LIFTING TROPICAL CURVES IN SPACE AND LINEAR SYSTEMS ON GRAPHS

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Abstract. Tropicalization is a procedure for associating a polyhedral complex to a subvariety of an algebraic torus. We study the question on which graphs arise from tropicalizing algebraic curves. By using Baker's technique of specialization of linear systems from curves to graphs, we are able to give a necessary condition for a balanced weighted graph to be the tropicalization of a curve. Our condition reproduces a generalization of Speyer's well-spacedness condition and also gives new conditions.

1. Introduction

Tropical geometry transforms questions in algebraic geometry to ones about polyhedral geometry. Let $\mathbb{K} = \mathbb{C}((t))$ be the field of formal Laurent series, and let $\mathcal{O} = \mathbb{C}[[t]]$ be its valuation ring, the ring of formal power series. To a subvariety in an algebraic torus $V \subset (\mathbb{K}^*)^n$, the method of tropicalization associates a polyhedral complex $\mathrm{Trop}(V) \subset \mathbb{R}^n$. It is a natural question to ask which polyhedral complexes arise in this fashion. In this case, we say that the polyhedral complex *lifts*. The case where V is a curve has been studied in papers of Speyer [18], Nishinou [15], Tyomkim [19], and in a forthcoming paper of Brugallé and Mikhalkin. These papers give necessary and sufficient conditions for a tropical curve to lift.

It is natural to enlarge the class of objects studied from curves in $(\mathbb{K}^*)^n$ to maps of curves by using the approach of Nishinou-Siebert [16]. Given a map of a smooth curve $f : C^* \to (\mathbb{K}^*)^n$, there is a parameterized tropicalization $\mathrm{Trop}(f) : \Sigma \to \mathbb{R}^n$, a map of a graph that is rational affine linear on edges. Here, the graph Σ is a certain kind of dual graph for a regular semi-stable model completing C^* over $\mathcal{O}$. To each edge e of Σ is associated a weight $m(e) \in \mathbb{N}$ that satisfies the balancing condition: for each vertex $v \in V(\Sigma)$ with adjacent edges $e_1, \ldots, e_k$ in primitive integer vector directions $w_1, \ldots, w_k$,

$$\sum m(e_i) w_i = 0.$$

Therefore, one should restrict one's attention to balanced weighted parameterized graphs $\Phi : \Sigma \to \mathbb{R}^n$. There is an important dichotomy introduced by Mikhalkin [12] between regular and super-abundant parameterized graphs depending on whether they move in a family of the expected dimension. In fact, one may do a dimension count of graphs that have the same combinatorial type and edge directions by varying the length of bounded edges subject to the constraint that loops must close up. More precisely, if $E(\Sigma)^\bullet$ is the set of bounded edges of Σ then there is a natural map

$$\mathbb{R}^{E(\Sigma)^\bullet} \to \mathrm{Hom}(H_1(\Sigma), \mathbb{R}^n)$$

taking a choice of edge-lengths to a function associating a cycle $\gamma \in H_1(\Sigma)$ to the total displacement when traveling around the edges of γ. The kernel of this map intersected with the positive orthant, $\mathbb{R}_+^{E(\Sigma)^\bullet}$ is the space of graphs with the same combinatorial type and slopes. If this map is surjective then (Σ, Φ) is said to be *regular*. Otherwise, it is said to be superabundant. If a curve is regular and the residue field $\mathbf{k}$ has characteristic 0, then it lifts by a theorem proved by Speyer [17] (see also the discussion in [13] and proofs by Nishinou [15] and Tyomkin [19] using deformation theory). A particular case where a curve is superabundant is when a cycle Γ is mapped to a proper affine subspace H of $\mathbb{R}^n$. In this case, the map

$$\mathbb{R}^{E(\Sigma)^\bullet} \to \mathrm{Hom}(H_1(\Sigma), \mathbb{R}^n) \to \mathrm{Hom}(H_1(\Gamma), \mathbb{R}^n)$$

is not surjective. We say the curve is *planar-superabundant*. In the case of genus 1 curves, every superabundant curve is planar-superabundant.

In the case of genus 1 curves, there is a necessary and sufficient condition due to Speyer [18] for a superabundant curve to lift. Let Γ be the cycle in $\Gamma' = \mathrm{Trop}(f)^{-1}(H)$. Let $x_1, \ldots, x_s$ be the boundary points of the connected component of Γ' containing Γ. Then the minimum lattice distance $\mathrm{dist}(x_i, \Gamma)$ must be achieved twice. Speyer proves this using uniformization of curves. Nishinou [15] later gave a proof of this condition and its higher genus analog using log deformation theory. A paper giving an analytic proof of the higher genus analog is in progress by Brugallé-Mikhalkin.

In this paper, we give a necessary condition for planar-superabundant curves to lift which is equivalent to Speyer's and Nishinou's conditions when they apply and which gives a new condition in additional cases. Our technique is to use the specialization of linear systems from curves to graphs developed by Baker [1]. One considers a particular log 1-form on C, $\omega_m = f^* \frac{\mathrm{d}z^m}{z^m}$ where z^m is a character on $(\mathbb{K}^*)^n$. By standard results from log geometry, this 1-form extends to a regular log 1-form on a regular semi-stable model $\mathcal{C}$ over $\mathcal{O}$ completing C^*. Σ is the dual graph of the central fiber $\mathcal{C}_0$. The degree of vanishing of ω_m on components C_v of the central fiber $\mathcal{C}_0$ defines a piecewise-linear function $\tilde{\varphi}_m : \Sigma \to \mathbb{R} \cup \{\infty\}$ which is an element of the linear system associated to the canonical bundle, $L(K_\Sigma)$. The values of $\tilde{\varphi}_m$ constrains the poles and zeros of the restriction of $(\omega_m)_v = \frac{\omega_m}{t^h}|_{C_v}$ for $v \in \Gamma$ and a suitable value of h. For certain v depending on m, $\frac{\omega_m}{t^h}|_{C_v}$ is an exact 1-form. This gives very strong conditions on $\tilde{\varphi}_m$. For example, we make use of the facts that an exact 1-form does not have simple poles and that a non-zero exact 1-form on a cycle of rational curves (which is a degenerate elliptic curve) must have two poles counted with multiplicity. In the case where $\mathcal{C}_0$ has all components rational and Σ is trivalent, the constraints are purely combinatorial. The main result of this paper is the following:

Theorem 1.1. *Let $f : C^* \to (\mathbb{K}^*)^n$ is a map of a smooth curve with parameterized tropicalization $\mathrm{Trop}(f) : \Sigma \to \mathbb{R}^n$. After possibly subdividing each edge of Σ into l congruent segments, for any $m \in M = \mathrm{Hom}((\mathbb{K}^*)^n, \mathbb{K}^*)$, there exists a non-negative function $\tilde{\varphi}_m \in L(K_\Sigma)$ such that the map $m \mapsto \varphi_m$ gives a tropical homomorphism of M to $L(K_\Sigma)$. For each $m \in M$, φ_m satisfies the following properties:*

(1) *For $e \in E(\Sigma)$ with $m \cdot e \neq 0$, $\varphi_m = 0$ on e,*
(2) *φ_m never has slope 0 on any edges e with $m \cdot e = 0$*

In addition, for any $c \in \mathbb{R}$, $H = \{x | \langle m, x\rangle = c\}$, let $\Gamma' = \mathrm{Trop}(f)^{-1}(H)$, considered as a subspace of Σ. Let Γ be a bounded, connected subgraph contained in the interior of Γ'. Then φ_m is $\mathcal{C}_0$-ample on Γ in Γ'

The definitions of the terms in the statement are given in the body of the paper. The function φ_m is a certain perturbation of $\tilde{\varphi}_m$ whose use simplifies the statement of the theorem.

We may rephrase this theorem as an obstruction to a balanced weighted integral graph Σ' in $\mathbb{R}^n$ to be $\mathrm{Trop}(C)$ for a curve $C \subset (\mathbb{K}^*)^n$ in terms of *tropical parameterizations* defined in Definition 4.17.

Corollary 1.2. *If* $\Sigma' = \mathrm{Trop}(C)$ *then there is a tropical parameterization* $p : \Sigma \to \Sigma'$ *with a tropical homomorphism* $M \to L(K_\Sigma)$ *given* $m \mapsto \varphi_m$ *satisfying the properties above.*

Given a balanced weighted integral graph Σ' in $\mathbb{R}^n$, there are only finitely many possible tropical parameterization $p : \Sigma \to \Sigma'$ of a given genus g, so only finitely many cases need to be checked to see if this condition prevents a graph from being the tropicalization of a curve of genus g.

We conjecture the following partial converse:

Conjecture 1.3. *If* $\mathrm{Trop}(f) : \Sigma \to \mathbb{R}^n$ *is a parameterized trivalent tropical curve and there is an association* $m \mapsto \varphi_m$ *satisfying the above properties, then there exists a map of a smooth curve* $f : C^* \to (\mathbb{K}^*)^n$ *with parameterized tropicalization* $\mathrm{Trop}(f)$.

Our goal in this paper is to study the combinatorial content of obstruction theory. In a certain sense, our approach is very similar to that of Nishinou and Tyomkin. We were curious about how the non-vanishing of an obstruction at a certain order in obstruction theory would manifest itself combinatorially. In fact, we unwound the definition of the obstruction in the log obstruction group $H^1(\mathcal{C}_0, \mathcal{H}\mathrm{om}(f^*\Omega_{X^\dagger}, \Omega_{\mathcal{C}_0^\dagger}))$. The obstructions that we were seeing initially looked very much like the conditions on the rank of a linear system on a graph as developed by Baker-Norine [2]. Ultimately, however, it did not exactly fit into that framework and instead had to do with whether a particular combinatorially defined divisor D_φ on a nodal curve had a non-trivial linear system. While the statement of our condition is unwieldy, we believe it to be natural. We hope that our techniques can be used in other problems.

