DEWITT-SCHWINGER RENORMALIZATION OF $\langle\phi^2\rangle$ IN d DIMENSIONS

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A compact expression for the DeWitt-Schwinger renormalization terms suitable for use in even-dimensional space-times is derived. This formula should be useful for calculations of $\langle\phi^2(x)\rangle$ and $\langle T_{\mu\nu}(x)\rangle$ in even dimensions.

1. Introduction

A major impediment to using semi-classical general relativity is calculating the renormalized expectation value of the stress tensor. Properly renormalized values for $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ provide information on particle production and spontaneous symmetry breaking, and are also required to calculate backreaction. Since in general relativity energy density is itself a source of curvature, great care must be taken in deciding what may be dismissed as 'unphysical'. Fortunately, there are several generally accepted renormalization schemes for curved space-times.[1] Our purpose is to present a compact formula for the renormalization terms that may be applied to $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ calculations in arbitrary black hole space-times of even dimension.

2. Connection to Green's Functions

Calculating $\langle T_{\mu\nu}\rangle$ for a general d-dimensional black hole space-time is difficult. For a scalar field, $T_{\mu\nu} \propto \phi^2$ and its derivatives, so we start with the simpler problem of calculating $\langle\phi^2\rangle = \langle H|\phi^2|H\rangle$, where $|H\rangle$ is the Hartle-Hawking vacuum. Note that $\langle\phi^2\rangle$ is the coincidence limit of the two point function $\langle\phi^2\rangle = \lim_{x\to x'}\langle\phi(x)\phi(x')\rangle$, and so may be expressed in terms of Green's functions. In particular, the Feynman Green's function is related to the time ordered propagator, $iG_{\rm F}(x,x') = \langle T\left(\phi(x)\phi(x')\right)\rangle$. A Wick rotation allows us to work in Euclidean space where $G_{\rm F}(t,x;t',x') = -iG_{\rm E}(i\tau,x;i\tau',x')$. The Euclidean Green's function, $G_{\rm E}$, now obeys

$$\left(\Box_{\rm E} - m^2 - \xi R(x)\right) G_{\rm E}(x,x') = -|g(x)|^{-1/2}\delta^d(x-x'), \tag{1}$$

where $\Box_{\rm E}$ is the Laplace-Beltrami operator in d-dimensional curved Euclidean space. To solve for $G_{\rm E}$, start with the Euclidean metric for a static space-time in d dimensions with line element

$$ds^2 = f(r)d\tau^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2. \tag{2}$$

Here τ is the Euclidean time, $\tau = -it$, r is a radial coordinate, and Ω represents a $(d-2)$-dimensional angular space. The only restriction for this method is that the metric must be diagonal. If the scalar field is at temperature T, then the Green's function is periodic in $\tau - \tau'$ with period T^{-1}. Assuming a separation of variables, standard Green's function techniques lead to the formal solution

$$G_{\rm E}(x,x') = \frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty} e^{i\kappa\varepsilon n}\sum_{\ell}\sum_{\{\mu_j\}} Y_{\ell,\{\mu_j\}}(\Omega)Y^*_{\ell,\{\mu_j\}}(\Omega')\chi_{n\ell}(r,r'), \tag{3}$$

where $\kappa = 2\pi T$, and $Y_{\ell,\{\mu_j\}}(\Omega)$ are eigenfunctions of the Helmholtz equation obtained from the from the angular part of Eq. (1). For black holes with spherical topology these are equivalent to the set of hyperspherical harmonics. The radial function $\chi_{n\ell}(r,r')$ obeys a complicated differential equation obtained by putting the above expression into Eq. (1). This expression is divergent in the sum over n.

3. DeWitt-Schwinger Renormalization

The $\langle\phi^2\rangle$ computation has been reduced to computating the coincidence limit of the Green's function – a divergent quantity. To assign physical meaning to $\langle\phi^2\rangle$ it must be rendered finite via some renormalization process, and the standard approach is to renormalize the expression for $G_{\rm E}(x,x')$ via Christensen's point splitting method applied to the DeWitt-Schwinger expansion of the propagator.[2–4] In d dimensions, the adiabatic DeWitt-Schwinger expansion of the Euclidean propagator is[3]

$$G^{\rm DS}_{\rm E}(x,x') = \frac{\pi\triangle^{1/2}}{(4\pi i)^{d/2}}\sum_{k=0}^{\infty} a_k(x,x')\left(-\frac{\partial}{\partial m^2}\right)^k\left(-\frac{z}{2im^2}\right)^{1-d/2} H^{(2)}_{d/2-1}(z). \tag{4}$$

