max_{\phi}\left\{u_t + [rx + \phi^\top(b - r\mathbb{1})]\,u_x + \frac{1}{2}\,\phi^\top\sigma\sigma^\top\phi\,u_{xx}\right\} = 0, \tag{3.16}$$

on $[0, T) \times (0, \infty)$, as well as the terminal condition

$$u(T, x) = U(x), \text{ for all } x > 0. \tag{3.17}$$

The first-order condition for the maximality in (3.16) *yields* (3.15)*. It, however, remains to verify the polity is optimal if it is derived from HJB equation. Here, we have just proved that the optimal portfolio policy is in the feedback form* (3.15) *by the martingale approach, following the essentially same argument as in Karatzas and Shreve (1998) Section 3.8.*

It is well known that

$$u_x(t,x) > 0 \text{ and } u_{xx}(t,x) < 0, \text{ for all } (t,x) \in [0,T] \times (0,\infty).$$

Then

$$f(t,x) > 0, \text{ for all } (t,x) \in [0,T] \times (0,\infty).$$

$f(t,x)$ is the absolute risk tolerance of the indirect utility function. In this paper, we call it the *indirect absolute risk tolerance function*. Accordingly, we call $\frac{f(t,x)}{x}$ the *indirect relative risk tolerance function*.

In view of (3.15), the vector of optimal portfolio proportions is

$$\frac{\hat{\phi}(t,x)}{x} = \frac{f(t,x)}{x}(\sigma(t)\sigma(t)^\top)^{-1}(b(t) - r(t)\mathbb{1}), \tag{3.18}$$

whose components represent the proportions of total wealth held in the risky assets. In particular, for logarithmic utility function $U(x) = \log x$, whose relative risk tolerance is constant and equals 1, it is well know (see, e.g., Merton (1969)) that the indirect relative risk tolerance $\frac{f(t,x)}{x} = 1$, for all $(t,x) \in [0,T] \times (0,\infty)$, and hence, by (3.18), the vector of optimal portfolio proportions is

$$(\sigma(t)\sigma(t)^\top)^{-1}(b(t) - r(t)\mathbb{1}),$$

which, hereafter, is called the *vector of log-optimal portfolio proportions*.

Notice that, for any von Neumann-Morgenstern utility function U, the vector of optimal portfolio proportions is given by the vector of log-optimal portfolio proportions multiplied by the indirect relative risk tolerance. This means effectively that it is enough for any agent to replace investments in all assets with investments in the risk-free asset and a single "mutual fund", whose portfolio is log-optimal. Different agents would have different weights between the log-optimal portfolio and the risk-free asset, depending on their indirect relative risk tolerance. The weight of total wealth invested in the log-optimal portfolio equals the indirect relative risk tolerance. The larger the indirect relative risk tolerance is, the larger weight is invested in the log-optimal portfolio. Accordingly, the amount of wealth invested in the log-optimal portfolio equals the indirect absolute risk tolerance. The larger the indirect absolute risk tolerance is, the more wealth is invested in the log-optimal portfolio.

4. Indirect Absolute Risk Tolerance Functions

Now we investigate the indirect absolute risk tolerance function f.

By (3.14), the first-order condition (3.4) can be rewritten as

$$U'(\hat{X}^{t,x}(T)) = u_x(t,x)H_t^T. \tag{4.1}$$

Moreover, the budget constraint is binding:

$$\mathbb{E}_t\left[\hat{X}^{t,x}(T)\,H_t^T\right] = x. \tag{4.2}$$

Lemma 4.1. *Under Assumptions 2.1 and 3.2, we have for any $t \in [0,T)$ that*

$$\mathbb{E}_t\left[\frac{\partial \hat{X}^{t,x}(T)}{\partial x}\,H_t^T\right] = 1, \ \textit{for all } x > 0. \tag{4.3}$$

Proof: We can obtain (4.3) by differentiating formally the both sides of (4.2) with respect to x. For a rigorous proof, see Appendix B. □

Differentiating (4.1) with respect to x yields

$$U''(\hat{X}^{t,x}(T))\frac{\partial \hat{X}^{t,x}(T)}{\partial x} = u_{xx}(t,x)H_t^T,$$

and consequently, by (4.1) once again,

$$-\frac{U'(\hat{X}^{t,x}(T))}{U''(\hat{X}^{t,x}(T))} = -\frac{u_x(t,x)}{u_{xx}(t,x)}\frac{\partial \hat{X}^{t,x}(T)}{\partial x} = f(t,x)\frac{\partial \hat{X}^{t,x}(T)}{\partial x}. \tag{4.4}$$

Proposition 4.2. *Under Assumptions 2.1 and 3.2, for all $t \in [0,T]$ and $x > 0$,*

$$f(t,x) = \mathbb{E}_t\left[-\frac{U'(\hat{X}^{t,x}(T))}{U''(\hat{X}^{t,x}(T))}H_t^T\right] \tag{4.5}$$

$$= \mathbb{E}_t\left[-\frac{U'\left(I\left(\lambda(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda(t,x)H_t^T\right)\right)}H_t^T\right]. \tag{4.6}$$

Moreover, $f(t,x)$ is uniformly linearly growing in x, that is, there exists a constant $c' > 0$ such that :

$$f(t,x) \le c'(1+x), \ \textit{for all } (t,x) \in [0,T]\times(0,\infty). \tag{4.7}$$

Proof: We can obtain (4.5) from (4.4) and Lemma 4.1. From (3.5), we get (4.6) as well. Moreover, by (4.5) and Assumption 3.2,

$$f(t,x) \le c\,\mathbb{E}_t\left[(1+\hat{X}^{t,x}(T))H_t^T\right] = ce^{-\int_t^T r(s)ds} + cx,$$

which with Assumption 2.1 clearly leads to (4.7). □

Remark 4.3. *Gollier (2001) obtained the same conclusion as in the preceding proposition, for static models.*

Proposition 4.4. *Under Assumptions 2.1 and 3.2,*

$$\{f(s,\hat{X}^{t,x}(s))H(s),\ s\in[t,T]\}$$

is a martingale, for every $t\in[0,T)$ and every $x>0$.

Proof: By (4.6), for all $s\in[t,T]$,

$$\begin{aligned}
& f(s,\hat{X}^{t,x}(s)) \\
= \ & \mathbb{E}_s\left[-\frac{U'\left(I\left(\lambda\left(s,\hat{X}^{t,x}(s)\right)H_s^T\right)\right)}{U''\left(I\left(\lambda\left(s,\hat{X}^{t,x}(s)\right)H_s^T\right)\right)}H_s^T\right] \quad (\text{by } \hat{X}^{t,x}(s)\in\mathcal{F}_s) \\
= \ & \mathbb{E}_s\left[-\frac{U'\left(I\left(\lambda(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda(t,x)H_t^T\right)\right)}H_s^T\right] \quad (\text{by (3.13)}) \\
= \ & \mathbb{E}_s\left[-\frac{U'(\hat{X}^{t,x}(T))}{U''(\hat{X}^{t,x}(T))}H_s^T\right] \quad (\text{by (3.5)}) \\
= \ & \mathbb{E}_s\left[f(T,\hat{X}^{t,x}(T))\frac{H(T)}{H(s)}\right].
\end{aligned}$$

Thus $\{f(s,\hat{X}^{t,x}(s))H(s), s\in[t,T]\}$ is a martingale. □

Remark 4.5. *It was first pointed out by Cox and Leland (1982) that*

$$\{f(s,\hat{X}^{t,x}(s))H(s),\ s\in[t,T]\}$$

is a local martingale when the risky asset price process is a geometric Brownian motion. By differentiating HJB equations, He and Huang (1994) observed it is a general property of an optimal consumption-portfolio policy. Here, we have shown it is a martingale, by a simple and pure probabilistic method, based on Proposition 4.2.

Based on the martingale property in the preceding proposition, the indirect absolute risk tolerance function f turns out to satisfy a nonlinear parabolic PDE, as shown in the next proposition.

Proposition 4.6. *Under Assumptions 2.1 and 3.2,*

$$f \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)),$$

and f satisfies PDE

$$\frac{1}{2}|\theta|^2 f^2 f_{xx} + rxf_x + f_t - rf = 0, \tag{4.8}$$

on $[0,T) \times (0,\infty)$, as well as the terminal condition

$$f(T,x) = -\frac{U'(x)}{U''(x)}, \text{ for all } x > 0. \tag{4.9}$$

PROOF: See Appendix B. □

Remark 4.7. *Just as in the present paper, the indirect absolute risk tolerance function also plays a fundamental role in the analysis in Kramkov and Sîrbu (2006, 2007), Musiela and Zariphopoulou (2006, 2008), Zariphopoulou and Zhou (2007).*

5. Comparisons Across Individuals

Apart from an agent with (von Neumann-Morgenstern) utility function U as described in the previous sections, we consider another agent whose (von Neumann-Morgenstern) utility function is V. Just like the arguments in the previous sections, we will use the following assumption:

Assumption 5.1. *The absolute risk tolerance function of V satisfies the linear growth condition, that is, there is a constant $c > 0$ such that*

$$-\frac{V'(x)}{V''(x)} \leq c(1+x), \text{ for all } x > 0.$$

The agent whose (von Neumann-Morgenstern) utility function is V solves the following dynamic investment problem:

$$\max_{(\psi(s))} \mathbb{E}_t[V(X(T))] \tag{5.1}$$

subject to

$$\begin{cases} dX(s) = \Big[X(s)r(s) + \psi^\top(s)(b(s) - r(s)\mathbb{1})\Big]ds + \psi^\top(s)\sigma(s)dB(s), \\ X(s) \geq 0, \text{ for } s \in [t,T], \\ X(t) = x, \end{cases} \tag{5.2}$$

where $\psi(s) = (\psi^1(s), \cdots, \psi^n(s))^\top$ is the vector of values of wealth invested in the risky assets at time $s \in [t, T]$.