We believe that the functions φ_m give an additional structure on tropical curves that arise as tropicalizations. In future work, we hope to explore the analogous structure on higher dimensional tropicalizations. We expect this work to fit into the log geometry framework of Gross-Siebert [4].

A slightly weakened for our main theorem holds for valuation fields $\mathbb{K}$ of equicharacteristic 0. For a given $m \in M$, one may produce φ_m satisfying the conditions of our theorem on some subdivision of Σ. This subdivision however may depend on m. The choice of $\mathbb{K} = \mathbb{C}((t))$ allows us to pick a subdivision that works for all $m \in M$ by using the fact that for some l, the log 1-forms ω_m all have $\mathbb{K}[t^{1/l}]$-rational zeroes.

Our method of understanding the behavior of a function by looking at the restriction to the central fiber of a related 1-form is very similar to Coleman's method of effective Chabauty in the bad reduction case as explored by Lorenzini-Tucker [10] and McCallum-Poonen [11]. Since Abelian varieties have a theory of toric degenerations [14], it may be possible to use our method to get bounds on the number of rational points that specialize to certain components in a degeneration of a curve in its Jacobian.

We give an outline of the paper. Background on specialization of linear systems is given in section 2. In section 3, we establish some new results involving specializing sections of line-bundles. We assemble background on toric schemes, log structures, tropicalization in section 4. The notation in Theorem 1.1 is defined in section 5. Section 6 gives the proof of the theorem while section 7 shows that the theorem implies a generalization of Speyer's well-spacedness condition. Section 8 gives an example that does not lift to a curve of genus 3 by our obstruction but which is not obstructed by any other known condition.

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2. Specialization from Curves to Graphs

We review some results on specialization of linear systems from curves to graphs from [1]. A semistable family of curves $\mathcal{C}$ over $\operatorname{Spec}\mathcal{O}$ is a family of curves such that the generic fiber $\mathcal{C}_{\mathbb{K}}$ is smooth and the central fiber $\mathcal{C}_0$ is reduced with only ordinary double points as singularities. By blowing up the singular points, one can ensure that $\mathcal{C}$ is regular. A node of $\mathcal{C}_0$ is formal locally parameterized by $\mathcal{O}[x,y]/(xy-t^l)$. The family is regular near the node if and only if $l=1$. A marked semistable family of curves is a family $\mathcal{C}$ together with sections, $\sigma_1,\dots,\sigma_n : \operatorname{Spec}\mathcal{O}\to\mathcal{C}^{\mathrm{sm}}$ valued in the smooth locus.

Definition 2.1. *The* dual graph *of a marked semistable curve $\mathcal{C}$ is a graph with bounded and labelled unbounded edges (called leaves) whose vertices correspond to irreducible components of $\mathcal{C}_0$, whose edges correspond to nodes connecting components, and whose leaves correspond to marked points. To each vertex v is associated the genus $g(v)$ of the corresponding irreducible component C_v on $\mathcal{C}$.*

The genus of the dual graph is defined to be

$$g(\Sigma)=h^1(\Sigma)+\sum_v g(v).$$

In general, $g(\Sigma)$ is equal to the genus of $\mathcal{C}_{\mathbb{K}}$. $\mathcal{C}$ is said to be maximally degenerate if all components of $\mathcal{C}_0$ are rational. In this case, $g(\mathcal{C}_{\mathbb{K}})=h^1(\Sigma)$.

Let $\mathcal{C}$ be a semistable regular family of curves over $\mathcal{O}$ with dual graph Σ. Let $V(\Sigma)$, $E(\Sigma)^\bullet$, $E(\Sigma)^\circ$ be the vertices, bounded edges, and leaves of Σ. For $v\in V(\Sigma)$, let C_v be the irreducible component of $\mathcal{C}_0$ corresponding to v. For an edge $e\in E(\Sigma)^\bullet$ between v_1,v_2, let p_e be the corresponding node in $\mathcal{C}_0$ between C_{v_1} and C_{v_2}. For an unbounded edge $e'\in E(\Sigma)^\circ$, let $\sigma_{e'}$ be the corresponding marked point.

A divisor on Σ is an element of the free abelian group on $V(\Sigma)$. We write a divisor as $D=\sum_{v\in V(\Sigma)}a_v(v)$. The group of all divisors is denoted by $\operatorname{Div}(\Sigma)$. We say a divisor D is non-negative and write $D\geq 0$ if $a_v\geq 0$ for all $v\in V(\Sigma)$. We write $D\geq D'$ if $D-D'\geq 0$. An important divisor in the following will be *the canonical divisor* on Σ is

$$K_\Sigma=\sum_{v\in V(\Sigma)}(\deg(v)+2g(v)-2)(v).$$

In the maximal degenerate case, this is equal to the canonical divisor of [2]. The specialization map $\rho:\operatorname{Div}(\mathcal{C})\to\operatorname{Div}(\Sigma)$ is given by for $\mathcal{D}\in\operatorname{Div}(\mathcal{C})$,

$$\rho(\mathcal{D})=\sum_{v\in\Gamma}(C_v\cdot\mathcal{D})(v).$$

In particular if $\varphi : V(\Sigma) \to \mathbb{Z}$,

$$\rho(\sum_v \varphi(v)C_v) = -\Delta(\varphi)$$

where $\Delta(\varphi)$ is the Laplacian of φ given by

$$\Delta(\varphi) = \sum_{v\in V(\Sigma)} \sum_{e=vw} (\varphi(v) - \varphi(w))(v).$$

Definition 2.2. *Let Λ be a divisor on Σ. If φ is a piecewise-linear function on Σ with $\Delta(\varphi) + \Lambda \geq 0$, we write $\varphi \in L(\Lambda)$ to denote that φ is in the linear system given by Λ.*

3. Specialization of Line Bundles

Definition 3.1. *Let $\mathcal{L}$ be a line bundle over a regular semistable family $\mathcal{C}$ with dual graph Σ. Let s a rational section of $\mathcal{L}$. Define the* vanishing function *of s to be $\varpi_s : \Sigma \to \mathbb{R} \cup \{\infty\}$ by setting $\varpi_s(v)$ to be the multiplicity of C_v in the divisor (s). We extend ϖ_s linearly on bounded edges and as a constant on unbounded edges. If $s = 0$, set $\varpi_s = \infty$.*

The divisor of s as a section of $\mathcal{L}$ satisfies

$$(s) = \sum \varpi_s(v)C_v + H$$

where H is a horizontal divisor. Note that $\varpi_s(v)$ is the order of vanishing of s on the generic point of C_v in the central fiber. If s is a regular section of $\mathcal{L}$, then $\varpi_s \geq 0$. In the case where s is a section of the restriction of $\mathcal{L}$ to the generic fiber, we define ϖ_s by extending s as a rational section on $\mathcal{C}$.

Let $\pi : \tilde{\mathcal{C}}_0 \to \mathcal{C}_0$ be the normalization morphism. We use C_v to refer to a connected component of $\tilde{C}_0$. Let Λ be the divisor on Σ given by

$$\Lambda = \sum_{v\in V(\Sigma)} \deg(\pi^*\mathcal{L}|_{C_v})(v).$$

Definition 3.2. *For s a regular section of $\mathcal{L}$, we say that s has* $\mathbb{K}$-rational zeroes *if the divisor (s) on $C_{\overline{\mathbb{K}}}$ is supported on $C(\mathbb{K})$.*

Note that $\mathbb{K}$-rational points of $\mathcal{C}$ specialize to smooth points of the central fiber $\mathcal{C}_0$.

Let s be a regular section of $\mathcal{L}$. For $v \in V(\Sigma)$, let $s_v = \pi^*\left(\frac{s}{t^{\varpi(v)}}\right)|_{C_v}$. For $e = v_1v_2$, let $\operatorname{ord}_{p_e}(s_{v_1})$ be the order of vanishing of s_{v_1} at $p_e \in C_{v_1}$. We relate the poles of s_v to ϖ_s.

Lemma 3.3. *Suppose s has $\mathbb{K}$-rational zeroes. Let $e = v_1v_2$ be an edge in Σ. Then,*

$$\operatorname{ord}_{p_e}(s_{v_1}) = \varpi_s(v_2) - \varpi_s(v_1).$$

Proof. Write $(s) = \overline{D} + \sum_v \varpi(v)C_v$ on $\mathcal{C}$ where $\overline{D}$ is a horizontal divisor. Then $\frac{s}{t^{\varpi_s(v_1)}}$ has divisor $\overline{D} + \sum_v(\varpi_s(v) - \varpi_s(v_1))C_v$. Since p_e is a node, no component of $\overline{D}$ intersects the central fiber in p_e. Consequently, near p_e, $\frac{s}{t^{\varpi_s(v_1)}}$ has the divisor $(\varpi_s(v_2) - \varpi_s(v_1))C_{v_2}$. By restricting to C_{v_1}, we see $\operatorname{ord}_{p_e}(s_{v_1}) = \varpi_s(v_2) - \varpi_s(v_1)$. ☐

We relate the vanishing functions of two sections and their sum. For $a, b, c \in \mathbb{R}$, we write $a \oplus b \oplus c = 0$ if the minimum of $\{a, b, c\}$ is achieved twice. For real valued functions, we write $f \oplus g \oplus h = 0$ if $f(x) \oplus g(x) \oplus h(x) = 0$ for all x.

Lemma 3.4. *Given sections s_1, s_2 such that s_1, s_2, s_1+s_2 have $\mathbb{K}$-rational zeroes then $\varpi_{s_1} \oplus \varpi_{s_2} \oplus \varpi_{s_1+s_2} = 0$.*

Proof. We first check on vertices. If $\varpi_{s_1}(v) \neq \varpi_{s_2}(v)$ then $\varpi_{s_1+s_2}(v) = \min(\varpi_{s_1}(v), \varpi_{s_2}(v))$. If $\varpi_{s_1}(v) = \varpi_{s_2}(v)$ then $\varpi_{s_1+s_2}(v) \geq \min(\varpi_{s_1}(v), \varpi_{s_2}(v))$. The case of unbounded edges follows trivially.

