Instability of the planetary orbits for space-time dimensions higher than four

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Abstract

It is shown that in a Minkowski space of total space-time dimension $D = d + 1$, the orbits of the planetary motion are stable only if the total dimension of space-time is $D \leq 4$. The proof is performed in a fully didactic way.

1 INTRODUCTION

Bertrand's theorem establishes that the only central potentials with stability are the harmonic oscillator and the Coulomb's one [2]. It has been related the closeness of the classical orbits to their symmetries [3]. The existence of different kinds of factorization operators for the radial Schrödinger equation corresponding to the Kepler and harmonic oscillator problems has been attributed to the classical stability of these systems according to the Bertrand's theorem [4]. Also, a generalization of this result has been developed when the potential has suitable extra angular momentum [5]. More subtle points has been studied like the connection between closed orbits and their stable symmetries for small perturbations [6]. It has been found that when all eight symmetries are stable the orbits are closed, but not necessarily if only three symmetries of them are stable, as it happens in the relativistic case where the perihelion of the orbit advances [6].

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Recently it has been studied the circular orbits stability for the Schwarzschild geometry ($D = 4$) in commutative and noncommutative spaces [7]. Schwarzschild solution in commutative space possesses stable circular orbits for a radial distance $r > 6GM$ and unstable ones for $3GM < r < 6GM$, where M is the mass of the attractor center and G is the gravitational constant. The space non-commutativity increases the radius of stable circular orbits [7]. Gravitational classical instability in higher dimensions ($D > 4$) has been explored for various static and non static spacetimes: black holes, brane worlds, generalized Schwarzschild, AdS black strings, "bubbles of nothing" [8] and hyperbolic spaces [9].

The aim of the present paper is to show that the planetary circular orbits in a Minkowski spacetime of total dimension $D = d + 1$, are stable only when $D \leq 4$. To obtain this result the Bertrand's theorem plays a central role which allow us to give an elegant and didactic proof. In section 2, we obtain the gravitational force in d spacial dimensions due to a particle of mass M. In section 3, we sketch the Bertrand's theorem, and it is applied to constraint the spatial dimensions in order to obtain the stability of circular orbits. In section 4, we give the concluding remarks.

2 GRAVITATIONAL FORCE IN A d-DIMENSIONAL SPACE

The magnitude of the attractive gravitational force between two point particles m and M in the three-dimensional space is given by

$$\vec{F} = -G^{(3)} \frac{mM}{r^2} \hat{r}, \tag{1}$$

which satisfies the Gauss' law (setting $m = 1$) in three dimensions

$$\int_{S^2(r)} \vec{F} \cdot d\vec{a} = -4\pi G^{(3)} M_{enc}, \tag{2}$$

where M_{enc} is the enclosed mass by the Gaussian sphere S^2 and $G^{(3)} \equiv G$.

We are interested in generalizing expression (1) to a d-dimensional space. In order to obtain this, we demand that the gravitational force satisfies the Gauss' law (2) for any d value, remaining fixed the value of the right hand

side. This means that the flux of the gravitational force on any sphere $S^{d-1}(r)$ is always equal to the enclosed mass times $4\pi G^{(3)}$ [1].

Some remarks about $d\vec{a}$ are necessary. If we consider the spheres S^1 and S^2, their areas are equal to $2\pi r$ and $4\pi r^2$, respectively. The area of the sphere S^3 is some alike a three-dimensional volume. To higher-dimensional spaces the area of a sphere S^{d-1} of radius r is known as the volume, $a \equiv (\text{vol}(S^{d-1}(r)))$. In what follows we use this terminology to denote areas for any d value. Hence, the volumes for S^1 and S^2 are

$$\text{vol}(S^1(r)) = 2\pi r, \tag{3}$$
$$\text{vol}(S^2(r)) = 4\pi r^2. \tag{4}$$

Thus, the factor 4π in equation (2) is equal to the volume of the unit sphere S^2. Equations (3) and (4), lead to infer that for a sphere in a d-dimensional space the equality $\text{vol}(S^{(d-1)}(r)) = r^{d-1}\text{vol}(S^{d-1})$ must hold, being $\text{vol}(S^{d-1})$ the volume of the unit sphere. Moreover, the volume differentials (r fixed) are

$$d[\text{vol}(S^1(r))] = rd\theta = rd[\text{vol}(S^1)], \tag{5}$$
$$d[\text{vol}(S^2(r))] = r^2\sin(\theta)d\theta d\phi = r^2 d[\text{vol}(S^2)], \tag{6}$$

from which we conclude that in general, for a d-dimensional sphere, the equality

$$da = d[\text{vol}(S^{d-1}(r))] = r^{d-1}d[\text{vol}(S^{d-1})] \tag{7}$$

holds.

In order to the gravitational force flux on the sphere $S^{d-1}(r)$ be the fixed constant $4\pi G^{(3)}$, according to the result (7), the force between two massive point particles must depend on $1/r^{d-1}$. We propose it has the form

$$\vec{F} = -G^{(d)}\frac{mM}{r^{d-1}}\hat{r}. \tag{8}$$

Now we focus our attention on the right hand side of equation (2). The mass M as an intrinsic matter property must not depend on the space dimension. Hence, the product $4\pi G^{(3)} = \text{vol}(S^2)G^{(3)}$ must have a fixed value for any d. Since in higher dimensions the area (volume) of the unit sphere depends on the space dimension, the gravitational constant must *depend* on the dimension too. Hence, $\text{vol}(S^{d-1})G^{(d)} = 4\pi G^{(3)}$. Above remarks allows to write the

generalized Gauss' law as

$$\int_{S^{d-1}(r)} \vec{F} \cdot d\vec{a} = -\text{vol}(S^{d-1})G^{(d)}M_{enc}. \tag{9}$$

which is satisfied by the gravitational force

$$\vec{F} = -\frac{4\pi G^{(3)}}{\text{vol}(S^{d-1})} \frac{mM}{r^{d-1}}\hat{r}, \tag{10}$$

for $m = 1$.