Let $v(t, x)$ denote the indirect utility function for problem (5.1)-(5.2). From Lemma 3.4, we know

$$v \in C^{1,\infty}([0, T) \times (0, \infty)) \cap C([0, T] \times (0, \infty)),$$

provided Assumptions 2.1 and 5.1 are satisfied. The corresponding indirect absolute risk tolerance function is

$$g(t, x) = -\frac{v_x(t, x)}{v_{xx}(t, x)},$$

and the optimal portfolio policy $\hat{\psi}$ is in the following feedback form:

$$\hat{\psi}(t, x) = g(t, x)(\sigma(t)\sigma(t)^\top)^{-1}(b(t) - r(t)\mathbb{1}). \tag{5.3}$$

Given two utility functions U and V, we say U is more risk-averse than V, or U is less risk-tolerant than V, if $-\frac{U''(x)}{U'(x)} \geq -\frac{V''(x)}{V'(x)}$, that is, $-\frac{U'(x)}{U''(x)} \leq -\frac{V'(x)}{V''(x)}$, for every $x > 0$. It is well known that U is more risk-averse than V if and only if there exists an increasing concave function F such that $U(x) = F(V(x))$, for all x; that is, U is a concave transformation of V (In other words, U is "more concave" than V.)

Remark 5.2. *Obviously, if V satisfies Assumption 5.1 and U is more risk-averse that V, then U satisfies Assumption 3.2.*

Now we are ready to report the main result of this paper.

Theorem 5.3. *Under Assumptions 2.1 and 5.1, assume further that U is more risk-averse than V, then $f(t, x) \leq g(t, x)$, for all $t \in [0, T)$ and $x > 0$.*

PROOF: See Appendix D. □

Remark 5.4. *It seems that we could apply the techniques of maximum principles for parabolic PDEs to prove $f \leq g$. However, due to the nonlinearity of PDE (4.8) and the unboundedness of the domain $[0, T) \times (0, \infty)$, the techniques of maximum principles can not be directly applied to f and g. In order to overcome this point, we approximate f with a sequence $\{f^{(m)}, m \geq 2\}$ of functions, which satisfy the PDEs with bounded domains. For these PDEs with bounded domains, we can use the techniques of maximum principles to show $f^{(m)} \leq g$, then by approximation, we have $f \leq g$. The approximating sequence is constructed in Appendix C, and the proof of the theorem is completed in Appendix D. Although what we want is to prove a comparison theorem of PDE, the method here is mainly probabilistic.*

The preceding theorem shows: If the agent with utility function $U(x)$ is more risk-averse than the agent with utility function $V(x)$, then, for each time $t \in [0, T)$, the corresponding indirect utility function $u(t, x)$ is more risk-averse than $v(t, x)$ as well. According to the discussion at the end of Section 3, at each time $t \in [0, T)$, if the agents have the same value of wealth, then the former agent invests less in the log-optimal portfolio (and hence, less in absolute value of wealth in the risky assets) than the later. So, we have established a continuous-time version of the comparative statics of the optimal portfolios across individuals.

Remark 5.5. *The conclusion of Theorem 5.3 also holds for incomplete markets with deterministic coefficients, see Section 7.*

6. Comparisons Across Wealth Levels

In this section, we recover the conclusions in Borell (2007), based on Proposition 4.2 and Theorem 5.3.

A utility function U is called to exhibit decreasing absolute risk aversion (henceforth, DARA) [resp. increasing absolute risk aversion (henceforth, IARA)], if $-\frac{U''(x)}{U'(x)}$ is decreasing [resp. increasing] with respect to x. Accordingly, U is called to exhibit decreasing relative risk aversion (henceforth, DRRA) [resp. increasing relative risk aversion (henceforth, IRRA)] if $-\frac{xU''(x)}{U'(x)}$ is decreasing [resp. increasing] with respect to x.

Theorem 6.1. *Under Assumptions 2.1 and 3.2, if U exhibits DARA, then for each t, $u(t, \cdot)$ exhibits DARA, namely, $f(t, x)$ is increasing with respect to x.*

Proof: We have known that I is strictly decreasing, and for each t, $\lambda(t, x)$ is strictly decreasing with respect to x. Then the assertions follows from (4.6). □

Remark 6.2. *The preservation of DARA has already been reported by Borell (2007), for a continuous-time complete model; see (Gollier, 2001, pp.209-210), for a static complete model. The method used here is same to that of Gollier (2001). For a static complete model, Gollier (2001) has also showed the preservation of IARA. But in our settings, as observed by (Borell, 2007, p.144), the assumption $U'(0) = \infty$ totally eliminates utility functions U exhibiting IARA (see also Lemma A.1). We believe, in our continuous-time setting, the preservation of IARA can be proved as well, by considering the utility functions defined on the whole real line $(-\infty, \infty)$, instead of the positive real line $(0, \infty)$.*

We can see from the preceding theorem and the discussion at the end of Section 3 that, if the utility function of the agent exhibits DARA, then the amount of wealth invested in the log-optimal portfolio is increasing as the total wealth rises.

As for the relative risk aversion, we have the following theorem, whose conclusion has already been reported by Borell (2007). The methodology here, however, is different from there. According to the method of Borell (2007), the IRRA case is much more complicate than the DRRA case. According to the method here, however, both cases can be easily dealt with, based on Theorem 5.3.

Theorem 6.3. *Under Assumptions 2.1 and 3.2, we have the following two assertions:*

(i) *If U exhibits DRRA, then for each t, $u(t,\cdot)$ exhibits DRRA, namely, $\frac{f(t,x)}{x}$ is increasing with respect to x;*

(ii) *If U exhibits IRRA, then for each t, $u(t,\cdot)$ exhibits IRRA, namely, $\frac{f(t,x)}{x}$ is decreasing with respect to x.*

Proof: Suppose U exhibits DRRA. For any constant $\gamma > 1$, consider the utility function V defined by $V(x) = U(\gamma x)$, for all $x > 0$. Obviously, we have

$$-\frac{xV''(x)}{V'(x)} = -\frac{\gamma x U''(\gamma x)}{U'(\gamma x)} \leq -\frac{xU''(x)}{U'(x)},$$

which yields U is more risk averse than V. Then by Theorem 5.3,

$$f(t,x) \leq g(t,x), \text{ for all } (t,x) \in [0,T] \times (0,\infty). \tag{6.1}$$

Moreover, it is easy to see $v(t,x) = u(t,\gamma x)$, for all $t \in [0,T]$ and $x > 0$. By computation, $g(t,x) = \frac{f(t,\gamma x)}{\gamma}$. Thus we get from (6.1) that $\frac{f(t,x)}{x} \leq \frac{f(t,\gamma x)}{\gamma x}$, for all $t \in [0,T]$ and $x > 0$. By the arbitrariness of $\gamma > 1$, we have proved assertion (i). Assertion (ii) can be proved by letting $\gamma \in (0,1)$ and by the same way. □

We can see from the preceding theorem and the discussion at the end of Section 3 that, if the utility function of the agent exhibits DRRA (resp. IRRA), then the weight of wealth invested in the log-optimal portfolio is increasing (resp. decreasing) as the total wealth rises.

7. Extension to an Incomplete Market

In a complete market, the optimal portfolio is given by (3.15). If the excess expected rate of return of asset i is zero, that is, $b^i(t) - r(t) = 0$, then no wealth will be invested in asset i. Based on this observation, the extension to an incomplete market with deterministic coefficients is straightforward.

Instead of a financial market with n risky assets and driven by an n-dimensional standard Brownian motion, we now consider an incomplete financial market with n risky assets and driven by a d-dimensional standard Brownian motion, $d > n$. The i-th risky asset's price process $S^i(t)$ satisfies the following equation:

$$dS^i(t) = S^i(t)\left(b^i(t)dt + \sum_{j=1}^{d} \sigma^{ij}(t)dB^j(t)\right), \; S^i(0) > 0, \; 1 \le i \le n.$$

Here $B(t) = (B^1(t), \cdots, B^d(t))^\top$ is a d-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Set $b(t) = (b^1(t), \cdots, b^n(t))^\top$ and $\sigma(t) = (\sigma^{ij}(t))_{1\le i\le n, 1\le j\le d}$. We assume the coefficients $b(t)$ and $\sigma(t)$ are deterministic, and $\operatorname{rank}(\sigma(t)) = n$.

Now we consider other $(d-n)$ risky assets whose price processes satisfies

$$dS^i(t) = S^i(t)\left(r(t)dt + \sum_{j=1}^{d} \sigma^{ij}(t)dB^j(t)\right), \; S^i(0) > 0, \; n+1 \le i \le d,$$

where the expected instantaneous rate of return of each asset is the risk-free rate $r(t)$ and the coefficient $(\sigma^{ij}(t))_{n+1\le i\le d, 1\le j\le d}$ makes the enlarged matrix

$$(\sigma^{ij}(t))_{1\le i\le d, 1\le j\le d}$$

non-singular. If we add these $(d-n)$ risky assets into the original market, then we have a larger market with d risky assets. Obviously, the larger market is complete. The larger market will not make an agent better off than the original market: no wealth will be invested in asset i, $i = n+1, \ldots, d$, since the expected instantaneous rate of return of each asset i, $i = n+1, \ldots, d$, is the risk-free rate $r(t)$. In the larger market, an agent will have the same optimal portfolio as in the original market. Consequently, the conclusion of Theorem 5.3 also holds for incomplete market with deterministic coefficients.