Now, we check bounded edges. Let $e = vw$ be a bounded edge. Since $\varpi_s = \varpi_{-s}$, we may treat $\{s_1, s_2, s_1+s_2\}$ symmetrically. First consider the case that $\varpi_{s_1}(v) = \varpi_{s_2}(v) = \varpi_{s_1+s_2}(v)$. We may also suppose $\varpi_{s_1}(w) = \varpi_{s_2}(w) \leq \varpi_{s_1+s_2}(w)$. Then the conclusion immediately follows. Now suppose that $\varpi_{s_1}(v) = \varpi_{s_2}(v) < \varpi_{s_1+s_2}(v)$. Then

$$(s_1)_v + (s_2)_v = \left(\frac{s_1+s_2}{t^{\varpi_{s_1}(v)}}\right)\Big|_{C_v} = 0.$$

Therefore, $\operatorname{ord}_{p_e}((s_1)_v) = \operatorname{ord}_{p_e}((s_2)_v)$. By Lemma 3.3, $\varpi_{s_1}(w) = \varpi_{s_2}(w)$. Since the result is true on vertices, $\varpi_{s_1}(w) = \varpi_{s_2}(w) \leq \varpi_{s_1+s_2}(w)$. □

Lemma 3.5. *If s is a section of $\mathcal{L}$ that is regular on the generic fiber C then $\Delta(\varpi_s) + \Lambda \geq 0$. Moreover, if*

$$Z_v = \{\sigma_i \in C_v | s_v(\sigma_i) = 0\}$$

are the marked points on C_v on which s_v vanishes then $\Delta(\varpi_s) + \Lambda \geq \sum_v (\#Z_v)(v)$.

Proof. Write $(s) = D$ for an effective divisor D on C. On $\mathcal{C}$ we have

$$(s) = \overline{D} + \sum_v \varpi_s(v) C_v$$

where s is extended as a section of $\mathcal{L}$. Consequently, $\rho((s)) = \Lambda$. On the other hand,

$$\rho(\overline{D} + \sum_v \varpi_s(v) C_v) = \rho(\overline{D}) - \Delta(\varpi_s) \geq -\Delta(\varpi_s).$$

We claim that $Z_v \subset \overline{D}$. Let $\sigma_i \in Z_v$. σ_i is a zero of $\frac{s}{t^{\varpi_s(v)}}$ which does not vanish on C_v. Therefore σ_i belongs to a component of the zero locus of s that intersects the generic fiber. Consequently $Z_v \subseteq \overline{D} \cap \mathcal{C}_0$ and so

$$\Delta(\varpi_s) + \Lambda = \rho(\overline{D}) \geq \sum_v (\#Z_v)(v).$$

□

In particular, the above lemma says that ϖ_s is an element of the linear system $L(\Lambda)$.

We will need the following lemma to find an algebraic extension $\mathbb{K}'$ of $\mathbb{K}$ to ensure that the zeroes of all elements of a linear system are $\mathbb{K}'$-rational.

Lemma 3.6. *Let L be a line bundle on C defined over $\mathbb{K}$. Let $V = \Gamma(C, L)$ be the sections of L. Then there exists a finite field extension $\mathbb{K}'/\mathbb{K}$ such that all zeroes of any $s \in V_{\mathbb{K}}$ are $\mathbb{K}'$-rational.*

Proof. L has finite degree on C, say d. If $s \neq 0$, $(s)_{\overline{\mathbb{K}}} \subset C(\overline{\mathbb{K}})$ consists of at most d points. This gives a homomorphism

$$\hat{\mathbb{Z}} = \operatorname{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \operatorname{Aut}((s)_{\overline{\mathbb{K}}}).$$

Since each Galois-orbit of a zero of s has at most d elements, every element of $\mathrm{Aut}((s)_{\overline{\mathbb{K}}})$ has order dividing $d!$. Consequently, the subgroup $d!\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$ acts trivially on $(s)_{\overline{\mathbb{K}}}$. But $d!\hat{\mathbb{Z}}$ has fixed field $\mathbb{K}' = \mathbb{K}[t^{\frac{1}{d!}}]$. Consequently, the Galois group $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}')$ acts trivially on $(s)_{\overline{\mathbb{K}}}$. It follows that the zeroes of s are $\mathbb{K}'$-rational. This choice of $\mathbb{K}'$ was independent of s. □

In general, given any field $\mathbb{K}$ and a finite number of sections, we can set $\mathbb{K}'$ to be a field containing the fields of definition of the zeroes of the section. In the above lemma, we have infinitely many sections and had to make use of the fact that $\mathbb{K} = \mathbb{C}((t))$.

4. Toric Schemes, Log Structures, and Tropicalization

In this section, we review the construction of the toric scheme $\mathcal{P}$ over $\mathcal{O}$ from a rational polyhedral subdivision Ξ of $\mathbb{R}^n$ [9, 16, 17], background about log structures [6, 7], and a suitable notion of parameterized tropicalization [16].

Recall that $\mathcal{O} = \mathbb{C}[[t]]$. We will use $\mathcal{O}_l$ to denote $\mathcal{O}/(t^{l+1})$. For a scheme $\mathcal{X}$ over $\mathcal{O}$, $\mathcal{X}_l$ will denote $\mathcal{X} \times_{\mathcal{O}} \mathcal{O}_l$. Let $(\mathbb{K}^*)^n$ be an algebraic torus and $M = \mathrm{Hom}((\mathbb{K}^*)^n, \mathbb{K}^*)$ be its character lattice and $N = \mathrm{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n)$ be its one-parameter subgroup lattice. Let $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ be $M \otimes \mathbb{R}$ and $N \otimes \mathbb{R}$, respectively.

Definition 4.1. *A complete rational polyhedral complex in $\mathbb{R}^n$ is a collection Ξ of finitely many convex rational polyhedra $P \subset \mathbb{R}^n$ whose minimal faces are vertices in $\mathbb{Q}^n$ such that*

- *If $P \in \Xi$ and P' is a face of P, then P' is in Ξ,*
- *If $P, P' \in \Xi$ then $P \cap P'$ is a face of both P and P', and*
- *The union $\bigcup_{P\in\Xi} P$ is equal to $\mathbb{R}^n$.*

Given a Ξ as above, we can construct a fan $\tilde{\Xi}$ in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ as follows: for each $P \in \Xi$ let $\tilde{P}$ be the closure in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ of the set

$$\{(x, a) \subset \mathbb{R}^n \times \mathbb{R}_{>0} : \frac{x}{a} \in P\}.$$

Then $\tilde{P}$ is a rational polyhedral cone in $\mathbb{R}^n \times \mathbb{R}_{>0}$. Its facets come in two types:

- cones of the form $\tilde{P}'$, where P' is a facet of P, and
- the cone $P_0 = \tilde{P} \cap (\mathbb{R}^n \times \{0\})$,

We let $\tilde{\Xi}$ be the collection of cones of the form $\tilde{P}$ and P_0 for P in Ξ. It is a rational polyhedral fan in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ by Cor 3.12 of [3]. Note that $\Xi = \tilde{\Xi} \cap (\mathbb{R}^n \times \{1\})$. On the other hand, the fan Ξ_0 given by $\tilde{\Xi} \cap (\mathbb{R}^n \times \{0\})$ is the limit as a approaches zero of the polyhedral complexes $a\Xi$.

Let $X(\tilde{\Xi})$ be the toric variety associated to the fan $\tilde{\Xi}$. Projection from $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ induces a map of fans from $\tilde{\Xi}$ to the fan $\{0, \mathbb{R}_{\geq 0}\}$ associated to $\mathbb{A}^1$. This gives rise to a flat morphism of toric varieties $X(\tilde{\Xi}) \to \mathbb{A}^1$. Let $\iota : \mathrm{Spec}\, \mathcal{O} \to \mathbb{A}^1$ be the inclusion induced by $\mathbb{Z}[t] \to \mathcal{O}$. We will use $\mathcal{P} = X(\Xi)$ denote the scheme over $\mathcal{O}$ given by $X(\tilde{\Xi}) \times_{\mathbb{A}^1} \mathcal{O}$.

We summarize results of [16] concerning this construction:

- The general fiber $X(\tilde{\Xi}) \times_{\mathrm{Spec}\, \mathcal{O}} \mathrm{Spec}\, \mathbb{K}$ is isomorphic to the toric variety over $\mathbb{K}$ associated to Ξ_0.

- If Ξ is *integral*, i.e. the vertices of every polyhedron in Ξ lie in $\mathbb{Z}^n$, then the central fiber $X(\widetilde{\Xi})_0 = X(\widetilde{\Xi}) \times_{\operatorname{Spec} \mathcal{O}} \operatorname{Spec} \mathbf{k}$ is reduced.
- There is an inclusion-reversing bijection between closed torus orbits in $X(\widetilde{\Xi})_{\mathbf{k}}$ and polyhedra P in Ξ; the irreducible components of $X(\widetilde{\Xi})_{\mathbf{k}}$ correspond to vertices in Ξ; the intersection of a collection of irreducible components corresponds to the smallest polyhedron in Ξ containing all of their vertices.

$\mathcal{P}_0$ is a union of a toric strata corresponding to the subdivision Ξ. For a cell $P \in \Xi$, $\mathcal{U}_P = \operatorname{Spec} \mathbb{Z}[\tilde{P}^\vee] \times_{\mathbb{A}^1} \mathcal{O}$ is a toric open set of $\mathcal{P}$. If P lies in the affine hyperplane $\{x | \langle m, x\rangle = c\}$, then $(m, c) \in \tilde{P}^\perp$ and $t^{-c}z^m$ is invertible in $\mathbb{Z}[\tilde{P}^\vee]$. Consequently, $t^{-c}z^m$ is a unit on $\mathcal{U}_P$ and therefore on $(\mathcal{U}_P)_l \subset \mathcal{P}_l$.

