A SYMMETRIZED CONJUGACY SCHEME FOR ORTHOGONAL EXPANSIONS

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Abstract. We establish a symmetrization procedure in a context of general orthogonal expansions associated with a second order differential operator L, a 'Laplacian'. Combined with a unified conjugacy scheme furnished in our earlier article it allows, via a suitable embedding, to associate a differential-difference 'Laplacian' $\mathbb{L}$ with the initially given orthogonal system of eigenfunctions of L, so that the resulting extended conjugacy scheme has the natural classical shape. This means, in particular, that the related 'partial derivatives' decomposing $\mathbb{L}$ are skew-symmetric in an appropriate L^2 space and they commute with Riesz transforms and conjugate Poisson integrals. The results shed also some new light on the question of defining higher order Riesz transforms for general orthogonal expansions.

1. Introduction

The seminal article of Muckenhoupt and E. M. Stein [2] initiated the investigation of conjugacy for discrete and continuous nontrigonometric orthogonal expansions. In the recent years a considerable activity could be observed in studying conjugacy, or better Riesz transforms, for orthogonal expansions in one-dimensional and multi-dimensional settings related to general second order differential operators.

A variety of papers has been devoted to the study of objects being ingredients of conjugacy notions defined by different authors in many particular situations. In connection to a dynamic development of investigation of conjugacy problem in different settings, a natural demand appeared on a general and universal definition of Riesz transforms. The authors' paper [3] was an attempt to provide a reasonable answer to this important demand by offering a unified conjugacy scheme that includes definitions of Riesz transforms and conjugate Poisson integrals for a broad class of expansions. The postulated definitions were supported by a "good" L^2-theory, existence of Cauchy-Riemann type equations, and numerous examples existing in the literature which are covered by the scheme.

There is, however, a shortcoming of this unified conjugacy scheme manifested in a lack of symmetry in the decomposition

$$L = A + \sum_{j=1}^{d} \delta_j^* \delta_j,$$

of a given second order partial differential operator L, a 'Laplacian', acting on functions on a d-dimensional domain $\mathcal{X} = (b,c)^d$, $\infty \le b < c \le \infty$. Here $A \ge 0$ is a constant, δ_j are 'partial derivatives' associated to L (first order partial differential operators, δ_j acts on the jth coordinate), and δ_j^* are their formal adjoints in an appropriate L^2 sense.

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Riesz transforms of first order defined in [3] are formally given by $R_j = \delta_j L^{-1/2}$ (or by $R_j = \delta_j L^{-1/2}\Pi_0$, see [3] for details), but a replacement of δ_j by δ_j^* in this definition is, in general, inappropriate since it may result in an operator taking L^2 functions out of L^2.

Asymmetry of the decomposition of L has, in fact, a deep impact onto the whole conjugacy scheme postulated in [3]. To be precise, taking into account existing examples, it seems that the case of an operator L acting on the whole space $\mathbb{R}^d$ is not really affected by this asymmetry. Therefore, in what follows we consider only the case $(b,c) \neq \mathbb{R}$, and assume, without any loss of generality, that $0 = b < c \le \infty$. Then a possible way of overcoming the lack of symmetry is provided by a symmetrization procedure, which is the purpose and the main achievement of the paper. This procedure is to some extent inspired by a situation of certain orthogonal systems appearing in the theory of Dunkl operators, see Section 3 for more comments. 'Partial derivatives' emerging from the symmetrization procedure, contrary to δ_j's, are skew-symmetric as it happens in many classical cases including, in particular, the usual Euclidean partial derivatives, Dunkl operators, left-invariant vector fields on Lie groups, etc.

Throughout the paper we use a fairly standard notation. The symbols Δ and dx will always refer to the Euclidean Laplacian, $\Delta = \sum_{i=1}^d \partial_{x_i}^2$, and Lebesgue measure acting, or considered, on an appropriate domain in $\mathbb{R}^d$ like, for instance, $\mathbb{R}^d_+ = (0,\infty)^d$. The symbol $\mathbb{N}$ is used to denote the set of nonnegative integers, $\mathbb{N} = \{0,1,2,\ldots\}$. Finally, $\langle\cdot,\cdot\rangle_\mu$ denotes the canonical inner product in an appropriate L^2 space, where μ is a given measure.

2. Initial situation

Our starting point is the situation discussed in [3, Section 2], where the concept of studying conjugacy for orthogonal expansions is based on the existence of a second order differential operator playing a similar role to that of the standard Laplacian in the classical harmonic analysis. Below, $d \ge 1$ will always denote the dimension.

We first consider the following one-dimensional objects, which in a while will serve as building blocks of d-dimensional product structure:

- an open (possibly unbounded) interval $X \subset \mathbb{R}$;
- a system $\{\mu_i : i = 1,\ldots,d\}$ of absolutely continuous measures on X, $\mu_i(dx_i) = w_i(x_i)dx_i$ with strictly positive densities $w_i \in C^2(X)$;
- a system $\{L_i : i = 1,\ldots,d\}$ of second order differential operators defined on $C_c^2(X)$.

Here, in this paper, we exclude the case $X = \mathbb{R}$ (see the comment in Section 1), so without any loss of generality we may assume that $X = (0,c)$, $0 < c \le \infty$. For each of the operators L_i, in order to ensure existence of the associated 'derivative', we assume the decomposition

$$L_i = a_i + \delta_i^*\delta_i, \tag{2.1}$$

where a_i is a nonnegative constant, and δ_i is a first order differential operator (a 'derivative') of the form

$$\delta_i = p_i(x_i)\frac{\partial}{\partial x_i} + q_i(x_i)$$

with real coefficients $p_i \in C^2(X)$, $q_i \in C^1(X)$, $p_i(x_i) \neq 0$ for $x_i \in X$; here δ_i^* represents the formal adjoint of δ_i in $L^2(X,\mu_i)$,

$$\delta_i^* = -p_i(x_i)\frac{\partial}{\partial x_i} + q_i(x_i) - p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} - p_i'(x_i)$$

determined by the identity

$$\langle \delta_i \varphi, \psi \rangle_{\mu_i} = \langle \varphi, \delta_i^* \psi \rangle_{\mu_i}, \qquad \varphi, \psi \in C_c^1(X).$$

Notice that $\delta^* \neq -\delta$. Thus, a posteriori, each L_i is a linear operator with continuous real-valued coefficients and negative leading term coefficient,

$$L_i = -p_i^2(x_i) \frac{\partial^2}{\partial x_i^2} - \left[2p_i(x_i)p_i'(x_i) + p_i^2(x_i) \frac{w_i'(x_i)}{w_i(x_i)} \right] \frac{\partial}{\partial x_i} + q_i^2(x_i) - \big(p_i(x_i)q_i(x_i)\big)' - p_i(x_i)q_i(x_i) \frac{w_i'(x_i)}{w_i(x_i)} + a_i.$$

Moreover, (2.1) implies that each L_i is symmetric and nonnegative on $C_c^2(X) \subset L^2(X, \mu_i)$.

