Post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem

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Abstract

We investigate the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. It is concluded that the equilateral triangular configuration can satisfy the post-Newtonian equation of motion in general relativity, if and only if all three masses are equal.

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I. INTRODUCTION

The three-body problem in the Newton gravity represents classical problems in astronomy and physics (e.g, [1–3]). In 1765, Euler found a collinear solution for the restricted three-body problem that assumes one of three bodies is a test mass. Soon later, his solution was extended for a general three-body problem by Lagrange, who also found an equilateral triangle solution in 1772. Now, the solutions for the restricted three-body problem are called Lagrange points L_1, L_2, L_3, L_4 and L_5, which are well known and described in textbooks of classical mechanics [1].

Lagrange points have recently attracted renewed interests for relativistic astrophysics, where they have discussed the gravitational radiation reaction on L_4 and L_5 analytically [4] and by numerical methods [5, 6].

As a pioneering work, Nordtvedt pointed out that the location of the triangular points is very sensitive to the ratio of the gravitational mass to the inertial one [7]. Along this course, it is interesting as a gravity experiment to discuss the three-body coupling terms at the post-Newtonian order, because some of the terms are proportional to a product of three masses as $M_1 \times M_2 \times M_3$. Such a triple product can appear only for relativistic three (or more) body systems but cannot for a relativistic compact binary nor a Newtonian three-body system.

The relativistic periastron advance of the Mercury is detected only after much larger shifts due to Newtonian perturbations by other planets such as the Venus and Jupiter are taken into account in the astrometric data analysis. In this sense, effects by the three-body coupling are worthy to investigate. Nevertheless, most of post-Newtonian works have focused on either compact binaries because of our interest in gravitational waves astronomy or N-body equation of motion (and coordinate systems) in the weak field such as the solar system (e.g. [8]). Actually, future space astrometric missions such as Gaia [9, 10] require a general relativistic modeling of the solar system within the accuracy of a micro arc-second [11]. Furthermore, a binary plus a third body have been discussed also for perturbations of gravitational waves induced by the third body [12–15].

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time. Hence it is important to take account of general relativistic effects on three-body configurations. The figure-eight configuration that was found decades ago [16, 17] has been numerically studied at the first post-Newtonian [18] and also the second post-Newtonian orders [19]. According to their numerical investigations, the solution remains true with a slight change in the figure-eight shape because of relativistic effects.

On the other hand, the post-Newtonian collinear configuration has been recently obtained as a relativistic extension of *Euler's collinear one*, where three bodies move around the common center of mass with the same orbital period and always line up [20]. It may offer a useful toy model for relativistic three-body interactions, because it is tractable by hand without numerical simulations. The uniqueness of the collinear configuration has been also proven [21].

Lagrange's equilateral triangular solution has also a practical importance, since it is stable for some cases. Lagrange's points L_4 and L_5 for the Sun-Jupiter system are stable and indeed the Trojan asteroids are located there. Clearly it is of greater importance to investigate Lagrange's equilateral triangular solution in the framework of general relativity. Do the post-Newtonian effects admit such a triangular solution? Although no explicit calculations have been done, nobody has doubted whether it were possible so far. We shall study this issue in this paper. The main purpose of this paper is to show that the equilateral triangular configuration can satisfy the post-Newtonian equation of motion, if and only if all three masses are equal. Throughout this paper, we take the units of $G = c = 1$.