Appendix A: Supplementary Data for Section 3

Lemma A.1. *If* $U'(0) = \infty$*, then* $\liminf_{x \downarrow 0} -\frac{U'(x)}{U''(x)} = 0$.

Proof: Suppose otherwise that $\liminf_{x \downarrow 0} -\frac{U'(x)}{U''(x)} = A > 0$, then there exists a $x_0 \in (0, 1)$ such that, $-\frac{U'(x)}{U''(x)} > \frac{A}{2}$, for all $x \in (0, x_0)$. Then we have

$$(\log U'(x))' = \frac{U''(x)}{U'(x)} > -\frac{2}{A}, \text{ for } x \in (0, x_0),$$

and therefore,

$$\log U'(1) - \log U'(x) = \int_x^1 \frac{U''(z)}{U'(z)}\, dz > \frac{2(x-1)}{A}, \text{ for } x \in (0, x_0).$$

Thus $U'(x) < U'(1)e^{\frac{2(1-x)}{A}}$, for $x \in (0, x_0)$, which is impossible, since $U'(0) = \infty$. □

Lemma A.2. *For any von Neumann-Morgenstern utility function* U*, there exist constants* $c_0 > 0$ *and* $c_1 > 0$ *such that*

(A.1) $$|U(I(y))| \leq \max\{c_0 + y, c_0 + c_1 I(y)\}, \text{ for all } y > 0.$$

Proof: The concavity of U implies

(A.2) $$U(I(y)) \leq U(1) + U'(1)(I(y) - 1), \text{ for all } y > 0.$$

On the other hand, it is well know that $U(I(y)) - yI(y) = \sup_{x>0}[U(x) - yx]$. Then

(A.3) $$-U(I(y)) \leq -yI(y) - U(1) + y \leq -U(1) + y, \text{ for all } y > 0.$$

Finally, a combination of (A.2) and (A.3) yields the assertion. □

Lemma A.3. *If von Neumann-Morgenstern utility function* U *satisfies Assumption 3.2, then we have*

(A.4) $$U'(x) \leq U'(1)\left(\frac{1+x}{2}\right)^{-\frac{1}{c}}, \text{ for all } x > 1,$$

(A.5) $$I(y) \leq 2(U'(1))^c y^{-c} - 1, \text{ for all } y \in (0, U'(1)).$$

Proof: By Assumption 3.2,

$$(\log U'(x))' = \frac{U''(x)}{U'(x)} \le -\frac{1}{c(1+x)}, \text{ for all } x > 0.$$

Then for all $x > 1$,

$$\begin{aligned}\log U'(x) - \log U'(1) &= \int_1^x (\log U'(z))' dz \\ &\le -\int_1^x \frac{dz}{c(1+z)} \\ &= -\frac{1}{c} \log \frac{1+x}{2},\end{aligned}$$

yielding (A.4). Suppose $y < U'(1)$, then $I(y) > 1$, and therefore, by (A.4),

$$y = U'(I(y)) \le U'(1) \left(\frac{1 + I(y)}{2} \right)^{-\frac{1}{c}},$$

which implies (A.5). □

Lemma A.4. *If von Neumann-Morgenstern utility function U satisfies Assumption 3.2, then we have for all $a > 0$ that*

$$\int_{-\infty}^{\infty} e^z I(e^z)\, e^{-az^2}\, dz < \infty, \tag{A.6}$$

$$\int_{-\infty}^{\infty} |U(I(e^z))|\, e^{-az^2}\, dz < \infty. \tag{A.7}$$

Proof: By Lemma A.3, we have

$$I(e^z) \le \begin{cases} 2(U'(1))^c\, e^{-cz} - 1, & \text{if } e^z < U'(1); \\ 1, & \text{if } e^z \ge U'(1), \end{cases} \tag{A.8}$$

which obviously yields (A.6). A combination of (A.1) and (A.8) leads to (A.7). □

We refer to (Karatzas and Shreve, 1991, pp.254-255) for the following lemma:

Lemma A.5. *Suppose $k : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function satisfying the condition*

$$\int_{-\infty}^{\infty} |k(z)| e^{-az^2} dz < \infty,$$

for some $a > 0$. Set

$$\kappa(t,z) = \mathbb{E}[k(z+\sqrt{t}\,\xi)],\ \ (t,z) \in \left[0, \frac{1}{2a}\right) \times \mathbb{R},$$

where $\xi \sim \mathcal{N}(0,1)$, the standard normal distribution. Then κ has continuous derivatives of all orders, for all $t \in \left(0, \frac{1}{2a}\right)$ and $z \in \mathbb{R}$. Moreover, if k is continuous at $z_0 \in \mathbb{R}$, then κ is continuous at $(0, z_0)$. Particularly, if k is continuous on $\mathbb{R}$, then κ is continuous on $\left[0, \frac{1}{2a}\right) \times \mathbb{R}$.

Proof of Lemma 3.3:
Assertions (i) and (ii) are clear. It remains to show (3.9) and (3.10).

Let $k(z) = e^z I(e^z)$, for all $z \in \mathbb{R}$. Obviously, $k(z) > 0$, for all $z \in \mathbb{R}$. By Lemma A.4, $\int_{-\infty}^{\infty} k(z) e^{-az^2}\, dz < \infty$, for all $a > 0$. Set

$$\kappa(t,z) = \mathbb{E}[k(z+\sqrt{t}\,\xi)],\ \ (t,x) \in [0,\infty) \times \mathbb{R},$$

where $\xi \sim \mathcal{N}(0,1)$. Then by Lemma A.5, κ has continuous derivatives of all orders, for all $t \in (0,\infty)$ and $z \in \mathbb{R}$, and is continuous on $[0,\infty) \times \mathbb{R}$. Since

$$\log\left(yH_t^T\right) = \log y - \int_t^T r(s)ds - \frac{1}{2}\Theta(t) - \int_t^T \theta(s)^\top dB(s),$$

where

$$\Theta(t) = \int_t^T |\theta(s)|^2 ds,\ \text{ for all } t \in [0,T],$$

we can see from (3.7) that

$$\begin{aligned} \mu(t,y) &= \frac{1}{y}\,\mathbb{E}_t\left[k\left(\log\left(yH_t^T\right)\right)\right] \\ &= \frac{1}{y}\,\kappa\left(\Theta(t),\ \log y - \int_t^T r(s)ds - \frac{1}{2}\Theta(t)\right), \end{aligned}$$

and therefore, (3.9) follows. Here we have used the fact that Θ is continuously differentiable on $[0,T]$, which is obvious under Assumption 2.1.

Now we prove $\lambda \in C^{1,\infty}([0,T) \times (0,\infty))$, using the Implicit Function Theorem (see Zorich (2004)). To this end, we extent the definition of μ to a open domain containing $[0,T) \times (0,\infty)$. Let $\beta : (-\infty,T] \to [0,\infty)$ and $\gamma : (-\infty,T] \to \mathbb{R}$ be defined as follows:

$$\begin{aligned} \beta(t) &= \begin{cases} \Theta(t), & \text{for } t \in [0,T], \\ \Theta(0) - |\theta(0)|^2 t, & \text{for } t < 0, \end{cases} \\ \gamma(t) &= \begin{cases} \int_t^T r(s)ds, & \text{for } t \in [0,T], \\ \int_0^T r(s)ds - r(0)t, & \text{for } t < 0. \end{cases} \end{aligned}$$

Obviously, β and γ are continuously differentiable on $(-\infty, T]$. Let $\tilde{\mu}$ be defined by

$$\tilde{\mu}(t,y) = \frac{1}{y}\,\kappa\left(\beta(t), \log y - \gamma(t) - \frac{1}{2}\beta(t)\right),\ \text{for } (t,y) \in (-\infty, T] \times (0,\infty).$$

Then $\tilde{\mu} \in C^{1,\infty}((-\infty, T) \times (0,\infty))$. It is not difficult to see that, for each $t \in (-\infty, T]$,

$$\tilde{\mu}(t,y) = \mathbb{E}\left[e^{-\gamma(t)-\frac{1}{2}\beta(t)+\sqrt{\beta(t)}\,\xi} I\left(ye^{-\gamma(t)-\frac{1}{2}\beta(t)+\sqrt{\beta(t)}\,\xi}\right)\right]$$

is strictly decreasing with respect to y, on $(0,\infty)$. Let $\tilde{\lambda}$ be defined by

$$\tilde{\mu}(t, \tilde{\lambda}(t,x)) = x,\ \text{for } (t,x) \in (-\infty, T] \times (0,\infty).$$

Then by the Implicit Function Theorem, $\tilde{\lambda} \in C^{1,\infty}((-\infty, T)\times(0,\infty))$. Moreover, it is easy to see that $\tilde{\mu} = \mu$, on $[0,T] \times (0,\infty)$, and therefore, recalling (3.8), $\tilde{\lambda} = \lambda$, on $[0,T] \times (0,\infty)$. Thus we have $\lambda \in C^{1,\infty}([0,T) \times (0,\infty))$.