We now introduce log structures closely following the exposition of [6].

Definition 4.2. *A* pre-log structure *on a scheme X is a pair $(\mathcal{M}, \alpha)$ where $\mathcal{M}$ is a sheaf of monoids $\mathcal{M}$ and α is a homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$. If α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$, then we say $(\mathcal{M}, \alpha)$ is a* log structure. *A* log scheme *$(X, \mathcal{M}, \alpha)$ is a scheme X with a log structure $(\mathcal{M}, \alpha)$. We may denote such a log scheme by $X^\dagger$.*

A pre-log structure $(\mathcal{M}, \alpha)$ on X canonically induces a log structure $(\mathcal{M}^a, \alpha^a)$ on X by adjoining units to $\mathcal{M}$.

Definition 4.3. *A* morphism of log schemes *$f : (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ is a pair (f, ϕ) where f is a morphism of schemes $f : X \to Y$ and ϕ is a homomorphism of sheaves of monoids on X, $\phi : f^{-1}\mathcal{N} \to \mathcal{M}$ such that the following diagram commutes:*

$$\begin{array}{ccc} f^{-1}\mathcal{N} & \xrightarrow{\phi} & \mathcal{M} \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X. \end{array}$$

Log structures glue in a way similar to schemes.

Example 4.4. The trivial log structure on X is $\mathcal{M} = \mathcal{O}_X^*$ with $\alpha = 1_{\mathcal{O}_X}$.

Example 4.5. For a commutative ring A and a monoid P, there is a natural pre-log structure on $\operatorname{Spec} A[P]$ given by $\alpha : P \to A[P]$. The induced log scheme $(\operatorname{Spec} A[P], P^a, \alpha^a)$ is called the *canonical monoid log structure*. The construction is functorial: if $\phi : P \to Q$ is a homomorphism of monoids, there is an induced morphism of log schemes $\phi^* : (\operatorname{Spec} A[Q], Q^a) \to (\operatorname{Spec} A[P], P^a)$.

Example 4.6. For Δ, a rational fan in $\mathbb{R}^n$, there is a natural log structure on the toric variety $X(\Xi)$. For each cone $\sigma \in \Delta$, there is a toric affine open $U_\sigma = \operatorname{Spec} \mathbf{k}[M \cap \sigma^\vee]$ with a monoid log structure. These log structures glue to give a log structure on $X(\Xi)$.

Example 4.7. Since $\mathbb{A}^1$ is the toric variety associated to $\mathbb{R}_{\geq 0}$, it naturally has a log structure induced by $\alpha : \mathbb{N} \to \mathbf{k}[t]$ where $\alpha(n) = t^n$. By considering $\operatorname{Spec} \mathcal{O}$ as the formal local neighborhood of the origin of $\mathbb{A}^1$, we obtain a log structure on $\operatorname{Spec} \mathcal{O}$ given by $\alpha : \mathcal{O}^* \times \mathbb{N} \to \mathcal{O}$ where $\alpha(c, n) = ct^n$.

Example 4.8. Given a smooth point p on a curve C, there is a log structure on C that is trivial away from p and near p is induced by $\mathbb{N} \to \mathcal{O}_C$ taking $n \mapsto u^n$ where u is a uniformizer for p. This is called the *model structure for marked points.*

Example 4.9. If Ξ is a rational polyhedral complex in $\mathbb{R}^n$ then $X(\widetilde{\Xi})$ has a log structure. Moreover, the natural map $X(\widetilde{\Xi}) \to \operatorname{Spec} \mathcal{O}$ is a morphism of log schemes.

Example 4.10. Pick $l \in \mathbb{N}$ and let $P = (0, l) \subset \mathbb{R}$. Set $\sigma = \tilde{P} \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$. Let $\pi : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be projection on the second factor. The cone $\sigma^\vee$ is spanned by the vectors $(-1, l), (1, 0)$. The monoid $\sigma^\vee \cap M$ is generated by the elements $f_1 = (-1, l), f_2 = (1, 0), e = (0, 1)$ under the relation $f_1 + f_2 = l \cdot e$. If we write these generators x_1, x_2, t then we have an explicit description for the morphism $\pi^\vee : \mathbf{k}[\mathbb{R}_{\geq 0}^\vee] \to \mathbf{k}[\sigma^\vee \cap M]$ as the inclusion $\mathbf{k}[t] \hookrightarrow \mathbf{k}[x_1, x_2, t]/(x_1 x_2 - t^l)$. The log morphism is induced by

$$\begin{array}{ccc} \mathbb{N}e & \longrightarrow & \mathbb{N}f_1 \oplus \mathbb{N}f_2 \oplus \mathbb{N}e/(f_1 + f_2 = l \cdot e) \\ \downarrow & & \downarrow \\ \mathbf{k}[t] & \longrightarrow & \mathbf{k}[x_1, x_2, t]/(x_1 x_2 - t^l). \end{array}$$

By base-changing $\mathbf{k}[t]$ to $\mathcal{O}$, we get a formal local model for nodes over $\mathcal{O}$. This log structure is the *model structure for nodes.*

A marked semistable family of curves has a canonical log structure that is trivial away from nodes and marked points, has the model structure near nodes and the model structure for marked points.

Definition 4.11. *Let $f : X^\dagger = (X, \mathcal{M}) \to Y^\dagger = (Y, \mathcal{N})$ be a morphism of log schemes. The sheaf of log differentials of $X^\dagger$ over $Y^\dagger$ is*

$$\Omega^1_{X^\dagger/Y^\dagger} = [\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})]/\mathcal{K}$$

where $\mathcal{K}$ is the $\mathcal{O}_X$-submodule generated by

$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a) \text{ and } (0, 1 \otimes \phi(b))$$

for all $a \in \mathcal{M}$ and $b \in f^{-1}\mathcal{N}$.

One should view log differentials as adjoining to the ordinary differentials elements of the form $\mathrm{d}\log(a) = \frac{\mathrm{d}(\alpha(a))}{\alpha(a)}$ for $a \in \mathcal{M}$. Both $\mathcal{P}^\dagger \to \mathcal{O}^\dagger$ and $\mathcal{C}^\dagger \to \mathcal{O}^\dagger$ are log smooth morphisms [6, Ex 4.6,4.7]. Consequently, the log differentials $\Omega^1_{\mathcal{P}^\dagger/\mathcal{O}^\dagger}$ and $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ are locally free sheaves.

Example 4.12. Let $\mathcal{C}$ be a regular semistable family over $\mathcal{O}$. Give $\mathcal{C}$ the model log structure near nodes and marked points. We consider the log differentials in $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$. If u is a uniformizer of a marked point, the log differential include the 1-form $\frac{\mathrm{d}u}{u}$. If $\mathcal{O}[x_1, x_2]/(x_1 x_2 - t)$ is a local model for a node in $\mathcal{C}$, we have the log differentials in $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ given by $\frac{\mathrm{d}x_1}{x_1}, \frac{\mathrm{d}x_2}{x_2}$ subject to $\frac{\mathrm{d}x_1}{x_1} + \frac{\mathrm{d}x_2}{x_2} = 0$. In fact, $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ is an invertible sheaf. Near marked points and nodes, it is generated by $\frac{\mathrm{d}u}{u}$ and $\frac{\mathrm{d}x_1}{x_1}$, respectively. One can view log differentials on $\mathcal{C}_0$ as 1-forms that are allowed simple poles at marked points and simple poles at nodes such that the residues on the branches sum to 0. In fact, if $\mathcal{C}$ is a marked semistable curve, $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ is the relative dualizing sheaf twisted by the divisor of marked points.

Example 4.13. Let $\mathcal{P} = X(\Xi)$ be a toric scheme over $\mathcal{O}$. For m a character of $(\mathbb{K}^*)^n$, $\omega_m = \mathrm{d}log(z^m)$ is a regular log differential. Moreover, if P is a cell of Ξ lying in the affine hyperplane $H = \{x | \langle m, x \rangle = c\}$, then $t^{-c} z^m$ is a unit on $\mathcal{U}_P$. Consequently on $\mathcal{U}_P$, $\omega_m = \frac{\mathrm{d}(t^{-c} z^m)}{t^{-c} z^m}$.

The most important example of specialization of sections of line bundles is that of the logarithmic canonical bundle. Let C be a smooth marked curve over $\mathbb{K}$. After possible base-change, we may suppose that C has a regular semistable model $\mathcal{C}$ over $\mathcal{O}$. Put the model log structure on $\mathcal{C}$. Let Σ be the dual graph. Now the invertible sheaf, $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ restricts to a component C_v of the central fiber as an invertible sheaf of degree $\deg(v) - 2 + 2g(v)$. Consequently, $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ specializes to

$$K_\Sigma = \sum_v (\deg(v) - 2 + 2g(C_v))(v).$$

In the case where C is maximally degenerate, this restricts to Baker's canonical divisor.

Let ω be a global section of $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$ with vanishing function ϖ on Σ. Let $e = v_1v_2$ be an edge in Σ. Near p_e, $\mathcal{C}$ has a formal chart $\mathcal{O}[[x_1, x_2]]/(x_1x_2 - t)$ where x_i is a uniformizer for C_{v_i} near p_e. Then $\frac{dx_1}{x_1} = -\frac{dx_2}{x_2}$ is a regular non-vanishing local section for $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}}$. The order of a zero of a log differential on C_v at p_e is such that $\mathrm{ord}_{p_e}(\frac{dx_1}{x_1}) = 0$. Consequently if f is a rational function on C_v and if $\mathrm{ord}_{p_e}(f) < 0$ then $\mathrm{ord}_{p_e}(\mathrm{d}f) = \mathrm{ord}_{p_e}(f)$ where $\mathrm{d}f$ is considered as a log differential.

We need to make use of the following theorem due to Nishinou and Siebert [16] about completing families of maps of curves.