Now we are in a position to specify a d-dimensional setting that is suitable for further development. We equip the space $\mathcal{X} = X \times \ldots \times X$ (d times) with the product measure

$$\mu = \mu_1 \otimes \ldots \otimes \mu_d.$$

We consider the d-dimensional 'Laplacian' L (more precisely, L is a generalization of $-\Delta$) defined initially on $C_c^2(\mathcal{X})$ by

$$L = L_1 + \ldots + L_d,$$

where each L_i is understood as a one-dimensional operator acting on the ith axis. Note that in view of the previous assumptions, L admits the decomposition

$$L = A + \sum_{i=1}^{d} \delta_i^* \delta_i, \qquad \text{where} \quad A = \sum_{i=1}^{d} a_i \geq 0;$$

here the indices of δ and δ^* indicate also on which axes actions of these operators take place.

Next, we introduce an orthogonal system associated with L. With no loss of generality we may restrict to L^2-normalized systems. Assume that for each $i = 1, \ldots, d$, there exists an orthonormal and complete in $L^2(X, \mu_i)$ system $\{\varphi_{k_i}^{(i)} : k_i \in \mathbb{N}\}$ consisting of eigenfunctions of L_i, with the corresponding eigenvalues $\{\lambda_{k_i}^{(i)} : k_i \in \mathbb{N}\}$, i.e. $L_i \varphi_{k_i}^{(i)} = \lambda_{k_i}^{(i)} \varphi_{k_i}^{(i)}$. Here and below we assume for simplicity that $\varphi_{k_i}^{(i)} \in C^\infty(X)$, but in fact much less regularity is needed (we omit a discussion in this direction since it could affect the main line of thought of the paper). For a multi-index $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ we define

$$\varphi_k = \varphi_{k_1}^{(1)} \otimes \ldots \otimes \varphi_{k_d}^{(d)}.$$

Then $\{\varphi_k : k \in \mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathcal{X}, \mu)$ consisting of eigenfunctions of L,

$$L\varphi_k = \lambda_k \varphi_k, \qquad \text{where} \quad \lambda_k = \lambda_{k_1}^{(1)} + \ldots + \lambda_{k_d}^{(d)}.$$

In addition, $\varphi_k \in C^\infty(\mathcal{X})$.

We impose the following technical assumptions on the systems of eigenvalues and eigenfunctions, which seem to be unavoidable on the considered level of generality. For every $i = 1, \ldots, d$, we assume that the one-dimensional eigenvalues are indexed in the (strictly) ascending order, $\lambda_0^{(i)} < \lambda_1^{(i)} < \lambda_2^{(i)} < \ldots$, and $\lim_{k_i} \lambda_{k_i}^{(i)} = \infty$. Consequently, the set $\{\lambda_k : k \in \mathbb{N}^d\}$ of multi-dimensional eigenvalues may be arranged into an increasing and divergent sequence

$$\Lambda_0 < \Lambda_1 < \Lambda_2 < \ldots, \qquad \Lambda_m \to \infty \quad \text{when} \quad m \to \infty.$$

Moreover, we require the associated 'partial derivatives' δ_i to be L^2-consistent with the orthogonal system, i.e. for each i

$$\delta_i \varphi_{k_i}^{(i)} \in L^2(X, \mu_i), \qquad k_i \in \mathbb{N},$$

and

$$\langle \delta_i \varphi_{k_i}^{(i)}, \delta_i \varphi_{m_i}^{(i)} \rangle_{\mu_i} = \langle \delta_i^* \delta_i \varphi_{k_i}^{(i)}, \varphi_{m_i}^{(i)} \rangle_{\mu_i}, \qquad k_i, m_i \in \mathbb{N}.$$

All these assumptions are not too restrictive, as may be seen by various examples given in [3, Section 7].

In the situation described above it is not hard to check that the 'Laplacian' L is symmetric and nonnegative on $C_c^2(\mathcal{X}) \subset L^2(\mathcal{X}, \mu)$, and the constant A in the decomposition of L does not exceed the smallest eigenvalue, $A \le \Lambda_0$, cf. [3, Lemma 1]. However, from the conjugacy point of view, the following fact is essential (see [3, Lemma 2]): given $i = 1, \dots, d$, the 'differentiated' system $\{\delta_i \varphi_k : k \in \mathbb{N}^d\}$ is orthogonal in $L^2(\mathcal{X}, \mu)$; furthermore, $\|\delta_i \varphi_k\|_{L^2(\mathcal{X},\mu)}^2 = \lambda_{k_i}^{(i)} - a_i$. This fact leads to investigating also 'Laplacians' standing behind the systems $\{\delta_i \varphi_k\}$, $i = 1, \dots, d$, and this turns out to be a crucial point in constructing proper conjugacy scheme for general orthogonal expansions. Define

$$M_j = A + \delta_j \delta_j^* + \sum_{i \ne j} \delta_i^* \delta_i = L + [\delta_j, \delta_j^*], \qquad j = 1, \dots, d,$$

where $[\delta_j, \delta_j^*]$ is the commutator

$$[\delta_j, \delta_j^*] = \delta_j \delta_j^* - \delta_j^* \delta_j = 2p_j(x_j) q_j'(x_j) - p_j(x_j) \left[p_j(x_j) \frac{w_j'(x_j)}{w_j(x_j)} + p_j'(x_j) \right]'.$$

By the very definition it follows that each M_j is symmetric and nonnegative on $C_c^2(\mathcal{X}) \subset L^2(\mathcal{X}, \mu)$. Moreover, for each $j = 1, \dots, d$, the system $\{\delta_j \varphi_k : k \in \mathbb{N}^d\}$ is an orthogonal system of eigenfunctions of M_j, with the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}^d\}$, i.e.

$$M_j(\delta_j \varphi_k) = \lambda_k (\delta_j \varphi_k);$$

see [3, Lemma 5]. The operators M_j (or rather their suitable self-adjoint extensions) are used to generate so-called modified Poisson semigroups that play an important role in the conjugacy scheme for orthogonal expansions proposed in [3], see [3, Section 5] for details.

The main inconvenience of the theory postulated in [3] is a lack of symmetry in principal objects and relations, and this phenomenon has roots in the asymmetry between the 'derivatives' δ_j and their adjoints δ_j^*. In consequence, definitions and the conjugacy scheme established in [3] admit essential deviations from the classical shape, see [3, Sections 5,6].

The main idea of this paper is to overcome the problem by embedding the situation considered in [3] into a more general setting, where the associated derivatives are skew-symmetric and the related conjugacy scheme has precisely the classical shape. The price is, however, that the related extended 'Laplacian' and 'derivatives' are differential-difference operators rather than purely differential ones. It is remarkable that most definitions and relations in the setting of [3] may be then recovered by suitable 'projecting' from the extended situation. However, in some cases the projection procedure leads to different and seemingly even more natural definitions. This remark concerns especially higher order Riesz transforms.