It remains to show λ is continuous at (T,x), for all $x > 0$. Let sequence $(t_n, x_n) \in [0,T] \times (0,\infty)$ converge to (T,x), for some $x > 0$, we shall prove $\lim_{n\to\infty} \lambda(t_n, x_n) = U'(x)$. Suppose $\liminf_{n\to\infty} \lambda(t_n, x_n) < U'(x)$, then there exists some $y_0 \in (0, U'(x))$ such that $\liminf_{n\to\infty} \lambda(t_n, x_n) < y_0$, then

$$x = \lim_{n\to\infty} \mu(t_n, \lambda(t_n, x_n)) \geq \liminf_{n\to\infty} \mu(t_n, y_0) = \mu(T, y_0) = I(y_0) > x,$$

which leads to a contradiction. Thus $\liminf_{n\to\infty} \lambda(t_n, x_n) < U'(x)$ is impossible. Similarly, we can prove $\limsup_{n\to\infty} \lambda(t_n, x_n) > U'(x)$ is impossible. So we complete the proof. □

Lemma A.6. *Suppose Assumption 2.1 is satisfied and $q : (0,\infty) \to \mathbb{R}$ is a Borel-measurable function satisfying the condition*

$$\int_0^\infty |q(e^z)| \exp\left\{-\frac{z^2}{2(\Theta(0)+\varepsilon)}\right\} dz < \infty,\ \textit{for some } \varepsilon > 0,$$

where $\Theta(t) = \int_t^T |\theta(s)|^2 ds$, for all $t \in [0,T]$. Let ν be defined by

$$\nu(t,y) = \mathbb{E}_t\left[q\left(yH_t^T\right)\right],\ (t,y) \in [0,T] \times (0,\infty).$$

Then $\nu \in C^{1,\infty}([0,T)\times(0,\infty))$. Moreover, if q is continuous at $y_0 > 0$, then ν is continuous at (T, y_0). Particularly, if q is continuous on $(0,\infty)$, then $\nu \in C([0,T] \times (0,\infty))$.

PROOF: It is just similar to the proof of (3.9). □

Based on the previous lemmas, we are ready to prove Lemma 3.4.

PROOF OF LEMMA 3.4:
Let $q(y) = U(I(y))$ and

$$\nu(t,y) = \mathbb{E}_t\left[q\left(yH_t^T\right)\right] = \mathbb{E}_t\left[U\left(I\left(yH_t^T\right)\right)\right],$$

then by Lemmas A.4 and A.6,

$$\nu \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)).$$

Moreover, recalling (3.11), $u(t,x) = \nu(t, \lambda(t,x))$. Then by Lemma 3.3, we finish the proof. □

Appendix B: Supplementary Data for Section 4

PROOF OF LEMMA 4.1:
We can rewrite (4.1) as

$$\hat{X}^{t,x}(T) = I\left(u_x(t,x)H_t^T\right).$$

Differentiating the preceding equality with respect to x yields

$$\frac{\partial \hat{X}^{t,x}(T)}{\partial x} = I'\left(u_x(t,x)H_t^T\right)u_{xx}(t,x)H_t^T. \tag{B.1}$$

Obviously, $\frac{\partial \hat{X}^{t,x}(T)}{\partial x} > 0$ almost surely, for each $x > 0$. Let function l be defined by

$$l(t,y) = \mathbb{E}_t\left[I'\left(yH_t^T\right)\left(H_t^T\right)^2\right], \tag{B.2}$$

for all $(t,y) \in [0,T] \times (0,\infty)$. Then (see Lemma B.1 below)

$$l \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)).$$

Therefore, by Lemma 3.4 and (B.1), we have for every $t \in [0,T]$ that

$$\mathbb{E}_t\left[\frac{\partial \hat{X}^{t,x}(T)}{\partial x}H_t^T\right] = u_{xx}(t,x)l(t,u_x(t,x))$$

is continuous with respect to x, on $(0,\infty)$. Moreover, for any $z > z_0 > 0$, we have

$$\begin{aligned} & \int_{z_0}^{z} \mathbb{E}_t \left[\frac{\partial \hat{X}^{t,x}(T)}{\partial x} H_t^T \right] dx \\ = & \ \mathbb{E}_t \left[\int_{z_0}^{z} \frac{\partial \hat{X}^{t,x}(T)}{\partial x} H_t^T \, dx \right] \quad \text{(by Fubini's Theorem)} \\ = & \ \mathbb{E}_t \left[(\hat{X}^{t,z}(T) - \hat{X}^{t,z_0}(T)) H_t^T \right] \\ = & \ z - z_0, \quad \text{(by (4.2))} \end{aligned}$$

which leads to (4.3). □

Lemma B.1. *Under Assumptions 2.1 and 3.2, let function l be defined by* (B.2)*, then*

$$l \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)).$$

Proof: Obviously, $I'(y) < 0$, for all $y > 0$. For each $y > 0$,

$$-I'(y) = -\frac{1}{U''(I(y))} = -\frac{U'(I(y))}{yU''(I(y))} \leq c\frac{1 + I(y)}{y},$$

where the inequality follows from Assumption 3.2. Thus, for any $z \in \mathbb{R}$,

$$0 < -e^{2z} I'(e^z) \leq c(e^z + e^z I(e^z)).$$

By Lemma A.4, $\int_{-\infty}^{\infty} q(e^z)\, e^{-az^2}\, dz < \infty$, for all $a > 0$, where $q(y) = -y^2 I'(y)$. Since

$$l(t,y) = -\frac{\mathbb{E}_t \left[q\left(y H_t^T \right) \right]}{y^2},$$

the proof can be finished by using Lemma A.6. □

Proof of Proposition 4.6:
By (3.17), (4.9) is obvious. Lemma 3.4 clearly implies that $f \in C^{1,\infty}([0,T) \times (0,\infty))$. Now we show $f \in C([0,T] \times (0,\infty))$. Noting $I'(y) = \frac{1}{U''(I(y))}$, we have from (4.6) that

$$f(t,x) = \mathbb{E}_t \left[-\lambda(t,x) H_t^T I' \left(\lambda(t,x) H_t^T \right) H_t^T \right],$$

that is, $f(t,x) = -\lambda(t,x) l(t, \lambda(t,x))$, where l is defined by (B.2). Lemmas B.1 and 3.3 combined lead to $f \in C([0,T] \times (0,\infty))$. To conclude the proposition, it remains to show f satisfies PDE (4.8) on $[0,T) \times (0,\infty)$. For notational simplicity, we use $\hat{X}(s)$ to denote $\hat{X}^{0,x}(s)$, the optimal wealth process with initial time $t = 0$ and initial wealth x. We have known that the optimal portfolio policy is given by (3.15). Then the optimal wealth process $\{\hat{X}(s),\, s \in [0,T]\}$ satisfies the following SDE:

$$d\hat{X}(s) = \hat{X}(s)r(s)ds + f(s,\hat{X}(s))|\theta(s)|^2 ds + f(s,\hat{X}(s))\theta(s)^\top dB(s).$$

Applying Itô's formula to $f(s,\hat{X}(s))H(s)$ yields that

$$\begin{aligned} & d\left[f(s,\hat{X}(s))H(s)\right] \\ = \;& H(s)\Big[-f(s,\hat{X}(s))r(s) + f_s(s,\hat{X}(s)) + f_x(s,\hat{X}(s))\hat{X}(s)r(s) \\ & +\frac{1}{2}f_{xx}(s,\hat{X}(s))f^2(s,\hat{X}(s))|\theta(s)|^2\Big]\, ds + \text{diffusion part}. \end{aligned}$$

By Proposition 4.4, $f(s,\hat{X}(s))H(s)$ is a martingale. Thus its drift part must be zero. On the other hand, $H(s)$ is log-normally distributed, and then by Lemma 3.3, by (3.12) and by taking $t = 0$, $\mathbb{P}(\hat{X}(s) \in G) > 0$ for every open set $G \subset \mathbb{R}$ and every $s \in (0,T)$. Therefore, we know f satisfies (4.8) on $(0,T) \times (0,\infty)$. Moreover, by the fact that $f \in C^{1,\infty}([0,T) \times (0,\infty))$, we know f satisfies (4.8) on $[0,T) \times (0,\infty)$. □

Appendix C: Approximation

We construct the approximating sequence $\{f^{(m)},\, m \geq 2\}$ in this appendix, by considering a sequence of expected utility maximization problems with constraints.

C.1. Expected Utility Maximization with Constraints

Given a (von Neumann-Morgenstern) utility function U, for every $m \geq 2$ and for every $t \in [0,T)$, we consider the following problem with a constraint:

$$\max_{(\phi(s))} \; \mathbb{E}_t[U(X(T))] \tag{C.1}$$

subject to

$$\begin{cases} dX(s) = \left[X(s)r(s) + \phi^\top(s)(b(s) - r(s)\mathbb{1})\right] ds + \phi^\top(s)\sigma(s)dB(s), \\ \frac{1}{m} e^{-\int_s^T r(\tau)d\tau} \leq X(s) \leq m e^{-\int_s^T r(\tau)d\tau}, \text{ for } s \in [t,T], \\ X(t) = x, \end{cases} \tag{C.2}$$

where $\phi(s) = (\phi^1(s), \cdots, \phi^n(s))^\top$ is the vector of values of wealth invested in risky assets at time $s \in [t, T]$, and $\frac{1}{m} e^{-\int_t^T r(s)ds} < x < me^{-\int_t^T r(s)ds}$.