Definition 4.14. *A stable map $f : C \to X(\Xi)$ over a scheme S is* torically transverse *if for every closed point $s \in S$, $f_s : C_s \to X(\Xi)$ satisfies*

1. *$f_s^{-1}((\mathbb{G}_m)^n_s) \subset C$ is dense, and*
2. *$f_s(C_s) \subset X(\Xi)$ is disjoint from strata of codimension greater than 1.*

Theorem 4.15. [16] *Let $f : C^* \to (\mathbb{K}^*)^n$ be a map of a smooth curve to an algebraic torus. Then after a possible base-change $\mathcal{O}[t^{\frac{1}{N}}] \to \mathcal{O}$, there is a completion of $(\mathbb{K}^*)^n$ to a toric scheme $\mathcal{P} = X(\Xi)$, a completion of C^* to a proper stable family $\mathcal{C}$ over $\mathcal{O}$, and an extension $f : \mathcal{C} \to \mathcal{P}$ such that*

1. *$f_0 : \mathcal{C}_0 \to \mathcal{P}_0$ has the property that for every irreducible component $\mathcal{P}'_0 \subset \mathcal{P}_0$, $f_0 : f^{-1}(\mathcal{P}'_0) \to \mathcal{P}_0$ is a torically transverse stable map.*
2. *There exists disjoint sections $\sigma_1, \ldots, \sigma_k : \operatorname{Spec}\mathcal{O} \to \mathcal{C}^{sm}$ such that $f^{-1}(\partial\mathcal{P}_\mathbb{K}) = \coprod \sigma_i(\operatorname{Spec}\mathbb{K})$.*
3. *Near $\sigma_{e'}(\operatorname{Spec}\mathbf{k})$, $\mathcal{P}$ is formal locally modeled on $(\mathbb{G}_m)^{n-1} \times \mathbb{A}^1_\mathcal{O}$ and f is modeled on $z \mapsto cu^{\mu(e')}$ where z is a uniformizer for $\mathbb{A}^1$ at 0, u is a uniformizer for $\sigma_{e'}(\operatorname{Spec}\mathcal{O})$ in $\mathcal{C}$, $\mu(e') \in \mathbb{N}$ and c is a unit.*
4. *Near the intersection of two irreducible components of $\mathcal{P}_0$ that is equal to $f(p_e)$ for a node $p_e \in \mathcal{C}_0$, $\mathcal{P}$ is formal locally modeled on $(\mathbb{G}_m)^{n-1} \times \big(\operatorname{Spec}\mathcal{O}[w_1, w_2]/(w_1w_2 - t^{s(e)})\big)$ for $s(e) \in \mathbb{N}$, $\mathcal{C}$ is modeled near p_e on $\operatorname{Spec}\mathcal{O}[x_1, x_2]/(x_1x_2 - ct^{s(e)/\mu(e)})$, $\mu(e) \in \mathbb{N}$ and f is modeled on $f^*w_i = c_i x_i^{\mu(e)}$ where c_i's are units.*

Moreover, if $\mathcal{C}$ is given the model log structures near $p_e \in \mathcal{C}_0$ and near $\sigma_i(\operatorname{Spec}\mathcal{O})$ and the trivial log structure elsewhere, $f : \mathcal{C} \to \mathcal{P}$ is a log morphism where $\mathcal{P}$ is given the log structure of a toric scheme.

Proof. One uses Prop 6.3 of [16] to extend the map C^* to $\mathcal{C}$. The local models are produced in the proof of Thm 8.3 of [16]. $\square$

The base-change corresponds to rescaling $\mathbb{R}^n$ such that the vertices of $\mathrm{Trop}(C)$ have integral coordinates. Here, Ξ will be chosen to be a complete polyhedral subdivision of $\mathbb{R}^n$ such that $\mathrm{Trop}(C)$ is a union of polyhedra in Ξ. Such a Ξ exists by Thm 2.2.1 of [17]. The intersection of two irreducible components of $\mathcal{P}_0$ in (4) above corresponds to an edge of lattice length $s(e)$ in Ξ.

This allows us to construct a map of the dual graph $\mathrm{Trop}(f) : \Sigma \to N_{\mathbb{R}}$ that we call the parameterized tropicalization. This is essentially a rephrasing of Construction 4.4 of Nishinou-Siebert [16] and is also developed by Tyomkin [19]. Pick a vertex $v_0 \in V(\Sigma)$. Let $x \in \mathcal{C}$ be a $\mathbb{K}$-point specializing to a smooth point on the component C_v in the central fiber. Set $\mathrm{Trop}(f)(v_0) = v(x) \in N_{\mathbb{R}}$ where $v : (\mathbb{K}^*)^n \to N_{\mathbb{R}}$ is the valuation. For $e \in E(\Gamma)^\bullet$ such that $f(p_e)$ is mapped to a smooth point of $\mathcal{P}_0$, set $\mathrm{Trop}(f)$ to be constant on e. For a bounded edge $e \cong [0,1]$ from v_1 to v_2 that is mapped to a singular point of $\mathcal{P}_0$, we have a map $M \to \mathbb{Z}$ given by $m \mapsto \mathrm{res}_{p_e}(f^*\frac{dz^m}{z^m}\big|_{C_v})$. This gives an element $n(e) \in N_{\mathbb{R}}$. Define $\mathrm{Trop}(f)$ on e by

$$\mathrm{Trop}(f)(t) = \mathrm{Trop}(f)(v_1) + \left(\frac{s(e)}{\mu(e)} n(e)\right) t.$$

In the case where $\mathrm{Trop}(f)$ is constant on an edge e, we say the edge is *contracted*. Let σ_i be a marked point corresponding to $e \cong [0,\infty)$ such that $\sigma_i(\mathbf{k}) \in C_v$. We have a similar map $M \to \mathbb{R}$ taking the residue of $f^*\frac{dz^m}{z^m}$ at σ_i. This gives $n(i) \in N_{\mathbb{R}}$. Define $\mathrm{Trop}(f)$ on e by

$$\mathrm{Trop}(f)(t) = \mathrm{Trop}(f)(v) + n(i)t.$$

This map is well-defined by construction as it constructs $\mathrm{Trop}(f)(\Sigma)$ as supported on Ξ. In fact, the image of the parameterized tropicalization is the tropicalization of the curve $f(C^*)$, $\mathrm{Trop}(f)(\Sigma) = \mathrm{Trop}(f(C^*))$.

Lemma 4.16. $\mathrm{Trop}(f)$ *satisfies the following balancing condition: if v is a vertex of Σ with bounded edges $e_1, \dots, e_k$ and unbounded edges $e'_1, \dots, e'_l$ then*

$$\sum_{j=1}^{k} \frac{\mu(e)}{s(e)} (\mathrm{Trop}(f)|_{e_j}(1) - \mathrm{Trop}(f)(v)) + \sum_{j=1}^{l} (\mathrm{Trop}(f)|_{e'_j}(1) - \mathrm{Trop}(f)(v)) = 0.$$

Proof. The quantity on the left is an element of $N_{\mathbb{R}}$. Its evaluation on $m \in M$, is the sum of residues of $f^*\frac{dz^m}{z^m}$ on C_v. This vanishes by the residue theorem. □

In the case where $f : C^* \to (\mathbb{K}^*)^n$ is an inclusion and all the initial degenerations are reduced, we reproduce Speyer's notion of parameterized tropical curves. [18]

We perform $\left(\frac{s(e)}{\mu(e)} - 1\right)$ blow-ups at p_e to ensure that $\mathcal{C}$ is a regular semistable model. This has the effect of subdividing e into $s(e)/\mu(e)$ edges. Likewise, we may also have to blow up nodes that are mapped to smooth points of $\mathcal{P}$ and then subdivide. We call the map $f : \mathcal{C} \to \mathcal{P}$ the regular semi-stable map. We will work with it below.

We give Σ the structure of an abstract tropical curve. Let a bounded edge e in Σ be given multiplicity $\mu(e)$. Then the edge $\mathrm{Trop}(f)(e)$ points in the primitive integer direction $n/\mu(e)$. Therefore, if each edge of Σ is given the primitive integer direction along its image and multiplicity $\mu(e)$, by the above lemma, Σ is balanced. Now, we relate the multiplicities on Σ to those of $\mathrm{Trop}(f(C^*))$.

Definition 4.17. *A map from a weighted graph to a balanced weighted integral graph, $p : \Sigma \to \Sigma'$ is said to be a tropical parameterization if*

(1) *For each edge $e \in E(\Sigma)$ is assigned the primitive integer direction of $p(e)$, then Σ is a balanced graph,*
(2) *For any edge $e' \in E(\Sigma')$, we have*

$$\sum_{e \in p^{-1}(e')} m(e) = m(e'),$$

(3) *If $v \in V(\Sigma)$ is a vertex all of whose edges are contracted by p then the degree of v is at least 3.*

Lemma 4.18. *The map $\mathrm{Trop}(f) : \Sigma \to \mathrm{Trop}(f(C^*))$ is a parameterized tropicalization.*

Proof. Condition (1) is Lemma 4.16. Condition (2) follows from the definition of the multiplicity of an edge of a tropical curve as the length of the associated initial degeneration [17]. To prove Condition (3), note that any vertex $v \in V(\Sigma)$ with only contracted edges corresponds to a component C_v of $\mathcal{C}_0$ on which f is constant. For $\mathcal{C}$ to be stable, C_v must contain three nodes or marked points. □

Note that if $\mathrm{Trop}(f(C^*))$ is trivalent with all multiplicities 1, the only possible tropical parameterization with all components of genus 0 is the identity map. More generally, we say that a vertex $\mathrm{Trop}(f(C^*))$ is *indecomposable* if its star cannot be written as the union (with multiplicities) of two proper balanced subgraphs. If all multiplicities of $\mathrm{Trop}(f(C^*))$ are 1 and all vertices are indecomposable, then the only tropical parameterization with all components of genus 0 is the identity map since the pre-image under p of any edge is a single edge, and it is impossible to insert any contracted edges at an indecomposable vertex. In the general, case, a given balanced weighted integral graph has only finitely many parameterizations of a given genus.