Equation (4) introduces several new variables. Let $s(x,x')$ be the geodesic distance between x and x', then define $2\sigma(x,x') = s^2(x,x')$ and $z^2 = -2m^2\sigma(x,x')$. The $a_k(x,x')$ are called DeWitt coefficients, and $H^{(2)}_\nu(z)$ is a Hankel function of the second kind. Lastly, $\triangle(x,x') = \sqrt{g(x)}D(x,x')\sqrt{g(x')}$ is the Van Vleck–Morette determinant, where $g(x) = \det(g_{\mu\nu}(x))$ and $D(x,x') = \det(-\sigma_{;\mu\nu'})$. Using the derivative properties of Bessel functions, noting that $z = i|z|$ is purely imaginary in Euclidean space, and defining $\nu = d/2 - 1 - k$, Eq. (4) can be written as

$$G^{\rm DS}_{\rm E}(x,x') = \frac{-2i\triangle^{1/2}}{(4\pi)^{d/2}}\sum_{k=0} a_k(x,x')(2m^2)^\nu|z|^{-\nu}\Big[(-1)^\nu\pi I_\nu(|z|) + iK_\nu(|z|)\Big]. \tag{5}$$

The DeWitt-Schwinger expansion is a WKB expansion of the Euclidean propagator for a generic space-time when the point separation is small. For a particular space-time, this procedure does not give the correct results for the Green's function with finite point separation because it ignores global space-time properties that determine the Green's function – such as the effective potential around a black hole – but it should reproduce the same divergent terms in the coincidence limit. Therefore, if the divergent terms of the DeWitt-Schwinger expansion can be isolated, then subtracting these terms from $G_{\rm E}(x,x')$ will make it finite as $x \to x'$. Since we are working in Euclideanized space the physical renormalization terms come from the real part of Eq. (5). The asymptotic behavior of $K_\nu(|z|)$ as $z \to 0$ implies that only terms with $\nu \geq 0$ contribute divergences in the coincidence limit, so

$$G_{\rm div}(x,x') = \frac{2\triangle^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} a_k(x,x')(2m^2)^\nu |z|^{-\nu} K_\nu(|z|). \tag{6}$$

To renormalize $G_{\rm E}$, Eq. (6) must be made commensurate with Eq. (3). We have shown[5] that an integral representation of $K_\nu(z)$ for small z and integer-valued ν is

$$K_\nu(z) = \frac{(-1)^\nu \sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})} \left(\frac{z}{2}\right)^\nu \int_0^\infty dt \cos(zt)(t^2+1)^{\nu-1/2}. \tag{7}$$

Changing of variables and using the Plana sum formula to convert the integral to a sum, the renormalization terms for the d-dimensional space-time of Eq. (2) are[5]

$$\begin{aligned} G_{\rm div}(x,x') = \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \Bigg\{ \frac{[a_k]\,\kappa\sqrt{\pi}}{(-f)^\nu \Gamma(\nu+\frac{1}{2})} \Bigg[\sum_{n=1}^{\infty} \cos(\kappa\varepsilon n) \left(\kappa^2 n^2 + m^2 f\right)^{\nu-\frac{1}{2}} - \frac{1}{2}(\kappa^2 + m^2 f)^{\nu-\frac{1}{2}} \\ - i\int_0^\infty \frac{dt}{e^{2\pi t}-1} \left\{ \left[(1+it)^2\kappa^2 + m^2 f\right]^{\nu-1/2} - \left[(1-it)^2\kappa^2+m^2 f\right]^{\nu-1/2} \right\} \\ + (m^2 f)^{\nu-\frac{1}{2}} {}_2F_1\left(\frac{1}{2},\frac{1}{2}-\nu,\frac{3}{2},-\frac{\kappa^2}{m^2 f}\right) \Bigg] + [a_k]E_\nu + \sum_{n=1}^{\nu}\sum_{p=1}^{2n}\sum_{j=0}^{p} \frac{2^{2n-1}(-m^2)^{\nu-n}\Gamma(n)}{\Gamma(\nu-n+1)} \frac{a_k^j \Delta_{p-j}^{1/2}}{(\sigma^\rho\sigma_\rho)^n} \Bigg\} \end{aligned} \tag{8}$$

for a scalar field at nonzero temperature $T > 0$. In this expression the E_ν are terms depending on the metric function f and have been tabulated elsewhere,[5] while a_k^m and $\Delta_m^{1/2}$ represent the m^{th} term in an expansion in powers of σ^ρ. This expression generalizes previously known four-dimensional results.[6] The corresponding renormalization terms for a scalar field at zero temperature $T = 0$ are similarly found.[5]

4. Discussion

Semi-classical general relativity requires calculation of $\langle T_{\mu\nu}\rangle_{\rm ren}$ in complicated – possibly higher dimensional – space-times. The first step in calculating $\langle T_{\mu\nu}\rangle_{\rm ren}$ for a scalar field is calculating $\langle\phi^2\rangle_{\rm ren}$. We have presented a compact expression for the renormalization terms for $\langle\phi^2\rangle$ in even dimensional, static, black hole space-times.

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