We can see the dynamic problem (C.1)-(C.2) can be transformed into a static one:

$$\text{(C.3)} \qquad \begin{array}{ll} \max\limits_{\frac{1}{m} \le X(T) \le m} & \mathbb{E}_t[U(X(T))] \\ \text{subject to} & \mathbb{E}_t\left[H_t^T X(T)\right] \le x. \end{array}$$

By a similar discussion as in Section 3, the solution $\hat{X}^{(m),t,x}(T)$ is given by

$$\text{(C.4)} \qquad \hat{X}^{(m),t,x}(T) = \frac{1}{m} \vee I\left(\lambda^{(m)}(t,x) H_t^T\right) \wedge m,$$

where the Lagrangian multiplier $\lambda^{(m)}(t,x) > 0$ and we use the following notation:

$$\frac{1}{m} \vee x \wedge m = \begin{cases} \frac{1}{m}, & \text{for } x \le \frac{1}{m}; \\ x, & \text{for } x \in \left(\frac{1}{m}, m\right); \\ m, & \text{for } x \ge m. \end{cases}$$

Moreover, the static budget constraint is binding:

$$\mathbb{E}_t\left[H_t^T \hat{X}^{(m),t,x}(T)\right] = x,$$

that is,

$$\text{(C.5)} \qquad \mathbb{E}_t\left[H_t^T \left(\frac{1}{m} \vee I\left(\lambda^{(m)}(t,x) H_t^T\right) \wedge m\right)\right] = x.$$

For any $y > 0$, define

$$\text{(C.6)} \qquad \mu^{(m)}(t,y) = \mathbb{E}_t\left[H_t^T \left(\frac{1}{m} \vee I\left(y H_t^T\right) \wedge m\right)\right].$$

The independent increments of Brownian motion yield $\mu^{(m)}$ is a deterministic function defined on $[0,T] \times (0,\infty)$. Obviously, for any given $t \in [0,T)$, $\mu^{(m)}(t,y)$ is continuous and strictly decreasing with respect to y, on $(0,\infty)$, and

$$\lim_{y \downarrow 0} \mu^{(m)}(t,y) = me^{-\int_t^T r(s)ds}, \quad \lim_{y \uparrow \infty} \mu^{(m)}(t,y) = \frac{1}{m} e^{-\int_t^T r(s)ds}.$$

By (C.5) and the definition of $\mu^{(m)}$,

$$\text{(C.7)} \qquad \mu^{(m)}(t, \lambda^{(m)}(t,x)) = x,$$

for any $x \in \left(\frac{1}{m}\,e^{-\int_t^T r(s)ds},\, me^{-\int_t^T r(s)ds}\right)$. Therefore, $\lambda^{(m)}$ is a deterministic function defined on $\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$, where

$$\begin{aligned} \mathcal{D}^{(m)} &= \left\{(t,x):\ \frac{1}{m}\,e^{-\int_t^T r(s)ds} < x < me^{-\int_t^T r(s)ds},\, t \in [0,T)\right\}, \\ \mathcal{T}^{(m)} &= \left\{(T,x):\ \frac{1}{m} < x < m\right\}. \end{aligned}$$

Moreover, for any given $t \in [0,T)$, $\lambda^{(m)}(t,x)$ is continuous and strictly decreasing with respect to x, on $\left(\frac{1}{m}\,e^{-\int_t^T r(s)ds},\, me^{-\int_t^T r(s)ds}\right)$, and

$$\text{(C.8)} \qquad \lim_{x \downarrow \frac{1}{m}\,e^{-\int_t^T r(s)ds}} \lambda^{(m)}(t,x) = \infty, \qquad \lim_{x \uparrow me^{-\int_t^T r(s)ds}} \lambda^{(m)}(t,x) = 0.$$

In view of (C.4), the indirect utility function of problem (C.3) is given by

$$\text{(C.9)} \qquad u^{(m)}(t,x) = \mathbb{E}_t\left[U\left(\frac{1}{m} \vee I\left(\lambda^{(m)}(t,x)H_t^T\right) \wedge m\right)\right].$$

The independent increments of Brownian motion yield that $u^{(m)}$ is a deterministic function defined on $\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$.

Lemma C.1. *Under Assumption 2.1, for each $m \geq 2$, we have*

$$\text{(C.10)} \qquad \mu^{(m)} \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)),$$

$$\text{(C.11)} \qquad \lambda^{(m)} \in C^{1,\infty}(\mathcal{D}^{(m)}) \cap C(\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}),$$

$$\text{(C.12)} \qquad u^{(m)} \in C^{1,\infty}(\mathcal{D}^{(m)}) \cap C(\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}).$$

Proof: Firstly, for each $m \geq 2$, the function $\frac{1}{m} \vee I(y) \wedge m$ is bounded and continuous. Then a similar discussion as in the proof of Lemma 3.3 leads to (C.10) and (C.11). Finally, (C.12) can be obtained by the same method to prove Lemma 3.4. □

The Lagrangian multiplier $\lambda^{(m)}(t,x)$ turns out to be the agent's marginal utility of wealth at the optimum, as shown by the next proposition, whose proof is deferred after two technical lemmas.

Proposition C.2. *Under Assumption 2.1, for all $m \geq 2$, $\lambda^{(m)}(t,x) = u_x^{(m)}(t,x)$ on $\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$.*

In what follows, we use $\mathbf{1}_A$ to denote the indicator function of a set A. For each $m \geq 2$, let $\nu^{(m)}$ be defined by

$$\nu^{(m)}(t,y) = \mathbb{E}_t\left[q^{(m)}\left(yH_t^T\right)\right], \tag{C.13}$$

for all $(t,y) \in [0,T] \times (0,\infty)$, where

$$q^{(m)}(y) = -\frac{U'(I(y))}{U''(I(y))}\, y\, \mathbf{1}_{\left\{\frac{1}{m}<I(y)<m\right\}}, \tag{C.14}$$

for all $y > 0$.

Lemma C.3. *Under Assumption 2.1, for all $m \geq 2$,*

$$\nu^{(m)} \in C^{1,\infty}([0,T) \times (0,\infty)),$$

and $\nu^{(m)}$ is continuous at (T,y), for all $y \in (0,\infty) \setminus \left\{U'(m), U'\left(\frac{1}{m}\right)\right\}$.

Proof: Obviously, for each $m \geq 2$, $q^{(m)}$ is bounded on $(0,\infty)$ and is continuous on $(0,\infty) \setminus \left\{U'(m), U'\left(\frac{1}{m}\right)\right\}$. Then Lemma A.6 leads to the assertion. □

Lemma C.4. *Set*

$$\begin{aligned} &\alpha^{(m)}(t,x) \\ = \;& \lambda_x^{(m)}(t,x)\,\mathbb{E}_t\left[\left(H_t^T\right)^2 I'\left(\lambda^{(m)}(t,x)H_t^T\right)\mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\right]. \end{aligned}$$

Then under Assumption 2.1, $\alpha^{(m)}(t,x) = 1$, for every $m \geq 2$ and every $(t,x) \in \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$.

Proof: Let $q^{(m)}$ and $\nu^{(m)}$ be defined by (C.14) and (C.13), respectively. Then

$$q^{(m)}(y) = -y^2 I'(y)\mathbf{1}_{\left\{\frac{1}{m}<I(y)<m\right\}},$$

and therefore,

$$\nu^{(m)}(t,y) = -y^2\,\mathbb{E}_t\left[\left(H_t^T\right)^2 I'\left(yH_t^T\right)\mathbf{1}_{\left\{\frac{1}{m}<I\left(y\frac{H(T)}{H(t)}\right)<m\right\}}\right].$$

Thus

$$\alpha^{(m)}(t,x) = -\frac{\lambda_x^{(m)}(t,x)\nu^{(m)}(t,\lambda^{(m)}(t,x))}{(\lambda^{(m)}(t,x))^2}.$$

Lemmas C.1 and C.3 combined imply $\alpha^{(m)} \in C(\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)})$.

On the other hand, for any given $t \in [0,T]$, and any z_0 and z such that

$$\frac{1}{m} e^{-\int_t^T r(s)ds} < z_0 < z < m e^{-\int_t^T r(s)ds},$$

we have

$$\begin{aligned}
& \int_{z_0}^{z} \alpha^{(m)}(t,x)\,dx \\
= \; & \mathbb{E}_t\left[\int_{z_0}^{z} \lambda_x^{(m)}(t,x)\left(H_t^T\right)^2 I'\left(\lambda^{(m)}(t,x)H_t^T\right)\mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\,dx\right] \\
= \; & \mathbb{E}_t\left[H_t^T\int_{z_0}^{z} \mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\,dI\left(\lambda^{(m)}(t,x)H_t^T\right)\right] \\
= \; & \mathbb{E}_t\left[H_t^T\int_{I(\lambda^{(m)}(t,z_0)H_t^T)}^{I(\lambda^{(m)}(t,z)H_t^T)} \mathbf{1}_{\left\{\frac{1}{m}<y<m\right\}}\,dy\right] \\
= \; & \mathbb{E}_t\left[H_t^T\left(\frac{1}{m}\vee I\left(\lambda^{(m)}(t,z)H_t^T\right)\wedge m - \frac{1}{m}\vee I\left(\lambda^{(m)}(t,z_0)H_t^T\right)\wedge m\right)\right] \\
= \; & z - z_0,
\end{aligned}$$

where the first equality follows from the Fubini's theorem, and the last equality follows from (C.5). Therefore, the assertion follows. □

Proof of Proposition C.2:
For every $t \in [0,T]$, and any z_0 and z such that

$$\frac{1}{m} e^{-\int_t^T r(s)ds} < z_0 < z < m e^{-\int_t^T r(s)ds},$$

we have

$$\begin{aligned}
& u^{(m)}(t,z) - u^{(m)}(t,z_0) \\
= \; & \mathbb{E}_t\left[U\left(\frac{1}{m}\vee I\left(\lambda^{(m)}(t,z)H_t^T\right)\wedge m\right) - U\left(\frac{1}{m}\vee I\left(\lambda^{(m)}(t,z_0)H_t^T\right)\wedge m\right)\right] \\
= \; & \mathbb{E}_t\left[\int_{I(\lambda^{(m)}(t,z_0)H_t^T)}^{I(\lambda^{(m)}(t,z)H_t^T)} \mathbf{1}_{\left\{\frac{1}{m}<y<m\right\}}\,dU(y)\right] \\
= \; & \mathbb{E}_t\left[\int_{z_0}^{z} \mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\,dU\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)\right] \\
= \; & \mathbb{E}_t\left[\int_{z_0}^{z} \mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\lambda^{(m)}(t,x)\left(H_t^T\right)^2 I'\left(\lambda^{(m)}(t,x)H_t^T\right)\lambda_x^{(m)}(t,x)dx\right] \\
= \; & \int_{z_0}^{z} \lambda^{(m)}(t,x)\alpha^{(m)}(t,x)dx \\
= \; & \int_{z_0}^{z} \lambda^{(m)}(t,x)dx,
\end{aligned}$$

where the first equality follows from (C.9), the last two equalities follows from Fubini's theorem and Lemma C.4, respectively. Therefore, the assertion follows. □