5. Lifting Condition

In this section, we define all of the terms in the statement of Theorem 1.1. Let $f : C^* \to (\mathbb{K}^*)^n$ be a morphism of a smooth curve with regular semi-stable completion $f : \mathcal{C} \to \mathcal{P}$ and parameterized tropicalization $\mathrm{Trop}(f) : \Sigma \to N_{\mathbb{R}}$. We state a combinatorial condition that $\mathrm{Trop}(f)$ must satisfy. For Γ a subgraph of Σ, let $(\mathcal{C}_\Gamma)_0$ be the subcurve of $\mathcal{C}_0$ that is the union of components corresponding to vertices of Γ, that is,

$$(\mathcal{C}_\Gamma)_0 = \bigcup_{v \in \Gamma} C_v.$$

For $e \in E(\Sigma)$ and $m \in M$, let $m \cdot e$ denote the inner product $\langle m, \mathrm{Trop}(f)|_e(1) - \mathrm{Trop}(f)|_e(0) \rangle$.

Definition 5.1. *Let L be a set of $(-\infty, \infty]$-valued functions on a graph under closed $\oplus$. For G, an abelian group, a* tropical homomorphism *of G to L is a function $a : G \to L$ such that*

(1) *$a(e) = \infty$,*
(2) *For $g_1, g_2 \in G$, $a(g_1) \oplus a(g_2) \oplus a(g_1 + g_2) = 0$.*

Let φ be a piecewise linear function on Σ. For $e \in E(\Sigma)$ and $v \in e$, let $s_\varphi(v, e)$ be the slope of φ at v along e. We have

$$\Delta(\varphi)(v) = -\sum_{e} s_\varphi(v, e).$$

For a subgraph $\Gamma \subset \Sigma$, let $\partial\Gamma$ denote the boundary of Γ considered as a closed subset of Σ.

Definition 5.2. *Let $\Gamma \subseteq \Gamma'$ be subgraphs of Σ such that Γ is bounded, connected, and contained in the interior of Γ'. We say φ* is $\mathcal{C}_0$-ample on Γ in Γ' *if for*

$$\begin{aligned} h &= \min_{v \in \Gamma} \varphi(v) \\ E_v^\partial &= \{e \in E(\Gamma') \setminus E(\Gamma), e \ni v\}, \end{aligned}$$

the invertible sheaf $\mathcal{O}_{(C_\Gamma)_0}(D_\varphi)$ has a section that is non-constant exactly on components C_v for vertices v with $\varphi(v) = h$ where

$$D_\varphi = \sum_{v \in \partial\Gamma, \varphi(v)=h} \left(\sum_{e \in E_v^\partial, s_\varphi(v,e)<0} -s_\varphi(v,e)(p_e). \right)$$

In other words, there is a meromorphic function f on $(C_\Gamma)_0$ with $(f) + D_\varphi \geq 0$ that is constant exactly on the components corresponding to $v \in \Gamma$ with $\varphi(v) > h$.

If Σ is trivalent with all components of $\mathcal{C}_0$ rational, then $\mathcal{C}_0$ is determined by Σ and $\mathcal{C}_0$-ampleness is a combinatorial condition. Otherwise, the condition could depend on the curve $\mathcal{C}_0$. If $(\mathcal{C}_\Gamma)_0$ is a smooth rational curve, then the ampleness condition becomes $\deg(D_\varphi) \geq 1$.

Lemma 5.3. *Let $\Gamma \subset \Sigma$ be a 2-vertex connected graph with no 1-valent vertices. If φ is $\mathcal{C}_0$-ample on $\Gamma \subset \Gamma'$ then $\deg(D_\varphi) \geq 2$.*

Proof. If $\deg(D_\varphi) \leq 0$ then there is no global non-constant section of $\mathcal{O}(D_\varphi)$ so we may suppose $\deg(D_\varphi) = 1$. Then D_φ is supported on some component C_v. This implies that f can be interpreted as a degree 1 rational function on C_v and is constant on $C' = (\mathcal{C}_\Gamma)_0 \setminus C_v$ which is connected. Since C' meets C_v in at least two points, f must attain the same value several times on C_v which is impossible. □

The above lemma includes the case where Γ is a cycle. It can be thought of as the statement that a non-constant rational function on a curve with $g(C) \geq 1$ has at least two poles (counted with multiplicity).

6. Proof of Lifting Condition

We first construct $\tilde{\varphi}_m$ which we will modify to φ_m. For $m \in M$, define a log 1-form $\omega_m = f^*(\frac{dz^m}{z^m})$ on $\mathcal{C}$. By employing Lemma 3.6, we may replace $\mathbb{K}$ by a finite extension to ensure that each ω_m has $\mathbb{K}$-rational zeroes. This has the effect of rescaling $\mathrm{Trop}(f)$ by a factor of l for some $l \in \mathbb{N}$. We pick a model $\mathcal{C}$ for C according to Theorem 4.15.

Let $\tilde{\varphi}_m$ be the order of vanishing of ω_m on components of the central fiber of $\mathcal{C}$, that is, $\varphi_m = \varpi_{\omega_m}$ where ω_m is considered as a section of $\Omega^1_{\mathcal{C}^\dagger/\mathcal{O}^\dagger}$. Since ω_m is regular on $\mathcal{C}$, $\tilde{\varphi}_m \geq 0$. $\Delta(\tilde{\varphi}_m) + K_\Sigma \geq 0$.

Lemma 6.1. *For a bounded edge $e \in E(\Sigma)$ with $m \cdot e \neq 0$, $\tilde{\varphi}_m = 0$ on e.*

Proof. Let $e = v_1 v_2$. Let x_i be uniformizers of C_{v_i} near p_e. Then, by the definition of parameterized tropicalization, $f^* \frac{\mathrm{d}z^m}{z^m} = \langle m, n \rangle \frac{\mathrm{d}x_1}{x_1}$ at p_e where $n = \frac{\mu(e)}{s(e)}(\mathrm{Trop}(f)(v_2) - \mathrm{Trop}(f)(v_1))$. Therefore, on C_{v_1}, at p_e, $\omega_m = f^* \frac{\mathrm{d}z^m}{z^m} = \frac{\mu(e)}{s(e)}(m \cdot e)\frac{\mathrm{d}x_1}{x_1}$. Similarly $\omega_m = -\frac{\mu(e)}{s(e)}(m \cdot e)\frac{\mathrm{d}x_2}{x_2}$ on C_{v_2} at p_e. Consequently, ω_m does not vanish on C_{v_1} or C_{v_2}. A similar argument holds for unbounded edges. □

Lemma 6.2. *Let H be the hyperplane cut out by $\langle m,x\rangle = c$. Let $\Gamma' = \mathrm{Trop}(f)^{-1}(H)$ and Γ be a bounded, connected subgraph contained in the interior of Γ' (considered as a subspace of Σ). Then $\tilde{\varphi}_m$ is $\mathcal{C}_0$-ample on Γ in Γ'.*

Proof. The morphism $\mathcal{C}\to\mathcal{P}$ induces $\mathcal{C}_l\to\mathcal{P}_l$. Let $(\mathring{\mathcal{C}}_{\Gamma'})_0 = \mathcal{C}_0\setminus\cup_{v\notin\Gamma'}C_v$, the points that are contained only in curves C_v for $v\in\Gamma'$. Let $(\mathring{\mathcal{C}}_{\Gamma'})_l$ be the lth order thickening of $(\mathring{\mathcal{C}}_{\Gamma'})_0$ in $\mathcal{C}_l$ given by pulling back the structure sheaf of $\mathcal{C}_l$. The morphism $f:(\mathring{\mathcal{C}}_{\Gamma'})_l\to\mathcal{P}_l$ maps $(\mathring{\mathcal{C}}_{\Gamma'})_l$ into the union of $(\mathcal{U}_P)_l$ for $P\in\Xi$ with $P\subset H$. On these schemes $t^{-c}z^m$ is a unit.

Since $f^*(t^{-c}z^m)$ is a unit on $(\mathring{\mathcal{C}}_{\Gamma'})_0$, it is constant on complete components of $\overline{(\mathring{\mathcal{C}}_{\Gamma'})_0}$. These are of the form C_v for $v\in\Gamma'\setminus\partial\Gamma'$ where $\partial\Gamma'$ be the set of vertices in $\Gamma'\cap\overline{\Sigma\setminus\Gamma'}$. Since $\omega_m = \frac{\mathrm{d}f^*(t^{-c}z^m)}{f^*(t^{-c}z^m)}$ is the log differential of a constant function, it vanishes on C_v for $v\in\Gamma\subset\Gamma'\setminus\partial\Gamma'$. If we let $h=\min_{v\in\Gamma}\tilde{\varphi}_m(v)$, we must have $h>0$.

Now, we will subtract an appropriate constant from $f^*(t^{-c}z^m)$ and divide by t^h to find a rational function on $(\mathcal{C}_\Gamma)_0$ that has poles of the same order as $\frac{\omega_m}{t^h}$. This will give the desired section of $\mathcal{O}_{(\mathcal{C}_\Gamma)_0}(D_{\tilde{\varphi}_m})$. Because $\omega_m=0$ on $(\mathring{\mathcal{C}}_\Gamma)_{h-1}$, that is, ω_m vanishes to the $(h-1)^{\mathrm{st}}$ order on components C_v with $v\in\Gamma$, we may conclude that $f^*(t^{-c}z^m)$ is equal to a constant $\overline{L}\in\mathcal{O}_{h-1}$ on the completion, $(\mathcal{C}_\Gamma)_{h-1}$. Lift $\overline{L}$ to some $L\in\mathcal{O}_h$. By the exact sequence

$$0\longrightarrow t^h\mathcal{O}_{(\mathring{\mathcal{C}}_\Gamma)_h}\longrightarrow\mathcal{O}_{(\mathring{\mathcal{C}}_\Gamma)_h}\longrightarrow\mathcal{O}_{(\mathring{\mathcal{C}}_\Gamma)_{h-1}}\longrightarrow 0\ ,$$

$f^*(t^{-c}z^m)-L$ is a section of $t^h\mathcal{O}_{(\mathring{\mathcal{C}}_\Gamma)_h}$.