II. NEWTONIAN LAGRANGE'S EQUILATERAL TRIANGULAR SOLUTION

First, we consider the Newton gravity among three masses denoted as M_I $(I = 1, 2, 3)$. The location of each mass is written as $\boldsymbol{x}_I$. We choose the origin of the coordinates, so that

$$M_1\boldsymbol{x}_1 + M_2\boldsymbol{x}_2 + M_3\boldsymbol{x}_3 = 0. \tag{1}$$

We start by seeing whether the Newtonian equation of motion for each body can be satisfied if the configuration is an equilateral triangle. Let us put $R_{12} = R_{23} = R_{31} \equiv a$, where we define the relative position between masses as

$$\boldsymbol{R}_{IJ} \equiv \boldsymbol{x}_I - \boldsymbol{x}_J, \tag{2}$$

and $R_{IJ} \equiv |\boldsymbol{R}_{IJ}|$ for $I, J = 1, 2, 3$. Then, the equation of motion for each mass becomes

$$\frac{d^2\boldsymbol{x}_I}{dt^2} = -M\frac{\boldsymbol{x}_I}{a^3}, \tag{3}$$

where M denotes the total mass $\sum_I M_I$. Therefore, it is possible that each body moves around the common center of mass with the same orbital period. Eq. (3) gives

$$\omega_N^2 = \frac{M}{a^3}, \tag{4}$$

where ω_N denotes the Newtonian angular velocity.

Figure 1 shows an equilateral triangular configuration. Let $\boldsymbol{\ell}_I$ denote the relative position vector of each mass with respect to the common center of mass (but not the geometrical center of the triangle) in the corotating frame with the angular velocity ω_N. The angles between $(\boldsymbol{\ell}_2, \boldsymbol{\ell}_1)$, $(\boldsymbol{\ell}_3, \boldsymbol{\ell}_2)$ and $(\boldsymbol{\ell}_1, \boldsymbol{\ell}_3)$ are defined cyclically as θ_1, θ_2 and θ_3 respectively. They are constant with time. There is an identity as $\theta_1 + \theta_2 + \theta_3 = 2\pi$, which can be used to delete one of the angles. The orbital radius $\ell_I \equiv |\boldsymbol{\ell}_I|$ of each body with respect to the common center of mass is obtained as [2]

$$\ell_1 = \frac{a}{M}\sqrt{M_2^2 + M_2 M_3 + M_3^2}, \tag{5}$$

$$\ell_2 = \frac{a}{M}\sqrt{M_1^2 + M_1 M_3 + M_3^2}, \tag{6}$$

$$\ell_3 = \frac{a}{M}\sqrt{M_1^2 + M_1 M_2 + M_2^2}. \tag{7}$$

III. POST-NEWTONIAN EQUILATERAL TRIANGULAR SOLUTION

Next, we consider the post-Newtonian effects on the triangular configuration by employing the Einstein-Infeld-Hoffman (EIH) equation of motion as [22, 23]

$$\begin{aligned}
\frac{d\boldsymbol{v}_K}{dt} = {} & \sum_{A\neq K} \boldsymbol{R}_{AK} \frac{M_A}{R_{AK}^3} \left[1 - 4\sum_{B\neq K} \frac{M_B}{R_{BK}} - \sum_{C\neq A} \frac{M_C}{R_{CA}} \left(1 - \frac{\boldsymbol{R}_{AK}\cdot\boldsymbol{R}_{CA}}{2R_{CA}^2} \right) \right. \\
& \left. + v_K^2 + 2v_A^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_K - \frac{3}{2}\left(\boldsymbol{v}_A\cdot\boldsymbol{n}_{AK}\right)^2 \right] \\
& - \sum_{A\neq K} (\boldsymbol{v}_A - \boldsymbol{v}_K) \frac{M_A \boldsymbol{n}_{AK}\cdot(3\boldsymbol{v}_A - 4\boldsymbol{v}_K)}{R_{AK}^2} \\
& + \frac{7}{2} \sum_{A\neq K} \sum_{C\neq A} \boldsymbol{R}_{CA} \frac{M_A M_C}{R_{AK} R_{CA}^3},
\end{aligned} \tag{8}$$

where $\boldsymbol{v}_I$ denotes the velocity of each mass in an inertial frame and we define

$$\boldsymbol{n}_{IJ} \equiv \frac{\boldsymbol{R}_{IJ}}{R_{IJ}}. \tag{9}$$

Let us see whether the three masses at the apices of an equilateral triangle can satisfy the EIH equation of motion. For such an equilateral triangle case, the second-order-mass terms are easy to handle, because every R_{IJ} is the same as a. What we have to take care of is the velocity-dependent terms.