3. Symmetrization

We now describe the symmetrization procedure and the resulting symmetrized situation. The construction is motivated to some extent by the setting of the Dunkl harmonic oscillator with the underlying reflection group isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$; we refer to [4] for more details concerning the Dunkl setting. Recall that $X = (0,c)$ for some $0 < c \le \infty$ and $\mathcal{X} = X^d$, and consider the space $\mathbb{X} = X_{\mathrm{SYM}} \times \ldots \times X_{\mathrm{SYM}}$ (d-times), where $X_{\mathrm{SYM}} = (-c,0) \cup (0,c)$. Notice that $\mathcal{X}$ is isomorphic to each of the 'Weil chambers' generated in $\mathbb{X}$ by reflections perpendicular to coordinate axes. We extend the measure μ to $\mathbb{X}$ by even extension of the one-dimensional densities w_i, $w_i(-x_i) = w_i(x_i)$, $x_i > 0$; we keep using the same symbols for the extended objects. Further, we extend the coefficients of L by letting

$$p_i(-x_i) = p_i(x_i), \qquad q_i(-x_i) = -q_i(x_i), \qquad x_i > 0;$$

again the emerging extended objects, including L, M_j and δ_j defined by means of the extended coefficients, are denoted by still the same symbols.

Definition 3.1. *For a suitable function f on $\mathbb{X}$ define its 'partial derivatives'*

$$D_j f(x) = p_j(x_j)\frac{\partial f}{\partial x_j}(x) + q_j(x_j)\frac{f(x)+f(\sigma_j x)}{2} + \Big[p_j(x_j)\frac{w_j'(x_j)}{w_j(x_j)} + p_j'(x_j) - q_j(x_j)\Big]\frac{f(x)-f(\sigma_j x)}{2},$$

where σ_j denotes the reflection in $\mathbb{X}$ in the hyperplane orthogonal to the jth coordinate axis, $\sigma_j(x_1,\ldots,x_j,\ldots,x_d) = (x_1,\ldots,-x_j,\ldots,x_d)$.

The result below follows by integration by parts and some elementary manipulations.

Proposition 3.1. *The operators D_j, $j = 1,\ldots,d$, are skew-symmetric in $L^2(\mathbb{X},\mu)$, $D_j^* = -D_j$, in the sense that*

$$\langle D_j f, g\rangle_\mu = -\langle f, D_j g\rangle_\mu, \qquad f,g \in C_c^1(\mathbb{X}).$$

This motivates the definition of the extended 'Laplacian' $\mathbb{L}$ as

$$\mathbb{L} = A - \sum_{i=1}^d D_i^2. \tag{3.1}$$

Then each D_i commutes with $\mathbb{L}$, which is an important feature at this point.

To state the next result it is convenient to introduce the following terminology. Given $\varepsilon \in \mathbb{Z}_2^d$, we say that a function f on $\mathbb{X}$ is ε-symmetric if $f \circ \sigma_j = (-1)^{\varepsilon_j} f$, $j = 1,\ldots,d$. If f is ε-symmetric and $\varepsilon_{j_0} = 0$ ($\varepsilon_{j_0} = 1$) then f is said to be even (odd) with respect to the j_0th coordinate.

Proposition 3.2. *The operator $\mathbb{L}$ is symmetric and nonnegative on $C_c^2(\mathbb{X}) \subset L^2(\mathbb{X},\mu)$. Moreover, for any ε-symmetric function $f \in C^2(\mathbb{X})$, $\varepsilon \in \mathbb{Z}_2^d$, we have*

$$\mathbb{L}f = Af + \sum_{\{j:\varepsilon_j=0\}} \delta_j^*\delta_j f + \sum_{\{j:\varepsilon_j=1\}} \delta_j\delta_j^* f.$$

In particular, $\mathbb{L}f = Lf$ when f is even with respect to all coordinates, and $\mathbb{L}f = M_j f$ if f is odd with respect to the jth coordinate and even with respect to the remaining coordinates.

Proof. The first part follows from the decomposition of $\mathbb{L}$ in terms of the D_j. Justifying the remaining part may be done by computing the explicit form of $\mathbb{L}$, which is

$$\begin{aligned}\mathbb{L}f(x) = Af(x) - \sum_{i=1}^{d} \bigg\{ & p_i^2(x_i)\frac{\partial^2 f}{\partial x_i^2}(x) + \bigg[2p_i(x_i)p_i'(x_i) + p_i^2(x_i)\frac{w_i'(x_i)}{w_i(x_i)}\bigg]\frac{\partial f}{\partial x_i}(x) \\ & + \bigg(q_i(x_i)\bigg[p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} + p_i'(x_i)\bigg] - q_i^2(x_i)\bigg)f(x) + p_i(x_i)q_i'(x_i)\frac{f(x)+f(\sigma_i x)}{2} \\ & + p_i(x_i)\bigg[p_i(x_i)\frac{w_i'(x_i)}{w_i(x_i)} + p_i'(x_i) - q_i(x_i)\bigg]'\frac{f(x)-f(\sigma_i x)}{2}\bigg\}.\end{aligned}$$

The proof is finished by comparing the above expression with the explicit forms of $\delta_i^*\delta_i$ and $\delta_i\delta_i^*$, which may be read off from the explicit expressions for L and M_j, see Section 2. □

Next we extend the eigenfunctions φ_k to $\mathbb{X}$ by letting $\varphi_{k_i}^{(i)}(-x_i) = \varphi_{k_i}^{(i)}(x_i)$, $x_i > 0$, $i = 1, \dots, d$. Then, automatically, given $\varepsilon \in \mathbb{Z}_2^d$, the function $\delta_1^{\varepsilon_1} \dots \delta_d^{\varepsilon_d}\varphi_k$ is ε-symmetric by the way of extending the coefficients of δ_j. It turns out that these are eigenfunctions of $\mathbb{L}$.

Lemma 3.3. *Let $\varepsilon \in \mathbb{Z}_2^d$ be fixed. Then*

$$\mathbb{L}\big(\delta_1^{\varepsilon_1} \dots \delta_d^{\varepsilon_d}\varphi_k\big) = \lambda_k\big(\delta_1^{\varepsilon_1} \dots \delta_d^{\varepsilon_d}\varphi_k\big), \qquad k \in \mathbb{N}^d.$$

Proof. Combine Proposition 3.2 with the product structure of $\delta_1^{\varepsilon_1} \dots \delta_d^{\varepsilon_d}\varphi_k$ and the fact that in the one-dimensional setting φ_k is an eigenfunction of $L = a + \delta^*\delta$ and $\delta\varphi_k$ is an eigenfunction of $M = a + \delta\delta^*$ (to be precise, the last fact was already invoked from [3] in the non-extended setting, but it easily carries over to the extended situation by the way of extending the coefficients of δ and δ^*). □

Note that for a given $\varepsilon \in \mathbb{Z}_2^d$ it may happen that for some $k \in \mathbb{N}^d$ the function $\delta_1^{\varepsilon_1} \dots \delta_d^{\varepsilon_d}\varphi_k$ vanishes identically. This occurs precisely when there is an $i \in \{1, \dots, d\}$ such that $\varepsilon_i = 1$ and $a_i = \lambda_0^{(i)}$, and $k \in \mathbb{N}^d$ is such that $k_i = 0$.