Let $s=\frac{f^*(t^{-c}z^m)-L}{t^h}$. Extend s as a rational function to each component of $(\mathcal{C}_\Gamma)_0$. If $\tilde{\varphi}_m(v)>h$, then $\frac{ds}{t^{-c}z^m}=\frac{\omega_m}{t^h}=0$ on C_v so s is constant on C_v. For $v\in\Gamma$ with $\tilde{\varphi}_m(v)=h$ and e, an edge in Γ' containing v, from Lemma 3.3, we have $\mathrm{ord}_{p_e}(\frac{\omega_m}{t^h}|_{C_v})=s_{\tilde{\varphi}_m}(v,e)$. If this quantity is negative, from $\frac{\mathrm{d}s}{f^*(t^{-c}z^m)}=\frac{\omega_m}{t^h}$ and the fact that $t^{-c}z^m$ is a constant on C_v, we have $\mathrm{ord}_{p_e}(\frac{\omega_m}{t^h}|_{C_v})=\mathrm{ord}_{p_e}(s|_{C_v})$. In any case, we have

$$\mathrm{ord}_{p_e}(s|_{C_v})\geq\min(s_{\tilde{\varphi}_m}(v,e),0),$$

and so s is a section of $\mathcal{O}_{(\mathcal{C}_\Gamma)_0}(D_{\tilde{\varphi}_m})$. ☐

Let $Z_{\tilde{\varphi}_m}\in\mathrm{Div}(\Sigma)$ be the *unbounded edge divisor* given by

$$Z_{\tilde{\varphi}_m}=\sum_{v\in\ V(\Sigma)}\left(\#\{e\ni v|m\cdot e=0, e\text{ unbounded}\}\right)(v).$$

Lemma 6.3.

$$\Delta(\tilde{\varphi}_m)+K_\Sigma-Z_{\tilde{\varphi}_m}\geq 0.$$

Proof. By Lemma 3.5, it suffices to show $Z_{\tilde{\varphi}_m}\leq\sum_{v\in V(\Sigma)}(\#Z_v)(v)$. Let e is an unbounded edge at v in Σ. Let $c'=\langle m,\mathrm{Trop}(f)(v)\rangle$. e corresponds to a marked point $\sigma_e(\mathrm{Spec}\,\mathbf{k})$. On C_v, $f^*(t^{-c'}z^m)$ is equal to a constant to a constant $L\in\mathcal{O}_{\tilde{\varphi}_m(v)}$ up to order $\tilde{\varphi}_m(v)-1$. Set $s'_v=\frac{f^*(t^{-c'}z^m)-L}{t^{\tilde{\varphi}_m(v)}}$ and $(\omega_m)_v=\frac{\omega_m}{t^{\tilde{\varphi}_m(v)}}|_{C_v}$. Now s'_v is regular on $(\mathcal{C}_e)_0$ which contains σ_e. Since $(\omega_m)_v=\frac{\mathrm{d}s'_v}{f^*(t^{-c'}z^m)}|_{C_v}$, $(\omega_m)_v$ cannot have a simple pole at σ_e Consequently, $\mathrm{ord}_{\sigma_e}((\omega_m)_v)>0$ and $\sigma_e(\mathrm{Spec}\,\mathbf{k})$ contributes to Z_v ☐

Lemma 6.4. *$\tilde{\varphi}_m$ does not have slope zero on any bounded edge e with $m\cdot e=0$.*

Proof. Let $e = v_1v_2$ be a bounded edge on which $\tilde{\varphi}_m$ is contsant. e corresponds to a node p_e contained in C_{v_1} and C_{v_2}. Set $s = \frac{f^*(t^{-c'}z^m)-L}{t^{\tilde{\varphi}_m(v_1)}}$ and $(\omega_m)_{v_1} = \frac{\omega_m}{t^{\tilde{\varphi}_m(v_1)}}|_{C_{v_1}}$ as in the proof of the lemma above. Now s is regular on $(\mathcal{C}_e)_0$. Since $(\omega_m)_{v_1} = \frac{ds}{f^*(t^{-c'}z^m)}|_{C_{v_1}}$, $(\omega_m)_{v_1}$ does not have a simple pole at p_e. Consequently, $\mathrm{ord}_{\sigma_e}((\omega_m)_{v_1}) > 0$ which contradicts Lemma 3.3 □

We will now perturb $\tilde{\varphi}_m$ by adding a function to ensure that it is non-constant on unbounded edges e with $m \cdot e = 0$ to obtain the desired φ_m.

Let ψ_m be the piecewise linear function such that

(1) On unbounded edges $e \cong [0, \infty)$ with $m \cdot e = 0$ (where $\tilde{\varphi}_m$ is constant), $\psi_m(t) = t$,
(2) $\psi_m = 0$ elsewhere.

Then $\Delta(\psi_m) = -Z_{\varphi_m}$. Set $\varphi_m = \tilde{\varphi}_m + \psi_m$. Then by Lemma 6.3, $\Delta(\varphi_m) + K_\Sigma \geq 0$. Note that since adding ψ_m does not affect edges of negative slope adjacent to vertices of a cycle Γ, it does not change the $\mathcal{C}_0$-ampleness.

Lemma 6.5. *The map $M \to L(K_\Sigma)$ given by $m \mapsto \varphi_m$ is a tropical homomorphism.*

Proof. Since $\omega_0 = 0$, $\tilde{\varphi}_0$ and hence φ_0 is constantly ∞.

Given $m_1, m_2 \in M$, $\omega_{m_1+m_2} = \omega_{m_1} + \omega_{m_2}$ so $\tilde{\varphi}_{m_1} \oplus \tilde{\varphi}_{m_2} \oplus \tilde{\varphi}_{m_1+m_2} = 0$ by Lemma 3.4. We must verify $\varphi_{m_1} \oplus \varphi_{m_2} \oplus \varphi_{m_1+m_2} = 0$. This only needs to be verified on unbounded edges e with $m \cdot e = 0$.

Set $m_3 = m_1 + m_2$. Suppose $\min(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3})$ is achieved by φ_{m_i} on an edge e. Since $\varphi_{m_i} = \varphi_{-m_i}$, we may treat the expression symmetrically in m_i.

If $m_i \cdot e \neq 0$ for some i, then $m_j \cdot e \neq 0$ for some $j \neq i$. Therefore, $\min(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3}) = 0$ on e and this minimum is achieved twice.

Otherwise $m_i \cdot e = 0$ for all i. In that case $\psi_{m_1} = \psi_{m_2} = \psi_{m_3}$ and we are done. □

Note that $\Delta(\varphi_m) + K_\Sigma \geq 0$ together with the vanishing of φ_m on edges not orthogonal to m gives the following property of φ: if $v \in V(\Sigma)$ is a vertex with an adjacent edges e_1 with $m \cdot e_1 \neq 0$, then $\varphi_m(v) = 0$, the slope satisfies $1 \leq s_{\varphi_m}(v, e)$ for all e with $m \cdot e = 0$, and

$$\sum_{e | m \cdot e = 0} s_{\varphi_m}(v, e) \leq \deg(v) - 2.$$

In particular if v is trivalent, then there is at most one adjacent edge e with $m \cdot e = 0$ by balancing, and we have $s_{\varphi_m}(v, e) = 1$.

7. Comparison to Known Obstructions

In this section, we relate $\mathcal{C}_0$-ampleness of a function φ on Γ in Γ' in a graph Σ to the necessity of Speyer's well-spacedness condition for genus 1 curves to lift and its higher genus generalization by Nishnou [15]. We make the following assumptions:

(1) Σ is trivalent,
(2) $\Sigma \setminus \Gamma$ is a forest, and
(3) the genus of every component of Σ is 0.
(4) Γ be a bounded, 2-vertex connected subgraph of $\Gamma' \setminus \partial\Gamma'$ with no 1-valent vertices .

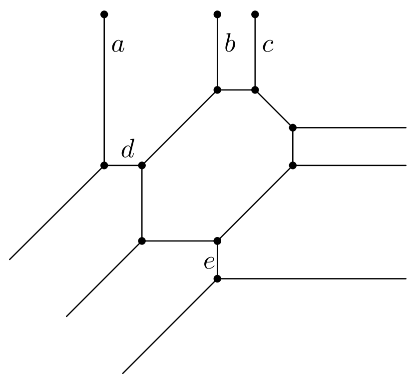
Assumption (2) is related to Nishinou's one-bouquet condition [15] which is required for his necessary and sufficient generalization of Speyer's well-spacedness condition to apply. Assumption (3) ensures that

$$K_\Sigma = \sum_{v \in V(\Sigma)} (\deg(v) - 2)(v).$$

The follow condition is equivalent to Speyer's well-spacedness condition [17] in the genus 1 case and generalizes it in higher genera:

Proposition 7.1. *Let* $\mathrm{Trop}(f) : \Sigma \to \mathbb{R}^n$ *be the parameterized tropicalization of a map of a smooth curve. Let* $m \in M$, $c \in \mathbb{R}$ *and* $H = \{x | x \cdot m = c\}$ *and* $\Gamma' = \mathrm{Trop}(f)^{-1}(H)$. *If* Γ *is as above then the minimum of* $\mathrm{dist}(w, \Gamma)$ *for* $w \in \partial\Gamma'$ *must be achieved for at least two values of* w.