We consider three masses in circular motion with the angular velocity ω, so that each ℓ_I can be a constant. The position and velocity of each body are expressed as

$$\boldsymbol{x}_1 = \ell_1 \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix}, \tag{10}$$

$$\boldsymbol{v}_1 = \ell_1\omega \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}, \tag{11}$$

$$\boldsymbol{x}_2 = \ell_2 \begin{pmatrix} \cos(\omega t + \theta_1) \\ \sin(\omega t + \theta_1) \end{pmatrix}, \tag{12}$$

$$\boldsymbol{v}_2 = \ell_2\omega \begin{pmatrix} -\sin(\omega t + \theta_1) \\ \cos(\omega t + \theta_1) \end{pmatrix}, \tag{13}$$

$$\boldsymbol{x}_3 = \ell_3 \begin{pmatrix} \cos(\omega t - \theta_3) \\ \sin(\omega t - \theta_3) \end{pmatrix}, \tag{14}$$

$$\boldsymbol{v}_3 = \ell_3\omega \begin{pmatrix} -\sin(\omega t - \theta_3) \\ \cos(\omega t - \theta_3) \end{pmatrix}, \tag{15}$$

where we used $\theta_1 + \theta_2 = 2\pi - \theta_3$, $|\boldsymbol{x}_I| = \ell_I$ and $|\boldsymbol{v}_I| = \ell_I\omega$.

For the later convenience, we compute the inner products between the velocity and relative position vectors as

$$\begin{aligned}
\boldsymbol{R}_{12} \cdot \boldsymbol{v}_1 &= -\ell_1 \ell_2 \omega \sin\theta_1, & (16)\\
\boldsymbol{R}_{12} \cdot \boldsymbol{v}_2 &= -\ell_1 \ell_2 \omega \sin\theta_1, & (17)\\
\boldsymbol{R}_{12} \cdot \boldsymbol{v}_3 &= \ell_3 \ell_1 \omega \sin\theta_3 - \ell_2 \ell_3 \omega \sin(\theta_3+\theta_1), & (18)\\
\boldsymbol{R}_{23} \cdot \boldsymbol{v}_1 &= \ell_1 \ell_2 \omega \sin\theta_1 - \ell_3 \ell_1 \omega \sin(\theta_1+\theta_2), & (19)\\
\boldsymbol{R}_{23} \cdot \boldsymbol{v}_2 &= -\ell_2 \ell_3 \omega \sin\theta_2, & (20)\\
\boldsymbol{R}_{23} \cdot \boldsymbol{v}_3 &= -\ell_2 \ell_3 \omega \sin\theta_2, & (21)\\
\boldsymbol{R}_{31} \cdot \boldsymbol{v}_1 &= -\ell_3 \ell_1 \omega \sin\theta_3, & (22)\\
\boldsymbol{R}_{31} \cdot \boldsymbol{v}_2 &= \ell_2 \ell_3 \omega \sin\theta_2 - \ell_1 \ell_2 \omega \sin(\theta_2+\theta_3), & (23)\\
\boldsymbol{R}_{31} \cdot \boldsymbol{v}_3 &= -\ell_3 \ell_1 \omega \sin\theta_3, & (24)\\
\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 &= \ell_1 \ell_2 \omega^2 \cos\theta_1, & (25)\\
\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 &= \ell_2 \ell_3 \omega^2 \cos\theta_2, & (26)\\
\boldsymbol{v}_3 \cdot \boldsymbol{v}_1 &= \ell_3 \ell_1 \omega^2 \cos\theta_3. & (27)
\end{aligned}$$