To construct an orthonormal system $\{\Phi_n\}$ associated with $\mathbb{L}$ and related to the original system $\{\varphi_k\}$, it is natural to consider first the one-dimensional case. Then the relevant multi-dimensional system will be obtained simply by taking tensor products. The construction below is partially motivated by the case of the classical trigonometric system. Let

$$\Phi_{n_i}^{(i)}(x_i) = \begin{cases} \frac{1}{\sqrt{2}}\varphi_{n_i/2}^{(i)}(x_i), & n_i \ \text{even}, \\ -\frac{1}{\sqrt{2}}\big(\lambda_{(n_i+1)/2}^{(i)} - a_i\big)^{-1/2}\,\delta_i\varphi_{(n_i+1)/2}^{(i)}(x_i), & n_i \ \text{odd}. \end{cases}$$

In this place it seems to be natural to require that for each $i = 1, \dots, d$, the derivative δ_i annihilates the first eigenfunction, $\delta_i\varphi_0^{(i)} \equiv 0$. This is equivalent to assuming that the constant a_i from the decomposition of L_i is equal to the first eigenvalue, $a_i = \lambda_0^{(i)}$. In the multi-dimensional setting the requirement means the equality $A = \Lambda_0$. We emphasize that this is indeed the case, up to a convention explained in a moment, of all the classical examples given in [3, Section 7]. Consequently, $\delta_i\varphi_0^{(i)}$ does not enter the definition of $\Phi_{n_i}^{(i)}$ above. On the other hand, notice that all the $\Phi_{n_i}^{(i)}$ are well-defined and non-vanishing. By the facts mentioned earlier (cf. [3, Lemma 2]) each of the systems $\{\Phi_{n_i}^{(i)} : n_i \in \mathbb{N}\}$,

$i = 1, \ldots, d$, is orthonormal in $L^2(X_{\mathrm{SYM}}, \mu_i)$. For a multi-index $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we define

$$\Phi_n = \Phi^{(1)}_{n_1} \otimes \ldots \otimes \Phi^{(d)}_{n_d}.$$

The multi-dimensional system $\{\Phi_n : n \in \mathbb{N}^d\}$ is orthonormal in $L^2(\mathbb{X}, \mu)$. Moreover, the Φ_n are eigenfunctions of $\mathbb{L}$, as stated below.

Lemma 3.4. *We have*

$$\mathbb{L}\Phi_n = \Big(\lambda^{(1)}_{\lfloor \frac{n_1+1}{2} \rfloor} + \ldots + \lambda^{(d)}_{\lfloor \frac{n_d+1}{2} \rfloor}\Big)\Phi_n, \qquad n \in \mathbb{N}^d,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function (the floor function).

Proof. Given $\varepsilon \in \mathbb{Z}_2^d$, notice that, up to a constant factor, $\Phi_{2k-\varepsilon}$ coincides with $\delta_1^{\varepsilon_1} \ldots \delta_d^{\varepsilon_d} \varphi_k$ whenever $2k - \varepsilon \in \mathbb{N}^d$. Then Lemma 3.3 gives the desired conclusion. □

In what follows, for multi-indices $n \in \mathbb{N}^d$ we will use the notation

$$\langle n \rangle = \bigg(\Big\lfloor \frac{n_1+1}{2} \Big\rfloor, \ldots, \Big\lfloor \frac{n_d+1}{2} \Big\rfloor \bigg),$$

and (in particular) $\langle n \rangle = \lfloor \frac{n+1}{2} \rfloor$ when n is a number. Then we may write shortly

$$\mathbb{L}\Phi_n = \lambda_{\langle n \rangle} \Phi_n, \qquad n \in \mathbb{N}^d.$$

The 'real' picture that emerges from the above procedure may be then turned into a 'complex' one. Indeed, define first in dimension one

$$\Psi^{(i)}_{n_i} = \frac{1}{\sqrt{2}} \Big(\Phi^{(i)}_{2|n_i|} + \mathrm{i} \operatorname{sgn} n_i \; \Phi^{(i)}_{2|n_i|-1} \Big), \qquad n_i \in \mathbb{Z},$$

and then for a multi-index $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$,

$$\Psi_n = \Psi^{(1)}_{n_1} \otimes \ldots \otimes \Psi^{(d)}_{n_d}.$$

An easy argument shows that the system $\{\Psi_n : n \in \mathbb{Z}^d\}$ is orthonormal in $L^2(\mathbb{X}, \mu)$ and consists of eigenfunctions of $\mathbb{L}$,

$$\mathbb{L}\Psi_n = \lambda_{|n|} \Psi_n, \qquad n \in \mathbb{Z}^d,$$

where $|n| = (|n_1|, \ldots, |n_d|)$.

We remark that the choice of signs in the construction of $\{\Phi_n\}$ is in principle arbitrary. Our particular choice is motivated by the fundamental example below.

Example 1. The basic example here (and in some sense a prototype) is the case of classical trigonometric expansions. Let $d = 1$ and consider the interval $\mathcal{X} = X = (0, \pi)$ equipped with the measure $\mu(dx) = \frac{1}{\pi} dx$. Further, consider the one-dimensional standard Laplacian $L = -\frac{d^2}{dx^2}$ on $(0, \pi)$ and the related orthonormal basis in $L^2(\mathcal{X}, \mu)$ of cosines,

$$\varphi_k(x) = \begin{cases} 1, & k = 0, \\ \sqrt{2} \cos kx, & k > 0. \end{cases}$$

Clearly, $L\varphi_k = \lambda_k \varphi_k$, where $\lambda_k = k^2$. In addition $M = L = -\frac{d^2}{dx^2}$. Applying the symmetrization procedure we arrive at the trigonometric systems

$$\{\Phi_n : n \in \mathbb{N}\} = \Big\{ \frac{1}{\sqrt{2}}, \sin x, \cos x, \sin 2x, \cos 2x, \ldots \Big\}$$

and

$$\{\Psi_n : n \in \mathbb{Z}\} = \Big\{\frac{1}{\sqrt{2}} \exp(inx) : n \in \mathbb{Z}\Big\}$$

on the interval $\mathbb{X} = (-\pi, \pi)$. These are orthonormal bases in $L^2((-\pi,\pi), \frac{1}{\pi}dx)$ of eigenfunctions of the Laplacian $\mathbb{L} = -\frac{d^2}{dx^2}$ considered on $(-\pi,\pi)$, $\mathbb{L}\Phi_n = \langle n\rangle^2\Phi_n$, $\mathbb{L}\Psi_n = n^2\Psi_n$. This example may be easily generalized to arbitrary dimension $d \geq 1$.

In the situation of Example 1 we could as well choose as the initial system $\{\varphi_k\}$ the system of sines. This, however, leads to a small obstacle since then the constant in the decomposition of L does not coincide with the first eigenvalue, as required above. On the other hand, the system of sines is commonly enumerated by $k = 1, 2, \ldots$, excluding $k = 0$. To overcome these problems, we introduce the following technical convention: in the case just described, and also in similar cases as those of Fourier-Bessel systems (see [3, Section 7.8]), we formally treat $\lambda_0 = 0$ and $\varphi_0 \equiv 0$ as the first eigenvalue and the corresponding eigenfunction, respectively. Then we are in a position to apply the symmetrization, which leads to an extended system $\{\Phi_n\}$ with $\Phi_0 \equiv 0$ to be neglected (thus, in fact, $\{\Phi_n\}$ is enumerated by $n = 1, 2, \ldots$, as is the initial system). Clearly, the convention just described in dimension one induces an analogous convention in the multi-dimensional situation.