We first go through an example to show how to derive well-spacedness in genus 1 from our condition. Consider a trivalent tropical elliptic curve in $\mathbb{R}^3$ such that Γ' looks like the following:



where the edges not terminating in a vertex are taken to be unbounded. Take Γ to be the cycle and orient the edges not in the cycle so that they point towards Γ. By Lemma 5.3, the divisor D_{φ_m} must have degree at least 2. The degree of D_{φ_m} is the sum of the positive slopes of edges coming into Γ at points where φ_m is minimized. φ_m is decreasing along unbounded edges so they cannot contribute to the divisor D_{φ_m} on the cycle. The slope along edges a, b, c is at most 1 and is non-increasing along them since $\Delta(\varphi_m) \geq 0$ on the interior of those edges. By the condition on $\Delta(\varphi_m)$, the slope on edge e must be non-positive. Similarly, the slope on edge d must be less than or equal to the smallest slope on a. Therefore, the only positive slopes entering Γ must come from edges d,b,c. At the points where those edges intersect Γ, φ_m must be less than or equal to the distance from those points to $\partial\Gamma'$. Equality is achieved if and only if the slope is 1 at those points. It follows that since the value of φ_m on Γ must be minimized at two of those points where the slope is 1, the minimum of $\{|a| + |d|, |b|, |c|\}$ must be achieved at least twice.

Now, we consider the general situation. Let $v \in \partial\Gamma$ and let T_v be the unique tree in $\Sigma \setminus (\Gamma \setminus \partial\Gamma)$ containing v. Direct the edges of T_v so that they point towards v. Let e be the edge of T_v adjacent to v. Write $T_v \cap \partial\Gamma' = \{w_1, \ldots, w_k\}$. Let $\gamma_1, \ldots, \gamma_k$ be the paths from $w_1, \ldots, w_k$ to v. Let l_i be the length of γ_i, $l(v) = \min(l_i)$, $S_v = \{i | l_i = l(v)\}$.

We will need to prune the graph Σ. Let $\Sigma' = \Sigma \setminus (T_v \setminus \{v\})$.

Lemma 7.2. *Suppose*

(1) $\Delta(\varphi_m) + K_\Sigma \geq 0$, *and*
(2) φ_m *is* $\mathcal{C}_0$*-ample with respect to* Γ *in* Γ'.

If $s(v,e) > 0$ *then* $\varphi_m|_{\Sigma'}$ *satisfies:*

(1) $\Delta(\varphi_m|_{\Sigma'}) + K_{\Sigma'} \geq 0$, *and*
(2) $\varphi_m|_{\Sigma'}$ *is* $\mathcal{C}_0$*-ample with respect to* Γ *in* $\Gamma' \setminus (T_v \setminus \{v\})$.

Proof. It is clear that $\varphi_m|_{\Sigma'}$ satisfies the $\mathcal{C}_0$-ampleness condition since the edge e does not contribute to D_{φ_m}.

If $s_{\varphi_m}(v,e) > 0$ then $\Delta(\varphi_m|_{\Sigma'})(v) \geq \Delta(\varphi_m)(v) + 1$ while $K_\Sigma(v) = K_{\Sigma'}(v) - 1$. Therefore,

$$\Delta(\varphi_m|_{\Sigma'}) + K_{\Sigma'} \geq \Delta(\varphi_m) + K_\Sigma \geq 0.$$

□

Observe that the slope of φ_m is non-increasing along smooth segments because if v' is a smooth point of an edge, $K_\Sigma(v') = 0$ and $\Delta(\varphi_m) \geq 0$.

The following lemma constrains the value of φ_m at points of $\partial\Gamma$ that could possibly contribute to D_{φ_m}.

Lemma 7.3. *Suppose* φ_m *is a non-negative piecewise linear function on* Γ' *satisfying* $\Delta(\varphi_m) + K_\Sigma \geq 0$ *with slope* 1 *near* $\partial\Gamma'$ *and* $\varphi_m|_{\partial\Gamma'} = 0$. *Suppose also that* φ_m *never has slope* 0 *on* $\Gamma' \setminus \partial\Gamma'$. *Let* $v \in \partial\Gamma$ *and* e *be the edge of* $\Gamma' \setminus \Gamma$ *adjacent to* v. *If* $s_{\varphi_m}(v,e) < 0$ *then* $\varphi_m(v) \geq l(v)$. *If, in addition,* $\#S_v = 1$, *then* $s_{\varphi_m}(v,e) = -1$ *and* $\varphi_m(v) = l(v)$.

Proof. Let v' be a vertex of $\Gamma' \setminus \partial\Gamma'$. The inequality $\Delta(\varphi_m) + K_\Sigma \geq 0$ implies

$$\sum_{e'} s_{\varphi_m}(v',e') \leq \deg(v') - 2,$$

and so there is an adjacent edge e' with $s_{\varphi_m}(v',e') < 0$. If $e = v_1 v$, then there is an edge $e_1 \neq e$ adjacent to v_1 with $s_{\varphi_m}(v_1,e_1) < 0$. Applying this argument repeatedly, we can find a path in Γ' from v to some w_i with all slopes negative. Since $\varphi_m(w_i) = 0$, $\varphi_m(v) \geq l_i \geq l(v)$.

Now suppose $\#S_v = 1$. Without loss of generality, suppose $S_v = \{1\}$. We claim that φ_m is linear with slope 1 along γ_1. Let v' be the first non-smooth vertex along γ_1. Since slopes are non-increasing along γ_1 from w_1 to v', $\varphi_m(v') \leq \mathrm{dist}(v',w_1)$. Let T' be a component of $T_v \setminus \{v'\}$ not containing w_1 but containing w_i for i in some set S. Let e' be the edge in T' adjacent to v'. If $s(v',e') < 0$ then by the first part of this lemma,

$$\varphi_m(v') \geq \min_{i \in S}(\mathrm{dist}(v',w_i)) > \mathrm{dist}(v',w_1) \geq \varphi_m(v').$$

This contradiction proves that $s(v,e') > 0$, and so by Lemma 7.2, we may remove T' from Γ'. By continuing this argument, we may eliminate all such trees T'. Therefore, we may suppose that v' is a smooth vertex of Γ'. Continuing this argument over γ_1, we may suppose that γ_1 is a smooth path. The slope of φ_m must be non-increasing along γ_1. Since it begins as 1 and ends as a positive slope, φ_m must be linear on γ_1. It follows that $\varphi_m(v) = l(v)$. □

The proof of Proposition 7.1 is completed by the following:

Lemma 7.4. *If* Γ *is a bounded* 2*-vertex connected graph with no* 1*-valent vertices and* φ_m *is* $\mathcal{C}_0$*-ample on* $\Gamma \subset \Gamma' \setminus \partial\Gamma'$ *then the minimum* $\min_{w \in \partial\Gamma'} \mathrm{dist}(w,\Gamma)$ *must be achieved at least twice.*

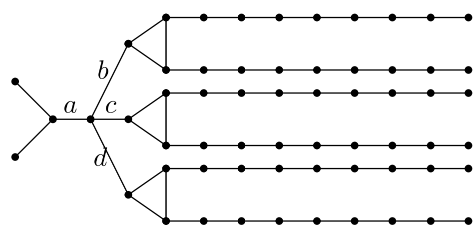
Proof. By Lemma 5.3, we must have $\deg(D_{\varphi_m}) \geq 2$. Let $l = \min_{v\in\partial\Gamma} l(v)$ and $F = \{v \in \partial\Gamma | l(v) = l\}$. If $\#F \geq 2$ then we are done, so we may suppose $F = \{v\}$. If $\#S_v \geq 2$, we are also done, so we may suppose $\#S_v = 1$. By the previous lemma, $s(v,e) = -1$ and $\varphi_m(v) = l(v) = l$. For $w \in \partial\Gamma \setminus \{v\}$ and $f \in E(\Gamma') \setminus E(\Gamma)$ adjacent to w, we have either $s_{\varphi_m}(w,f) > 0$ or

$$\varphi_m(w) \geq l(w) > l = \varphi_m(v).$$

In either case, we can conclude that w does not contribute to D_{φ_m}. Consequently, $\deg(D_{\varphi_m}) = 1$ which is a contradiction. □

8. Example

In this section, we give a tropical curve that does not satisfy our lifting condition but to which Proposition 7.1 does not apply. We claim that there is a balanced parameterized graph $h : \Sigma \to \mathbb{R}^3$ and that there is a rational hyperplane H such that $h^{-1}(H)$ is the following graph where all edges have multiplicity 1:



In fact, it is straightforward to embed this graph in a plane H so that it is balanced and the quadrivalent vertex is indecomposable. One may add pairs of unbounded edges to the boundary of this graph in $\mathbb{R}^3$ to ensure that it is balanced. This example does not satisfy Nishinou's one-bouquet condition which is required for his higher genus necessary and sufficient condition to apply.

We claim that such a graph cannot possibly be the tropicalization of a map of a smooth curve of genus 3. Since every edge is given multiplicity 1 and every vertex is indecomposable, the only parameterization of this graph is the identity. Moreover, any genus 3 lift of his curve must have maximally degenerate reduction. Let us suppose that we have a function φ_m meeting the conditions of Theorem 1.1. We now apply the following inequality to vertices v:

$$\sum_e s_{\varphi_m}(v,e) \leq \deg(v) - 2.$$

We orient the edges not part of any cycle towards the nearest cycle. Because the slope of φ_m on the two edges pointing towards edge a is at most 1, slope of φ_m on a is at most 3. The slopes of φ_m on b,c, and d must sum to at most 5. Consequently, we may suppose that φ_m has slope at most 1 on edge b. The other paths from the cycle Γ containing b to $\partial\Gamma'$ are too long for φ_m to have positive slope on them and for $\varphi_m|_\Gamma$ to be minimized along their intersection with Γ. It follows that they cannot contribute to D_{φ_m}, and D_{φ_m} must have degree at most 1 on Γ. Consequently, D_{φ_m} cannot have a non-constant section on $(C_\Gamma)_0$ for Γ.

This examples satisfies Proposition 7.1 (without the forest assumption) but it fails to meet our lifting condition.

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