In order to compute the orbital radius of each mass, the location of the mass center at the post-Newtonian order must be determined. It is expressed as [22, 23]

$$\boldsymbol{G}_{PN} = E^{-1} \sum_A M_A \boldsymbol{x}_A \left[1 + \frac{1}{2}\left(v_A^2 - \sum_{B \neq A} \frac{M_B}{R_{AB}}\right)\right], \tag{28}$$

where E is defined as

$$E = \sum_A M_A \left[1 + \frac{1}{2}\left(v_A^2 - \sum_{B \neq A} \frac{M_B}{R_{AB}}\right)\right]. \tag{29}$$

By using Eq. (28), the post-Newtonian orbital radius $\ell_{PN1} \equiv |\boldsymbol{x}_1 - \boldsymbol{G}_{PN}|$ is obtained as

$$\begin{aligned}
\ell_{PN1}^2 &= \ell_1^2 \\
&\quad + \frac{a}{2M^3}(-2M_1^2 M_2^2 - 2M_3^2 M_1^2 + 2M_1 M_2^3 + 2M_3^3 M_1 + M_2^3 M_3 - 2M_2^2 M_3^2 \\
&\quad + M_2 M_3^3 - 2M_1^2 M_2 M_3 + M_1 M_2^2 M_3 + M_1 M_2 M_3^2)\left(1 - \frac{a^3 \omega_N^2}{M}\right). \qquad (30)
\end{aligned}$$

By noting Eq. (4), we find that the second term in the R.H.S. of Eq. (30) vanishes and hence $\ell_{PN1} = \ell_1$. By the cyclic permutations, we find also $\ell_{PN2} = \ell_2$ and $\ell_{PN3} = \ell_3$.

As a consequence, the common center of mass for the equilateral solution remains unchanged. Without this unexpected thing, our calculations would become much more lengthy.

The above expressions for the inner products are substituted into the R.H.S. of Eq. (8). After straightforward calculations, the equation of motion for M_1 can be written as

$$\begin{aligned}
-\omega^2 \boldsymbol{x}_1 &= -\frac{M}{a^3}\boldsymbol{x}_1 + g_{PN1}\boldsymbol{x}_1 \\
&\quad + \frac{\sqrt{3}M}{16a^3}\boldsymbol{n}_{\perp 1}\frac{M_2M_3(M_2-M_3)}{M_2^2+M_2M_3+M_3^2} \\
&\qquad \times \left[10 + \frac{a^3}{M^2}\big(-4M_1+5M_2+5M_3\big)\omega^2\right],
\end{aligned} \tag{31}$$

where we used Eq. (4) for velocity-dependent terms, $\boldsymbol{n}_{\perp 1} = \boldsymbol{v}_1/\ell_1\omega$ is defined as the unit normal vector to $\boldsymbol{x}_1$, and g_{PN1} denotes the post-Newtonian terms defined as

$$\begin{aligned}
g_{PN1} &= \frac{1}{16a^4(M_2^2+M_2M_3+M_3^2)} \\
&\quad \times \Big[80(M_2^2+M_2M_3+M_3^2)M^2 - 2(8M_2^3+M_2^2M_3+M_2M_3^2+8M_3^3)M \\
&\qquad -\Big\{32(M_2^2+M_2M_3+M_3^2)M^2 + 12M_2M_3(M_2+M_3)M \\
&\qquad\quad -(16M_2^4+41M_2^3M_3+84M_2^2M_3^2+41M_2M_3^3+16M_3^4)\Big\}\frac{a^3}{M}\omega_N^2\Big].
\end{aligned} \tag{32}$$

Here, terms with ω_N^2 come from the velocity-dependent terms and may be reexpressed by using Eq. (4).