In what follows, we derive the feedback form of the optimal portfolio policy, following the same argument as in Section 3. We use $\{\hat{X}^{(m),t,x}(s),\, t \le s \le T\}$ to denote the corresponding optimal wealth process for problem (C.1)-(C.2). It is easy to see that

$$\{\hat{X}^{(m),t,x}(s)H(s),\, t \le s \le T\}$$

is a martingale. Then, for all $s \in [t,T]$,

$$\begin{aligned}
\hat{X}^{(m),t,x}(s) &= \frac{1}{H(s)}\mathbb{E}_s[H(T)\hat{X}^{(m),t,x}(T)] \\
&= \mathbb{E}_s\left[H_s^T\left(\frac{1}{m}\vee I\left(\lambda^{(m)}(t,x)H_t^T\right)\wedge m\right)\right] \\
&= \mathbb{E}_s\left[H_s^T\left(\frac{1}{m}\vee I\left(\lambda^{(m)}(t,x)H_t^s H_s^T\right)\wedge m\right)\right] \\
&= \mu^{(m)}\left(s,\lambda^{(m)}(t,x)H_t^s\right), \qquad \text{(C.15)}
\end{aligned}$$

where (C.15) follows from (C.6) and the fact that $\lambda^{(m)}(t,x)H_t^s$ is $\mathcal{F}_s$-measurable. Therefore, by (C.7) and (C.15),

$$\lambda^{(m)}\left(s,\hat{X}^{(m),t,x}(s)\right) = \lambda^{(m)}(t,x)H_t^s,\ \ s \in [t,T]. \qquad \text{(C.16)}$$

Based on (C.15)-(C.16) and Proposition C.2, by a similar way to (3.15), we can see that the optimal portfolio policy $\hat{\phi}^{(m)}$ for problem (C.1)-(C.2) is in the following feedback form:

$$\hat{\phi}^{(m)}(s,x) = f^{(m)}(s,x)(\sigma(s)\sigma(s)^\top)^{-1}(b(s)-r(s)\mathbb{1}), \qquad \text{(C.17)}$$

where

$$f^{(m)}(s,x) = -\frac{u_x^{(m)}(s,x)}{u_{xx}^{(m)}(s,x)},\ \text{for all } (s,x) \in \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}.$$

C.2. Indirect Absolute Risk Tolerance Functions $f^{(m)}$

We now derive the PDEs satisfied by indirect absolute risk tolerance functions $f^{(m)}$.

Lemma C.5. *Under Assumption 2.1, for each $m \geq 2$,*

$$\begin{aligned} & f^{(m)}(t,x) \\ \text{(C.18)} \qquad &= \mathbb{E}_t\left[-\frac{U'\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}H_t^T\mathbf{1}_{\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\}}\right] \\ \text{(C.19)} \qquad &= \mathbb{E}_t\left[-\frac{U'(\hat{X}^{(m),t,x}(T))}{U''(\hat{X}^{(m),t,x}(T))}H_t^T\mathbf{1}_{\{\frac{1}{m}<\hat{X}^{(m),t,x}(T)<m\}}\right], \end{aligned}$$

for all $(t,x) \in \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$.

Proof: In view of (C.4), we only need to prove (C.18). From the facts that $\lambda^{(m)}(t,x) = u_x^{(m)}(t,x)$ and that

$$\frac{U'\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)} = \lambda^{(m)}(t,x)H_t^T\, I'\left(\lambda^{(m)}(t,x)H_t^T\right),$$

by Lemma C.4, we can have (C.18). □

In analogy with Proposition 4.4, Lemma C.5 implies the following proposition, whose proof is omitted.

Proposition C.6. *Under Assumption 2.1, set*

$$Y^{(m),t,x}(s) = f^{(m)}(s, \hat{X}^{(m),t,x}(s)),\ \ s \in [t,T),$$

and

$$Y^{(m),t,x}(T) = -\frac{U'(\hat{X}^{(m),t,x}(T))}{U''(\hat{X}^{(m),t,x}(T))}\mathbf{1}_{\{\frac{1}{m}<\hat{X}^{(m),t,x}(T)<m\}}.$$

Then $\{Y^{(m),t,x}H(s),\ s \in [t,T]\}$ is a martingale, for every $(t,x) \in \mathcal{D}^{(m)}$.

Based on the preceding proposition, we can derive the PDEs for $f^{(m)}$ as shown in the next proposition, whose proof is deferred after two technical lemmas.

Proposition C.7. *Under Assumptions 2.1 and 3.2, for each $m \geq 2$,*

$$f^{(m)} \in C^{1,\infty}(\mathcal{D}^{(m)}) \cap C(\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}),$$

and satisfies PDE

$$\text{(C.20)} \qquad \frac{1}{2}|\theta|^2(f^{(m)})^2 f_{xx}^{(m)} + rxf_x^{(m)} + f_t^{(m)} - rf^{(m)} = 0,$$

on $\mathcal{D}^{(m)}$, subject to terminal condition

$$f^{(m)}(T,x) = -\frac{U'(x)}{U''(x)}, \text{ for } x \in \left(\frac{1}{m}, m\right), \tag{C.21}$$

and boundary conditions

$$\begin{cases} \lim_{(s,x)\to(t, me^{-\int_t^T r(s)ds})} f^{(m)}(s,x) = 0, & \text{for } t \in [0,T); \\ \lim_{(s,x)\to\left(t, \frac{1}{m} e^{-\int_t^T r(s)ds}\right)} f^{(m)}(s,x) = 0, & \text{for } t \in [0,T). \end{cases} \tag{C.22}$$

Moreover, we have

$$\limsup_{(s,x)\to\left(T,\frac{1}{m}\right)} f^{(m)}(s,x) = -\frac{U'\left(\frac{1}{m}\right)}{U''\left(\frac{1}{m}\right)}, \tag{C.23}$$

$$\limsup_{(s,x)\to(T,m)} f^{(m)}(s,x) = -\frac{U'(m)}{U''(m)}. \tag{C.24}$$

Remark C.8. (C.23) *and* (C.24) *imply that $f^{(m)}$ is upper semi-continuous on the closure $\overline{\mathcal{D}^{(m)}}$, which is essential in proving Lemma D.1 and hence Theorem 5.3.*

Lemma C.9. *Under Assumption 2.1, for each given $m \geq 2$ and $t \in [0,T)$,*

$$\lim_{(s,x)\to\left(t,\frac{1}{m} e^{-\int_t^T r(s)ds}\right)} \lambda^{(m)}(s,x) = \infty, \tag{C.25}$$

$$\lim_{(s,x)\to\left(t, me^{-\int_t^T r(s)ds}\right)} \lambda^{(m)}(s,x) = 0. \tag{C.26}$$

Proof: We only prove (C.26), since the proof of (C.25) is similar. Suppose, to the contrary, that

$$\limsup_{(s,x)\to\left(t, me^{-\int_t^T r(s)ds}\right)} \lambda^{(m)}(s,x) > 0, \tag{C.27}$$

then there exist a constant $\varepsilon > 0$ and a sequence $\{(s_k, x_k), k \geq 1\} \subset \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$ converging to $\left(t, me^{-\int_t^T r(s)ds}\right)$ such that $\lambda^{(m)}(s_k, x_k) > \varepsilon$, for all k. Thus, by relation (C.7),

$$x_k = \mu^{(m)}(s_k, \lambda^{(m)}(s_k, x_k)) < \mu^{(m)}(s_k, \varepsilon).$$

By (C.10), letting $k \to \infty$ yields

$$me^{-\int_t^T r(s)ds} \le \mu^{(m)}(t,\varepsilon) < me^{-\int_t^T r(s)ds},$$

which leads to a contradiction. So, (C.27) is impossible, and therefore, (C.26) is proved. □

Lemma C.10. *Under Assumption 2.1, for each given $m \ge 2$,*

$$\liminf_{(s,x)\to\left(T,\frac{1}{m}\right)} \lambda^{(m)}(s,x) \quad \ge \quad U'\left(\frac{1}{m}\right), \tag{C.28}$$

$$\limsup_{(s,x)\to(T,m)} \lambda^{(m)}(s,x) \quad \le \quad U'(m). \tag{C.29}$$

Proof: We only prove (C.29), since the proof of (C.28) is similar. Suppose, to the contrary, that

$$\limsup_{(s,x)\to(T,m)} \lambda^{(m)}(s,x) > U'(m), \tag{C.30}$$

then there exist a constant $\varepsilon > 0$ and a sequence $\{(s_k,x_k), k \ge 1\} \subset \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$ converging to (T,m) such that $U'(m)+\varepsilon < U'\left(\frac{1}{m}\right)$ and $\lambda^{(m)}(s_k,x_k) > U'(m)+\varepsilon$, for all k. Thus, by relation (C.7),

$$x_k = \mu^{(m)}(s_k, \lambda^{(m)}(s_k,x_k)) < \mu^{(m)}(s_k, U'(m)+\varepsilon).$$

By (C.10), letting $k \to \infty$ yields

$$m \le \mu^{(m)}(T, U'(m)+\varepsilon) = I(U'(m)+\varepsilon) < m,$$

which leads to a contradiction. So, (C.30) is impossible, and therefore, (C.29) is proved. □

Proof of Proposition C.7:
From Lemma C.1, we can see $f^{(m)} \in C^{1,\infty}(\mathcal{D}^{(m)})$. By Proposition C.6 and by a similar way to prove Proposition 4.6, we can show $f^{(m)}$ satisfies PDE (C.20), on $\mathcal{D}^{(m)}$, as well as terminal condition (C.21). So, it suffices to show $f^{(m)}$ is continuous at (T,x), for all $x \in \left(\frac{1}{m}, m\right)$, satisfies boundary conditions in (C.22), and satisfies (C.23) and (C.24).