Applying this convention to $\varphi_k(x) = \sqrt{2}\sin kx$, $k = 1, 2, \ldots$, and passing to symmetrization we arrive at the trigonometric system $\{\Phi_n\}$ as in Example 1, but with $\Phi_0 = 1/\sqrt{2}$ excluded. Notice that this time the extended system is not complete. This indicates that the symmetrization applied to the system of cosines provides a more natural way of embedding the system of sines into the extended symmetric situation. In other words, it is more natural to view the sines as the 'differentiated' system rather than the initial one.

4. Riesz transforms and conjugacy

In this section we investigate the symmetrized setting from the conjugacy point of view. We define Riesz transforms and conjugate Poisson integrals associated to the extended 'Laplacian' $\mathbb{L}$, and then show that these definitions fit into a consistent conjugacy scheme. This scheme, including Cauchy-Riemann type equations, has precisely the classical shape. When the convention described at the end of Section 3 is in force, then the results below should be understood accordingly.

Following a general concept, we define formally the Riesz transforms of order $N \geq 1$ by $R^l = D^l \mathfrak{L}^{-|l|/2}$, $|l| = N$; all necessary notions will be explained momentarily. To make this definition strict, we need to specify a suitable self-adjoint extension of $\mathbb{L}$. Consider the operator

$$\mathfrak{L}f = \sum_{n\in\mathbb{N}^d} \lambda_{\langle n\rangle} \langle f, \Phi_n\rangle_\mu \Phi_n \tag{4.1}$$

defined on the domain

$$\operatorname{Dom}\mathfrak{L} = \Big\{ f \in L^2(\mathbb{X},\mu) : \sum_{n\in\mathbb{N}^d} \big|\lambda_{\langle n\rangle}\langle f, \Phi_n\rangle_\mu\big|^2 < \infty \Big\}. \tag{4.2}$$

We denote by $\mathcal{N} = (\operatorname{span}\{\Phi_n : n \in \mathbb{N}^d, \lambda_{\langle n\rangle} \neq 0\})^\perp$ the null subspace of $\mathfrak{L}$. Note that $\mathcal{N}$ is not necessarily trivial. Independently of the case, we shall always write Π_0 for the orthogonal projection of $L^2(\mathbb{X},\mu)$ onto $\mathcal{N}^\perp$.

Lemma 4.1. *The inclusion $C^2_c(\mathbb{X}) \subset \operatorname{Dom}\mathfrak{L}$ holds, so that $\mathfrak{L}$ is a nonnegative self-adjoint extension of the operator $\Pi_0\mathbb{L}$ defined initially on $C^2_c(\mathbb{X})$. Moreover, the spectrum of $\mathfrak{L}$ satisfies*

$$\{\Lambda_0, \Lambda_1, \ldots\} \subset \sigma(\mathfrak{L}) \subset \{0\} \cup \{\Lambda_0, \Lambda_1, \ldots\}.$$

Proof. Here arguments are similar to those from the proofs of [3, Lemma 3] and [3, Lemma 6]. We omit the details. □

The spectral decomposition of $\mathfrak{L}$ may be written as

$$\mathfrak{L}f = \sum_{m=0}^{\infty} \Lambda_m \mathcal{P}_m f, \qquad f \in \operatorname{Dom}\mathfrak{L},$$

where the spectral projections are

$$\mathcal{P}_m f = \sum_{\{n\in\mathbb{N}^d \,:\, \lambda_{\langle n\rangle}=\Lambda_m\}} \langle f, \Phi_n\rangle_\mu \, \Phi_n, \qquad m \in \mathbb{N}.$$

Next we define more strictly the Riesz transforms of order $N \geq 1$ by

$$R^l = D^l \mathfrak{L}^{-|l|/2}\Pi_0, \qquad l = (l_1, \ldots, l_d) \in \mathbb{N}^d \backslash \{(0, \ldots, 0)\},$$

where $|l| = l_1 + \ldots + l_d = N$ is the order of the transform, and $D^l = D_1^{l_1} \ldots D_d^{l_d}$ (since the D_j commute, any composition of them may be written in such a form). Notice that for the order one, if $l = e_j$ (the jth coordinate vector), then $D^l = D_j$ and consequently, R^l coincides with $D_j\mathfrak{L}^{-1/2}\Pi_0$; in what follows we will denote these operators by R_j, $j = 1, \ldots, d$. If $N \geq 2$ and $|l| = N$, it is customary to call the operators R^l the Riesz transforms of *higher order*. To provide a fully rigorous definition of R^l we use the spectral series of $\mathfrak{L}$ and set

$$R^l f = \sum_{\lambda_{\langle n\rangle}\neq 0} \big(\lambda_{\langle n\rangle}\big)^{-|l|/2} \langle f, \Phi_n\rangle_\mu D^l \Phi_n, \qquad f \in L^2(\mathbb{X}, \mu); \tag{4.3}$$

(notice that $\lambda_{\langle n\rangle} = 0$ may happen only when $n = (0, \ldots, 0)$). To show that this formula indeed gives rise to L^2-bounded operators we first need to have a closer look at the action of the 'derivatives' on the eigenfunctions. Recall that $\lambda_0^{(j)} = a_j$, $j = 1, \ldots, d$, and so $\Lambda_0 = A$.

Lemma 4.2. *Given $j = 1, \ldots, d$ and $N \geq 1$, we have*

$$D_j^N \Phi_n = \begin{cases} (-1)^{N/2}\big(\lambda^{(j)}_{\langle n\rangle_j} - a_j\big)^{N/2} \, \Phi_n, & N \text{ even}, \\ (-1)^{n_j+1+(N-1)/2}\big(\lambda^{(j)}_{\langle n\rangle_j} - a_j\big)^{N/2} \, \Phi_{n-(-1)^{n_j} e_j}, & N \text{ odd}, \end{cases}$$

with the convention that $\Phi_n \equiv 0$ if $n \notin \mathbb{N}^d$.

Proof. Because of the product structure we may and do assume that $d = 1$ (thus k and n are nonnegative integers). Recall that

$$\Phi_{2k} = \frac{1}{\sqrt{2}}\varphi_k, \qquad \Phi_{2k-1} = \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{\lambda_k - a}}\delta\varphi_k.$$

Since $\varphi_k = \sqrt{2}\Phi_{2k}$ is an even function we have

$$D\Phi_{2k} = \delta\Big(\frac{1}{\sqrt{2}}\varphi_k\Big) = -\sqrt{\lambda_k - a}\,\Phi_{2k-1}, \qquad k \geq 0,$$

and since $-D^2 = \mathbb{L} - a$ and $(\mathbb{L} - a)\varphi_k = (\lambda_k - a)\varphi_k$ (see Lemma 3.3),

$$D\Phi_{2k-1} = -D^2\Big(\frac{1}{\sqrt{\lambda_k - a}}\Phi_{2k}\Big) = (\mathbb{L} - a)\Big(\frac{1}{\sqrt{\lambda_k - a}}\Phi_{2k}\Big) = \sqrt{\lambda_k - a}\Phi_{2k}, \qquad k \ge 1.$$