We should note that the third term in the R.H.S. of Eq. (31) is parallel to the velocity of M_1 and thus perpendicular to $\boldsymbol{x}_1$ for a circular motion case. Therefore, the mass M_1 can be in circular motion, if and only if the coefficient of the third term vanishes, that is $M_2 = M_3$. Likewise, the masses M_2 and M_3 can be in circular motion, if and only if $M_3 = M_1$ and $M_1 = M_2$, respectively. Hence, all the three masses can have a circular motion, if and only if $M_1 = M_2 = M_3$.

The remaining thing to do is to see whether orbital periods of the three masses are all the same in order to preserve the triangular shape if $M_1 = M_2 = M_3$. It is easy to see this, because one can obtain the post-Newtonian forces g_{PN2} and g_{PN3} from g_{PN1} by cyclic manipulations as $1 \to 2 \to 3 \to 1$, and finally by taking the equality of $M_1 = M_2 = M_3$, one can find $g_{PN1} = g_{PN2} = g_{PN3}$. Therefore, it is concluded that the equilateral triangular configuration remains true for the post-Newtonian equation of motion in general relativity, if and only if all three masses are equal.

Eq. (31) gives uniquely the post-Newtonian angular velocity as $\omega^2 = Ma^{-3} - g_{PN}$, where

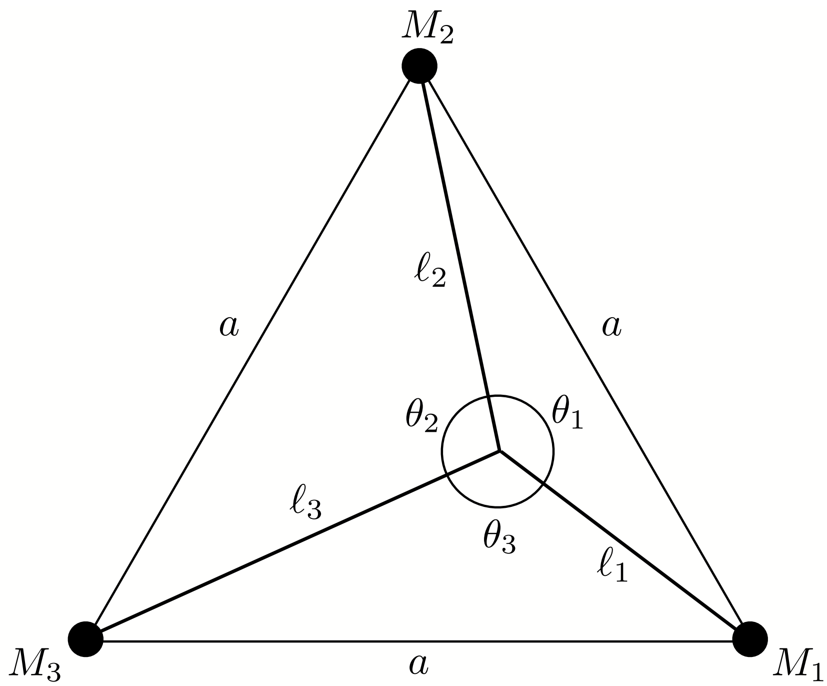

FIG. 1: Equilateral triangular configuration. Each mass is located at one of the apices. We define θ_I $(I = 1, 2, 3)$ with respect to the common center of mass.

$g_{PN} \equiv g_{PN1} = g_{PN2} = g_{PN3}$ for $M_1 = M_2 = M_3 = M/3$. Here, g_{PN} simply becomes

$$g_{PN} = \frac{M}{a^3}\left(\frac{57}{12}\frac{M}{a} - \frac{41}{24}a^2\omega_N^2\right). \tag{33}$$

One can show $g_{PN} > 0$ and hence $\omega < \omega_N$.

IV. SUMMARY

We investigated the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. We found that the equilateral triangular configuration can satisfy the post-Newtonian equation of motion in general relativity, if and only if all three masses are equal.

It is left as a future work to examine post-Newtonian perturbations to triangular configurations for general masses. The configuration may be non-equilateral or non-periodic.

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