Let $\nu^{(m)}$ be defined by (C.13), then by Lemma C.5,

$$f^{(m)}(t,x) = \frac{\nu^{(m)}(t,\lambda^{(m)}(t,x))}{\lambda^{(m)}(t,x)}, \text{ for all } (t,x) \in \mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}.$$

From (C.11) and Lemma C.3, we can see $f^{(m)}$ is continuous at (T, x), for all $x \in \left(\frac{1}{m}, m\right)$.

In view of (C.18), we can obtain (C.22) from a combination of Lemma C.9 and the Dominated Convergence Theorem.

From Lemma C.10, we have

$$\limsup_{(s,x)\to(T,m)} \lambda^{(m)}(s,x) H_s^T \le U'(m), \text{ almost surely,}$$

and therefore,

$$\liminf_{(s,x)\to(T,m)} I\left(\lambda^{(m)}(s,x) H_s^T\right) \ge m, \text{ almost surely.}$$

Recalling (C.4), we can see the preceding inequality implies

$$\lim_{(s,x)\to(T,m)} \hat{X}^{(m),s,x}(T) = m, \text{ almost surely.}$$

Then from (C.19) and the Fatou's Lemma, we can get

$$\limsup_{(s,x)\to(T,m)} f^{(m)}(s,x) \le -\frac{U'(m)}{U''(m)}.$$

Moreover, (C.21) yields

$$\lim_{x\uparrow m} f^{(m)}(T,x) = -\frac{U'(m)}{U''(m)}.$$

Thus (C.24) is obtained. The proof of (C.23) is similar. □

The sequence $f^{(m)}$ constructed above is indeed approximating f, as the next proposition shows, whose proof is deferred after two technical lemmas.

Proposition C.11. *Under Assumptions 2.1 and 3.2,*

$$\lim_{m\to\infty} f^{(m)}(t,x) = f(t,x),$$

for all $(t,x) \in [0,T) \times (0,\infty)$.

Lemma C.12. *Under Assumptions 2.1 and 3.2,* $\lim_{m\to\infty} \mu^{(m)}(t,y) = \mu(t,y)$, *for any* $(t,y) \in [0,T) \times (0,\infty)$.

PROOF: For any $m \ge 2$,

$$H_t^T \left(\frac{1}{m} \vee I\left(yH_t^T\right) \wedge m\right) \le H_t^T + H_t^T\, I\left(yH_t^T\right).$$

Then in view of (C.6) and (3.7), the assertion can be obtained from the Dominated Convergence Theorem. □

Lemma C.13. *Under Assumptions 2.1 and 3.2,* $\lim_{m\to\infty}\lambda^{(m)}(t,x)=\lambda(t,x)$*, for any* $(t,x)\in\mathcal{D}^{(m)}$.

Proof: Given $(t,x)\in\mathcal{D}^{(m)}$, suppose, to the contrary, that either

$$\liminf_{m\to\infty}\lambda^{(m)}(t,x)<\lambda(t,x)$$

or

$$\limsup_{m\to\infty}\lambda^{(m)}(t,x)>\lambda(t,x).$$

If $\liminf_{m\to\infty}\lambda^{(m)}(t,x)<\lambda(t,x)$, then there exist a constant $\varepsilon>0$ and a subsequence of $\{\lambda^{(m)}(t,x),m\geq 2\}$, which is still denoted by $\{\lambda^{(m)}(t,x),m\geq 2\}$, such that $\lambda(t,x)-\varepsilon>0$ and $\lambda^{(m)}(t,x)<\lambda(t,x)-\varepsilon$ for any m. Therefore, by relation (C.7),

$$x=\mu^{(m)}(t,\lambda^{(m)}(t,x))>\mu^{(m)}(t,\lambda(t,x)-\varepsilon).$$

By Lemma C.12, letting $m\to\infty$ yields $x\geq\mu(t,\lambda(t,x)-\varepsilon)>x$, which leads a contradiction. Thus $\liminf_{m\to\infty}\lambda^{(m)}(t,x)<\lambda(t,x)$ is impossible. By the same way, we can show $\limsup_{m\to\infty}\lambda^{(m)}(t,x)>\lambda(t,x)$ is also impossible. □

Proof of Proposition C.11:
First of all, for any given $(t,x)\in[0,T)\times(0,\infty)$, we can see from Lemma C.13 that

$$\begin{aligned}
&-\frac{U'\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}H_t^T\mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\\
\to\quad&-\frac{U'\left(I\left(\lambda(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda(t,x)H_t^T\right)\right)}H_t^T,\ \text{almost surely,}
\end{aligned}$$

as $m\to\infty$. Then by Assumption 3.2,

$$\begin{aligned}
&-\frac{U'\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)}H_t^T\mathbf{1}_{\left\{\frac{1}{m}<I\left(\lambda^{(m)}(t,x)H_t^T\right)<m\right\}}\\
\leq\quad& c\left(1+I\left(\lambda^{(m)}(t,x)H_t^T\right)\right)H_t^T\\
\leq\quad& c\left(1+I\left(y_0H_t^T\right)\right)H_t^T,
\end{aligned}$$

where $y_0=\inf_{m\geq 2}\lambda^{(m)}(t,x)>0$. Finally, (C.18) and the Dominated Convergence Theorem combined yield

$$f^{(m)}(t,x)\to\mathbb{E}_t\left[-\frac{U'\left(I\left(\lambda(t,x)H_t^T\right)\right)}{U''\left(I\left(\lambda(t,x)H_t^T\right)\right)}H_t^T\right]=f(t,x),$$

as $m \to \infty$. □

Appendix D: Proof of Theorem 5.3

In this section, we use the results in Appendix C to finish the proof of Theorem 5.3.

Under the conditions of Theorem 5.3, from Proposition 4.6, we know the indirect absolute risk tolerance function

$$g \in C^{1,\infty}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty)),$$

and satisfies PDE

$$\frac{1}{2}|\theta|^2 g^2 g_{xx} + rxg_x + g_t - rg = 0, \tag{D.1}$$

on $[0,T) \times (0,\infty)$, with terminal condition

$$g(T,x) = -\frac{V'(x)}{V''(x)}, \text{ for all } x > 0. \tag{D.2}$$

Since V satisfies Assumption 5.1 and U is more risk averse than V, we know U satisfies Assumption 3.2. Let the approximating sequence $\{f^{(m)}, m \geq 2\}$ of f be constructed as in Appendix C.

Lemma D.1. *Under Assumptions 2.1 and 5.1, if* $-\frac{U'(x)}{U''(x)} < -\frac{V'(x)}{V''(x)}$*, for all* $x > 0$*, then for each* $m \geq 2$*,* $f^{(m)}(t,x) \leq g(t,x)$*, on* $\mathcal{D}^{(m)}$*.*

Proof: For each $m \geq 2$, set

$$\begin{aligned} \mathcal{B}_0^{(m)} &= \left\{ \left(t, \frac{1}{m} e^{-\int_t^T r(s)ds}\right) : t \in [0,T] \right\}, \\ \mathcal{B}_1^{(m)} &= \left\{ (t, me^{-\int_t^T r(s)ds}) : t \in [0,T] \right\}, \end{aligned}$$

then

$$\overline{\mathcal{D}^{(m)}} = \mathcal{D}^{(m)} \cup \partial^* \mathcal{D}^{(m)},$$

where $\overline{\mathcal{D}^{(m)}}$ denotes the closure of $\mathcal{D}^{(m)}$ and

$$\partial^* \mathcal{D}^{(m)} = \mathcal{B}_0^{(m)} \cup \mathcal{T}^{(m)} \cup \mathcal{B}_1^{(m)}.$$

The function $f^{(m)}$ is originally defined on $\mathcal{D}^{(m)} \cup \mathcal{T}^{(m)}$. Now we extend its definition to $\overline{\mathcal{D}^{(m)}}$ as follows: $f^{(m)}(t,x) = 0$, for all $(t,x) \in \mathcal{B}_0^{(m)} \cup \mathcal{B}_1^{(m)}$ such that $t < T$, and

$$f^{(m)}\left(T, \frac{1}{m}\right) = -\frac{U'\left(\frac{1}{m}\right)}{U''\left(\frac{1}{m}\right)}, \quad f^{(m)}(T,m) = -\frac{U'(m)}{U''(m)}.$$

With this extension, we can see from Proposition C.7 that $f^{(m)}$ is an upper semi-continuous function on $\overline{\mathcal{D}^{(m)}}$ (cf. Appendix E).