Therefore,

$$\begin{aligned} D\Phi_n &= \begin{cases} -\sqrt{\lambda_{\langle n\rangle} - a}\,\Phi_{n-1}, & n \text{ even} \\ \sqrt{\lambda_{\langle n\rangle} - a}\,\Phi_{n+1}, & n \text{ odd} \end{cases} \\ &= (-1)^{n+1}\sqrt{\lambda_{\langle n\rangle} - a}\,\Phi_{n-(-1)^n}, \qquad n \in \mathbb{N}, \end{aligned}$$

with the convention that $\Phi_{-1} \equiv 0$. To finish the proof it is now sufficient to observe that a double application of D maps, up to a multiplicative constant, Φ_n onto itself,

$$D^2\Phi_n = -(\mathbb{L} - a)\Phi_n = -\big(\lambda_{\langle n\rangle} - a\big)\Phi_n.$$

□

Note that here, in contrast with the examples considered in [3, Section 7], D_j has proper invariant subspaces that decompose orthogonally the whole subspace $\Pi_0 L^2(\mathbb{X}, \mu) \subset L^2(\mathbb{X}, \mu)$. They are spanned by the pairs $\{\Phi_n, \Phi_{n-e_j}\}$, where n is such that $n_j > 0$ is even. Notice that D_j acts trivially on the subspace spanned by $\{\Phi_n : n_j = 0\}$.

Corollary 4.3. *Given $l \in \mathbb{N}^d \backslash \{(0, \dots, 0)\}$ we have*

$$D^l\Phi_n = (-1)^{|l|/2 + |(n+3/2)\widetilde{l}|}\bigg(\prod_{j=1}^d \big(\lambda^{(j)}_{\langle n\rangle_j} - a_j\big)^{l_j/2}\bigg)\Phi_{n-(-1)^n\widetilde{l}}, \qquad n \in \mathbb{N}^d,$$

where $\widetilde{l}$ is a multi-index such that $\widetilde{l}_j = 0$ if l_j is even and $\widetilde{l}_j = 1$ otherwise, $(-1)^n\widetilde{l} = ((-1)^{n_1}\widetilde{l}_1, \dots, (-1)^{n_d}\widetilde{l}_d)$ and $(n+3/2)\widetilde{l} = ((n_1+3/2)\widetilde{l}_1, \dots, (n_d+3/2)\widetilde{l}_d)$.

As a consequence of the above corollary and Bessel's inequality we get the following.

Proposition 4.4. *The series defining the Riesz transforms R^l converge in $L^2(\mathbb{X}, \mu)$ and for each order $N \ge 1$ the mapping*

$$f \mapsto \bigg(\sum_{|l|=N} |R^l f|^2\bigg)^{1/2}$$

is a (nonlinear) contraction in $L^2(\mathbb{X}, \mu)$. In particular, each R^l is a linear contraction.

It is remarkable that the present approach to the higher order Riesz transforms is considerably simpler than that in [3, Section 4]. This is due to the fact that in the symmetrized setting the subspace spanned by the orthogonal system is invariant under actions of the associated 'derivatives'. More comments in this connection will be given in Section 5.

We pass to defining conjugate Poisson integrals in the symmetrized setting. The Poisson semigroup $\{P_t\}_{t\ge 0}$ associated with $\mathfrak{L}$ is, by the spectral theorem, given on $L^2(\mathbb{X}, \mu)$ by

$$P_t f = \exp\big(-t\mathfrak{L}^{1/2}\big)f = \sum_{m=0}^\infty \exp(-t\Lambda_m^{1/2})\mathcal{P}_m f.$$

Clearly, each P_t, $t \geq 0$, is a contraction on $L^2(\mathbb{X},\mu)$. We now define the conjugate Poisson integrals U_t^j, $t \geq 0$, $j = 1,\dots,d$, as the contractions on $L^2(\mathbb{X},\mu)$ given by

$$U_t^j f = P_t R_j f, \qquad f \in L^2(\mathbb{X},\mu).$$

To rewrite this by means of the spectral series observe that by Lemma 4.2

$$R_j f = \sum_{\lambda_{\langle n\rangle}\neq 0} (-1)^{n_i+1} \left(\frac{\lambda^{(j)}_{\langle n\rangle_j} - a_j}{\lambda_{\langle n\rangle}} \right)^{1/2} \langle f, \Phi_n\rangle_\mu \Phi_{n-(-1)^{n_j} e_j}.$$

Since $\langle n - (-1)^{n_j} e_j\rangle = \langle n\rangle$, we see that

$$U_t^j f = \sum_{\lambda_{\langle n\rangle}\neq 0} (-1)^{n_i+1} \exp\left(-t\sqrt{\lambda_{\langle n\rangle}}\right) \left(\frac{\lambda^{(j)}_{\langle n\rangle_j} - a_j}{\lambda_{\langle n\rangle}} \right)^{1/2} \langle f, \Phi_n\rangle_\mu \Phi_{n-(-1)^{n_j} e_j}.$$

This, together with Bessel's inequality, shows that for each $t \geq 0$ also the mapping

$$f \mapsto \sqrt{|U_t^1 f|^2 + \ldots + |U_t^d f|^2}$$

is a contraction in $L^2(\mathbb{X},\mu)$.

Our definitions of Riesz transforms and conjugate Poisson integrals are well motivated by the following system of Cauchy-Riemann type equations.

Proposition 4.5. *Let f belong to the subspace of $L^2(\mathbb{X},\mu)$ spanned by the Φ_n's. Then*

$$D_i U_t^j f = D_j U_t^i f, \qquad i,j = 1,\dots,d,$$
$$D_j P_t f = -\frac{\partial}{\partial t} U_t^j f, \qquad j = 1,\dots,d.$$

If $A = 0$, then also

$$\sum_{j=1}^d D_j U_t^j f = \frac{\partial}{\partial t} P_t f;$$

for $A > 0$ the function f on the right-hand side above must be replaced by $f - A\mathfrak{L}^{-1}\Pi_0 f$. Moreover, we have the harmonicity relations

$$\Big(\frac{\partial^2}{\partial t^2} - \mathbb{L}\Big) P_t f = 0, \qquad \Big(\frac{\partial^2}{\partial t^2} - \mathbb{L}\Big) U_t^j f = 0, \qquad j = 1,\dots,d.$$

Proof. It is enough to restrict the situation to $f = \Phi_n$, $n \in \mathbb{N}^d$. Since

$$U_t^j \Phi_n = (-1)^{n_j+1} \exp\left(-t\sqrt{\lambda_{\langle n\rangle}}\right) \left(\frac{\lambda^{(j)}_{\langle n\rangle_j} - a_j}{\lambda_{\langle n\rangle}} \right)^{1/2} \Phi_{n-(-1)^{n_j} e_j}$$

and for $i \neq j$, $D_i \Phi_{n-(-1)^{n_j} e_j} = (-1)^{n_i+1}(\lambda^{(i)}_{\langle n\rangle_i} - a_i)^{1/2} \Phi_{n-(-1)^{n_i} e_i-(-1)^{n_j} e_j}$, the first identity follows. The second identity may be also easily justified because

$$D_j P_t \Phi_n = (-1)^{n_j+1} \exp\left(-t\sqrt{\lambda_{\langle n\rangle}}\right) \left(\lambda^{(j)}_{\langle n\rangle_j} - a_j\right)^{1/2} \Phi_{n-(-1)^{n_j} e_j} = -\frac{\partial}{\partial t} U_t^j \Phi_n.$$