Let $h = f^{(m)} - g$, then h is upper semi-continuous on $\overline{\mathcal{D}^{(m)}}$. Since $-\frac{U'(x)}{U''(x)} < -\frac{V'(x)}{V''(x)}$, for all $x > 0$, we know $h < 0$ on $\partial^*\mathcal{D}^{(m)}$. Obviously, $\partial^*\mathcal{D}^{(m)}$ is a compact set, and therefore, from Proposition E.2, we can see on $\partial^*\mathcal{D}^{(m)}$, h attains its maximum at some $(t_0, x_0) \in \partial^*\mathcal{D}^{(m)}$, which implies

$$h(t,x) \le -\varepsilon < 0, \text{ for all } (t,x) \in \partial^*\mathcal{D}^{(m)},$$

where $\varepsilon = -h(t_0, x_0) > 0$. Moreover, there exists a constant $\delta > 0$ such that

$$h(t,x) < -\frac{\varepsilon}{2} < 0, \text{ for all } (t,x) \in \mathcal{O}^{(m)}(\delta) \cap \overline{\mathcal{D}^{(m)}},$$

where

$$\begin{aligned}
\mathcal{O}^{(m)}(\delta) &= \mathcal{O}_1^{(m)}(\delta) \cup \mathcal{O}_2^{(m)}(\delta) \cup \mathcal{O}_3^{(m)}(\delta), \\
\mathcal{O}_1^{(m)}(\delta) &= \left\{(t,x): \ t \in [0,T], \ \frac{1}{m} e^{-\int_t^T r(s)ds} \le x < \frac{1}{m} e^{-\int_t^T r(s)ds} + \delta\right\}, \\
\mathcal{O}_2^{(m)}(\delta) &= (T-\delta, T] \times \left(\frac{1}{m}, m\right), \\
\mathcal{O}_3^{(m)}(\delta) &= \left\{(t,x): \ t \in [0,T], \ m e^{-\int_t^T r(s)ds} - \delta < x \le m e^{-\int_t^T r(s)ds}\right\}.
\end{aligned}$$

Otherwise, for each $k \ge 1$, there exists a $(t_k, x_k) \in \mathcal{O}^{(m)}(\frac{1}{k}) \cap \overline{\mathcal{D}^{(m)}}$ such that $h(t_k, x_k) \ge -\frac{\varepsilon}{2}$. It is not difficult to see that there is a subsequence of $\{(t_k, x_k), k \ge 1\}$, which is still denoted by $\{(t_k, x_k), k \ge 1\}$, converging to some $(s, y) \in \partial^*\mathcal{D}^{(m)}$. From the upper semi-continuity of h, we have $h(s,y) \ge \limsup_{k\to\infty} h(t_k, x_k) \ge -\frac{\varepsilon}{2}$, which is impossible, since $-\varepsilon$ is the maximum of h on $\partial^*\mathcal{D}^{(m)}$.

Obviously, $\mathcal{D}^{(m)} \setminus \mathcal{O}^{(m)}(\delta)$ is a compact set, then there exists a constant α such that

$$-\alpha + \frac{1}{2}|\theta|^2 (f^{(m)} + g) g_{xx} - r < 0, \quad \text{on } \mathcal{D}^{(m)} \setminus \mathcal{O}^{(m)}(\delta).$$

Suppose, to the contrary, that $f^{(m)} > g$ somewhere in $\mathcal{D}^{(m)}$. Consider function w defined by $w = he^{\alpha t}$, then $w < 0$ on $\mathcal{O}^{(m)}(\delta) \cap \overline{\mathcal{D}^{(m)}}$ and $w > 0$ somewhere in $\mathcal{D}^{(m)}$. Obviously, w is upper semi-continuous, and therefore, by Proposition E.2, w attains its positive maximum at $(t_1, x_1) \in \mathcal{D}^{(m)} \setminus \mathcal{O}^{(m)}(\delta)$. So, at (t_1, x_1), $w > 0$, $w_x = 0$, $w_{xx} \le 0$, and $w_t = h_t e^{\alpha t} + \alpha h e^{\alpha t} \le 0$.

Consequently, at (t_1, x_1), $h > 0$, $h_x = 0$, $h_{xx} \le 0$, and $h_t \le -\alpha h$. Thus, at (t_1, x_1), we have from (C.20) and (D.1) that

$$\begin{aligned} 0 &= \frac{1}{2}|\theta|^2[(f^{(m)})^2 f^{(m)}_{xx} - g^2 g_{xx}] + rx(f^{(m)}_x - g_x) \\ &\quad +(f^{(m)}_t - g_t) - r(f^{(m)} - g) \\ &= \frac{1}{2}|\theta|^2(f^{(m)})^2 h_{xx} + rxh_x + h_t + \left[\frac{1}{2}|\theta|^2(f^{(m)} + g)g_{xx} - r\right] h \\ &\le \left[-\alpha + \frac{1}{2}|\theta|^2(f^{(m)} + g)g_{xx} - r\right] h \\ &< 0, \end{aligned}$$

which leads to a contradiction. □

Lemma D.2. *Under Assumptions 2.1 and 5.1, if* $-\frac{U'(x)}{U''(x)} < -\frac{V'(x)}{V''(x)}$, *for all* $x > 0$, *then* $f(t, x) \le g(t, x)$, *on* $[0, T) \times (0, \infty)$.

Proof: Proposition C.11 and Lemma D.1 combined yield the assertion. □

Now we prove Theorem 5.3 as follows.

Proof of Theorem 5.3:
For any $\varepsilon > 0$, let $U^\varepsilon : (0, \infty) \to \mathbb{R}$ be a function such that its derivative

$$(U^\varepsilon)'(x) = U'(x)e^{-\varepsilon x},$$

for all $x \in (0, \infty)$. Obviously, U^ε is a utility function and

$$-\frac{(U^\varepsilon)'(x)}{(U^\varepsilon)''(x)} = \frac{U'(x)}{-U''(x) + \varepsilon U'(x)}, \text{ for all } x > 0,$$

which implies

$$\text{(D.3)} \qquad -\frac{(U^\varepsilon)'(x)}{(U^\varepsilon)''(x)} < -\frac{U'(x)}{U''(x)} \le -\frac{V'(x)}{V''(x)}, \text{ for all } x > 0.$$

Corresponding to utility function U^ε, the indirect absolute risk tolerance function is denoted by $f^\varepsilon(t, x)$. By Lemma D.2, we have

$$f^\varepsilon(t, x) \le g(t, x), \text{ for all } (t, x) \in [0, T) \times (0, \infty).$$

In order to complete the proof, it suffices to show $\lim_{\varepsilon \downarrow 0} f^\varepsilon(t, x) = f(t, x)$, for all $t \in [0, T)$ and $x > 0$.

Actually, it is clear that $(U^\varepsilon)'(x) \uparrow U'(x)$ and $-\frac{(U^\varepsilon)'(x)}{(U^\varepsilon)''(x)} \uparrow -\frac{U'(x)}{U''(x)}$ as $\varepsilon \downarrow 0$, for all $x > 0$. Let I^ε denote the inverse marginal utility function of U^ε, that is, $(U^\varepsilon)'(I^\varepsilon(y)) = y$ for all $y > 0$, then $I^\varepsilon(y) \uparrow I(y)$ as $\varepsilon \downarrow 0$, for all $y > 0$. Let $\mu^\varepsilon(t, y) = \mathbb{E}_t \left[I^\varepsilon \left(y H_t^T \right) H_t^T \right]$, then $\mu^\varepsilon(t, y) \uparrow \mu(t, y)$ as $\varepsilon \downarrow 0$, for all $(t, y) \in [0, T) \times (0, \infty)$. Let λ^ε be defined by

$$\mu^\varepsilon(t, \lambda^\varepsilon(t, x)) = x, \text{ for all } (t, x) \in [0, T) \times (0, \infty),$$

then $\lambda^\varepsilon(t, x) \uparrow \lambda(t, x)$ as $\varepsilon \downarrow 0$, for all $(t, x) \in [0, T) \times (0, \infty)$. Consequently, we have

$$-\frac{(U^\varepsilon)' \left(I^\varepsilon \left(\lambda^\varepsilon(t, x) H_t^T \right) \right)}{(U^\varepsilon)'' \left(I^\varepsilon \left(\lambda^\varepsilon(t, x) H_t^T \right) \right)} H_t^T \to -\frac{U' \left(I \left(\lambda(t, x) H_t^T \right) \right)}{U'' \left(I \left(\lambda(t, x) H_t^T \right) \right)} H_t^T$$

almost surely as $\varepsilon \downarrow 0$, for all $(t, x) \in [0, T) \times (0, \infty)$. Moreover, by Proposition 4.2, we have

$$f^\varepsilon(t, x) = \mathbb{E}_t \left[-\frac{(U^\varepsilon)' \left(I^\varepsilon \left(\lambda^\varepsilon(t, x) H_t^T \right) \right)}{(U^\varepsilon)'' \left(I^\varepsilon \left(\lambda^\varepsilon(t, x) H_t^T \right) \right)} H_t^T \right].$$

Then by the same way to prove Proposition C.11, using the Dominated Convergence Theorem, we can have $\lim_{\varepsilon \downarrow 0} f^\varepsilon(t, x) = f(t, x)$, for all $t \in [0, T)$ and $x > 0$. □

Appendix E: Upper Semi-Continuous Functions

Let $\mathbb{M}$ be a normed space, and $|\cdot|$ denote the norm. A function f defined on $\mathbb{M}$ is said to be upper semi-continuous at $x_0 \in \mathbb{M}$ if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x) - f(x_0) < \varepsilon$, for all $x \in \mathbb{M}$ such that $|x - x_0| < \delta$. A function f is said to be upper semi-continuous on $\mathbb{M}$ if it is upper semi-continuous at every point of $\mathbb{M}$.

We refer to (Luenberger, 1969, p.40) for the following two propositions:

Proposition E.1. *A function f defined on $\mathbb{M}$ is upper semi-continuous at x_0 if and only if* $\limsup_{x \to x_0} f(x) \le f(x_0)$.

Proposition E.2. *An upper semi-continuous function achieves its maximum on any compact subset.*

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