To verify the third identity, we observe that

$$\frac{\partial}{\partial t} P_t \Phi_n = -\sqrt{\lambda_{\langle n\rangle}} \exp\left(-t\sqrt{\lambda_{\langle n\rangle}}\right) \Phi_n$$

and since D_j commutes with $\mathbb{L}$, thus also with U_t^j, we have

$$\begin{aligned} D_j U_t^j \Phi_n = U_t^j(D_j\Phi_n) &= (-1)^{n_j+1}\sqrt{\lambda_{\langle n\rangle_j}-a_j}\, U_t^j \Phi_{n-(-1)^{n_j}e_j} \\ &= -\Big(\lambda^{(j)}_{\langle n\rangle_j}-a_j\Big)\exp\Big(-t\sqrt{\lambda_{\langle n\rangle}}\Big)\big(\lambda_{\langle n\rangle}\big)^{-1/2}\Phi_n. \end{aligned}$$

Here the last equality is obtained by recalling that $\langle n-(-1)^{n_j}e_j\rangle = \langle n\rangle$ and also noticing that $n-(-1)^{n_j}e_j-(-1)^{n_j-(-1)^{n_j}}e_j = n$. Finally, checking the harmonicity relations does not cause any problems. □

We remark that a suitable information on the growth of the eigenvalues $\lambda_{\langle n\rangle}$ and on the growth of the eigenfunctions Φ_n and their derivatives allows to show that the identities of Proposition 4.5 hold in fact for all $f\in L^2(\mathbb{X},\mu)$; see [3, Proposition 5].

Further support for the symmetrized conjugacy scheme is provided by the identity

$$\sum_{j=1}^d R_j^2 f = -f + A\mathfrak{L}^{-1}f, \qquad f\in \Pi_0 L^2(\mathbb{X},\mu);$$

notice that when $A=0$ the potential term above vanishes. This is an analogue of the well-known relation $\sum_j R_j^2 = -\mathrm{Id}$, satisfied by the classical Riesz transforms $R_j=\partial_j(-\Delta)^{-1/2}$.

A comment concerning the 'complex' picture from Section 3 is in order. Note that replacing the symbols $\mathbb{N}^d$, $\langle n\rangle$ and Φ_n in (4.1) and (4.2) by $\mathbb{Z}^d$, $|n|$ and Ψ_n, respectively, changes neither $\operatorname{Dom}\mathfrak{L}$ nor $\mathfrak{L}$. Further, replacing $\lambda_{\langle n\rangle}$ and Φ_n in (4.3) by $\lambda_{|n|}$ and Ψ_n, respectively, does not change the Riesz operators (in the one-dimensional setting, the action of D on Ψ_n is $D\Psi_n = \mathrm{i}\operatorname{sgn} n\sqrt{\lambda_{|n|}-a}\,\Psi_n$, $n\in\mathbb{Z}$, and similarly for D_j^N). Consequently, the Poisson semigroup and the conjugate Poisson integrals remain unchanged.

Finally, notice that in the context of Example 1 the Riesz transform given by (4.3) for $l=d=1$ results in the classic conjugacy mapping

$$\frac{a_0}{2}+\sum_{n=1}^\infty\big(a_n\cos nx + b_n\sin nx\big)\mapsto \sum_{n=1}^\infty\big(b_n\cos nx - a_n\sin nx\big)$$

($\sum_{n\in\mathbb{Z}} a_n e^{\mathrm{i}nx}\mapsto \sum_{n\in\mathbb{Z}}\mathrm{i}\operatorname{sgn} n\, a_n e^{\mathrm{i}nx}$ in the 'complex' picture).

5. Comments and examples

First we observe that the setting considered in [3] is naturally embedded in the symmetrized situation. Indeed, given a function f on $\mathcal{X}$, consider its extension $\widetilde{f}$ to $\mathbb{X}$ that is even with respect to all coordinates. Then the definitions and relations from the symmetrized scheme can be applied to $\widetilde{f}$, and this clearly induces analogous restricted definitions and relations related to the original space $\mathcal{X}$. In this way the general definitions of Riesz transforms of order one given in [3, Section 3] and conjugate Poisson integrals given in [3, Section 5] are contained in the symmetrized definitions from Section 4. In a similar manner the Cauchy-Riemann type equations and harmonicity relations [3, (5.3)-(5.6)] are 'projections' of the identities from Proposition 4.5. Moreover, by considering extensions of a function f that are odd with respect to one coordinate and even with respect to all remaining coordinates it can be seen that the 'supplementary' operators and relations established in [3, Section 6] are suitable 'projections' of the symmetrized counterparts from Section 4.

However, the definition of higher order Riesz transforms induced by the symmetrized scheme in the initial setting is essentially different from that postulated in [3, Section 4]. Nevertheless, it seems to be far more appropriate and natural. Observe, that the 'projection' from the symmetrized situation via considering functions that are even with respect to all coordinates leads to higher order derivatives in the initial setting that are of the form

$$\mathbb{D}^n = \big(\underbrace{\ldots\delta_1\delta_1^*\delta_1\delta_1^*\delta_1}_{n_1 \text{ components}}\big)\big(\underbrace{\ldots\delta_2\delta_2^*\delta_2\delta_2^*\delta_2}_{n_2 \text{ components}}\big)\ldots\big(\underbrace{\ldots\delta_d\delta_d^*\delta_d\delta_d^*\delta_d}_{n_d \text{ components}}\big),$$

where $n=(n_1,\ldots,n_d)$ is a multi-index. This obviously makes a contrast (when $n_j>1$ for some $j=1,\ldots,d$) with the derivatives

$$\delta^n = \delta_1^{n_1}\ldots\delta_d^{n_d}$$

used in [3] to define higher order Riesz transforms.

The definition of higher order Riesz transforms in the initial setting based on $\mathbb{D}^n$ (i.e. the 'even projection' of the symmetrized definition) seems to be more natural, in particular no complications occur in connection with showing L^2-boundedness of these operators, see [3, Section 4, Section 7.9]. The new light on understanding higher order derivatives and Riesz transforms in the initial setting should also have an important impact on developing the theory of Sobolev spaces related to orthogonal expansions. This subject remains to be investigated.

We conclude the paper with several concrete examples involving selected classical orthogonal expansions, where the symmetrization procedure can be easily traced explicitly. More exemplifications can be derived from those given in [3, Section 7]; in particular, we follow the notation from there. For the sake of clarity and simplicity, in Examples 2–4 below we assume that $d=1$.

Example 2. Let $\{h_n : n\in\mathbb{N}\}$ be the classical Hermite functions on $\mathbb{R}$ and consider the system $\varphi_k=\sqrt{2}h_{2k}$ on the half-line $\mathcal{X}=(0,\infty)$, $k\in\mathbb{N}$. This system is an orthonormal basis in $L^2(\mathcal{X},dx)$ consisting of eigenfunctions of the harmonic oscillator $L=-\frac{d^2}{dx^2}+x^2$ restricted to $(0,\infty)$. The related derivatives decomposing L are $\delta=\frac{d}{dx}+x$ and $\delta^*=-\frac{d}{dx}+x$, see [3, Section 7.4] (notice that δ is not skew-symmetric). Passing to the symmetrized situation we get the orthonormal system $\{\Phi_n\}$ in $L^2(\mathbb{R},dx)$, which coincides, up to signs, with the full system of Hermite functions. The symmetrized derivative is $Df=\frac{df}{dx}+x\check{f}$, where $\check{f}(x)=f(-x)$ is the reflection of f, and the symmetrized 'Laplacian' has the form

$$\mathbb{L}f=-\frac{d^2f}{dx^2}+x^2f+2f_{\text{odd}},$$

with $f_{\text{odd}}=(f-\check{f})/2$ being the odd part of f. Notice that $\mathbb{L}$ differs from the harmonic oscillator by the reflection term above. On the other hand, the derivative D is formally skew-adjoint in $L^2(\mathbb{R},dx)$.

Example 3. A natural generalization of the previous example is obtained by taking $\mathcal{X}=(0,\infty)$ equipped with the measure $\mu_\alpha(dx)=x^{2\alpha+1}dx$, $\alpha>-1$, and considering the system $\varphi_k=\ell_k^\alpha$ of Laguerre functions of convolution type, see [3, Section 7.6]. Here α is a parameter of type, and the value $\alpha=-1/2$ corresponds to the situation described in

Example 2. The related standard 'Laplacian' is

$$L = -\frac{d^2}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx} + x^2$$

and the associated derivatives are of the form $\delta = \frac{d}{dx} + x$, $\delta^* = -\frac{d}{dx} + x - \frac{2\alpha+1}{x}$. Passing to the symmetrized situation we arrive at the system $\{\Phi_n\}$ that coincides, up to signs, with the system of generalized Hermite functions emerging in the context of the Dunkl harmonic oscillator and the underlying group of reflections isomorphic to $\mathbb{Z}_2$, see [4]. However, the symmetrized 'Laplacian'

$$\mathbb{L}f = -\frac{d^2 f}{dx^2} - \frac{2\alpha+1}{x}\frac{df}{dx} + x^2 f + \frac{2\alpha+1}{x^2} f_{\mathrm{odd}} + 2f_{\mathrm{odd}}$$

differs from the Dunkl harmonic oscillator by the term $2f_{\mathrm{odd}}$ above. The symmetrized derivative $Df = \frac{df}{dx} + x\check{f} + \frac{2\alpha+1}{x} f_{\mathrm{odd}}$ is skew-symmetric, which is not the case of δ.

Example 4. Finally, consider an orthonormal basis $\{\varphi_k\}$ of $L^2(\mathcal{X}, \mu)$ consisting of eigenfunctions of a divergence form operator

$$Lf = -\frac{1}{w}\big(wf'\big)' + af = -\frac{d^2 f}{dx^2} - \frac{w'}{w}\frac{df}{dx} + af,$$

where w is the density of μ and $a \ge 0$ is a constant. We assume that all the technical assumptions from Section 2 are satisfied, in particular $\mathcal{X}$ is an interval of the form $(0, c)$, $0 < c \le \infty$. The derivatives decomposing L have the form $\delta = \frac{d}{dx}$, $\delta^* = -\frac{d}{dx} - \frac{w'}{w}$. Performing the symmetrization procedure, we find the symmetrized 'Laplacian'

$$\mathbb{L}f = -\frac{d^2 f}{dx^2} - \frac{w'}{w}\frac{df}{dx} + af - \Big(\frac{w'}{w}\Big)' f_{\mathrm{odd}}$$

and the associated derivative

$$Df = \frac{df}{dx} + \frac{w'}{w} f_{\mathrm{odd}},$$

which is skew-symmetric in $L^2(\mathbb{X}, \mu)$.

A special case of the situation just described occurs when φ_k are the (normalized) Hermite polynomials of successive even orders, $\mathcal{X} = (0, \infty)$, $w(x) = e^{-x^2}$, $a = 0$, and $L = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}$ is the classical Ornstein-Uhlenbeck operator restricted to the positive half-line. Passing to the symmetrized situation one receives the (normalized) system of Hermite polynomials of all successive orders and the symmetrized 'Laplacian' $\mathbb{L}$ which differs from the Ornstein-Uhlenbeck operator by the reflection term $2f_{\mathrm{odd}}$. The point, however, is that the associated derivative is skew-symmetric.

Another important special case is obtained by choosing φ_k to be the normalized Jacobi trigonometric polynomials considered on the interval $\mathcal{X} = (0, \pi)$ equipped with the measure $d\mu(\theta) = w(\theta)d\theta = (\sin\frac{\theta}{2})^{2\alpha+1}(\cos\frac{\theta}{2})^{2\beta+1}d\theta$. Here $\alpha, \beta > -1$ are parameters of type, and taking $\alpha = \beta = -1/2$ we recover the situation of cosine expansions already discussed in Example 1. The related 'Laplacian' is

$$L = -\frac{d^2\theta}{d\theta^2} - \frac{\alpha - \beta + (\alpha+\beta+1)\cos\theta}{\sin\theta}\frac{d}{d\theta} + \Big(\frac{\alpha+\beta+1}{2}\Big)^2$$

and the derivatives decomposing it have the form $\delta = \frac{d}{d\theta}$, $\delta^* = -\frac{d}{d\theta} - (\alpha + \frac{1}{2})\cot\frac{\theta}{2} + (\beta + \frac{1}{2}\tan\frac{\theta}{2})$. The symmetrized 'Laplacian' is then

$$\mathbb{L}f = -\frac{d^2 f}{d\theta^2} - \frac{\alpha - \beta + (\alpha+\beta+1)\cos\theta}{\sin\theta}\frac{df}{d\theta} + \Big(\frac{\alpha+\beta+1}{2}\Big)^2 f + \frac{(\alpha+\beta+1) + (\alpha-\beta)\cos\theta}{\sin^2\theta} f_{\mathrm{odd}}$$

and the associated skew-symmetric derivative is

$$Df = \frac{df}{d\theta} + \frac{\alpha - \beta + (\alpha+\beta+1)\cos\theta}{\sin\theta} f_{\mathrm{odd}}.$$

It is worth to note that this D coincides with the Jacobi-Dunkl operator on the interval $(-\pi, \pi)$, and the extended system in the 'complex' picture $\{\Psi_n\}$ consists of trigonometric polynomials called the Jacobi-Dunkl polynomials; see [1], for instance.

Developing widely understood harmonic analysis for orthogonal expansions is intimately connected with (sometimes implicit) choice of the associated 'Laplacian'. The results of this paper, and in particular the examples given above, show that in many cases there are reasonable and in some aspects more natural alternatives for standard 'Laplacians' related to various orthogonal systems appearing in the literature. From this point of view deleting the constant A in the decomposition (3.1) of $\mathbb{L}$ would lead, in some sense, to canonical 'Laplacian' associated to general orthogonal expansions, bringing the related harmonic analysis closer to the classical case.

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