The next step is to find the volume of a unit sphere of dimension $d - 1$. We proceed as in [10] and calculate the integral

$$I_d = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_d e^{-r^2} = \int_{R^d} e^{-r^2} dv \tag{11}$$

in *all space* in two ways.

First, since $r^2 = x_1^2 + x_2^2 + ... + x_d^2$, the integrand of (11) is separable and it reduces to a product of one-dimensional integrals. Thus,

$$I_d = \prod_{i=1}^{d} \int_{-\infty}^{\infty} dx_i e^{-x_i^2} \tag{12}$$

However, each integral of the product is well known from the probability theory, it is $\int_{-\infty}^{\infty} dx_i e^{-x_i^2} = \pi^{\frac{1}{2}}$. Therefore, $I_d = \pi^{\frac{d}{2}}$.

Second, the value of I_d is obtained by considering the space R^d as divided by thin spherical shells. At fixed r the space is the sphere $S^{d-1}(r)$ and the volume of the space between r and $r + dr$ is $dv = \text{vol}(S^{d-1}(r))dr$. This fact allows to write equation (11) as

$$I_d = \int_0^{\infty} \text{vol}(S^{d-1}(r))e^{-r^2} dr \tag{13}$$

$$= \text{vol}(S^{d-1}) \int_0^{\infty} r^{d-1} e^{-r^2} dr \tag{14}$$

$$= \frac{1}{2}\text{vol}(S^{d-1}) \int_0^{\infty} t^{\frac{d}{2}-1} e^{-t} dt, \tag{15}$$

where we have used the relation $\text{vol}(S^{(d-1)}(r)) = r^{d-1}\text{vol}(S^{d-1})$ and performed the variable change $t = r^2$. The last integral is identified with the integral representation of the gamma function defined by $\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt,$

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with $x > 0$. With this result we obtain that $I_d = \frac{1}{2}\text{vol}(S^{d-1})\Gamma(\frac{d}{2})$. Thus by equating our two results for I_d, we obtain

$$\text{vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{16}$$

Results (10) and (16) complete our purpose to find the gravitational force between two point particles in a d-dimensional space.

Bertrand's theorem ensures that the circular planetary orbits around the sun in our four-dimensional space-time are stable under small perturbations [2], and relates its stability to the values of the power in potentials with the form $V = r^\nu$. We notice that the force (10) due to one particle of mass M on a particle of mass m is proportional to $1/r^{d-1}$. This fact will allow us to apply Bertrand's theorem to find the values of the spatial dimension d in order to have stable planetary orbits in higher dimensions.

3 THE BERTRAND'S THEOREM AND STABILITY OF THE CIRCULAR ORBITS IN A d-DIMENSIONAL SPACE

We begin by reviewing the Bertrand's theorem proof. Since the total angular momentum is conserved, this allows us to set an inertial frame in the mass center of the system. Also, since the force is central, the angular momentum is conserved and restrain the orbit to be plane. Hence, the motion of the two particles is reduced to that of a single one with reduced mass $\mu = \frac{mM}{m+M}$ under the potential $V(r)$, being r the distance between particles. It is usual to set the z-axis in the angular momentum direction, orthogonal to the orbit plane. Also, since $V(r)$ does not depend on time t, the total energy E is conserved.

Since the lagrangian of the system is

$$L = T - V = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{17}$$

and does not depend on the variable θ, the angular momentum $\ell = \mu r^2\dot{\theta}$ is conserved. The total energy of the system is

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{18}$$

$$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + V(r) \tag{19}$$

$$\equiv \frac{1}{2}\mu\dot{r}^2 + V_{ef}, \tag{20}$$

from which is obvious the definition of the effective potential $V_{ef}(r)$. For the case in our study, the reduced mass is $\mu = \frac{mM}{m+M} \approx m$. Thus, it follows that

$$\frac{\partial V_{ef}}{\partial r} = \frac{\partial V(r)}{\partial r} - \frac{\ell^2}{mr^3}, \tag{21}$$

or

$$F_{ef} = F + \frac{\ell^2}{mr^3}, \tag{22}$$

where $F \equiv -\frac{\partial V}{\partial r}$.

The condition to have circular orbits is imposed by $\dot{r} = 0$, and they are stable only when the potential has an effective minimum at a distance $r = r_0$. This fact implies that for circular orbits, $F_{ef} = -\left.\frac{\partial V_{ef}}{\partial r}\right|_{r=r_0}$ must be equal to zero (whenever the case, a maximum or a minimum). Thus, equation (22) reduces to

$$F = -\frac{\ell^2}{mr_0^3}. \tag{23}$$

By demanding that the effective potential $V_{ef}(r)$ get a minimum at $r = r_0$, from equation (21) it must satisfy

$$\left.\frac{\partial^2 V_{ef}}{\partial r^2}\right|_{r=r_0} = \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} + 3\frac{\ell^2}{mr_0^4} > 0 \tag{24}$$

this implies

$$-\left.\frac{\partial F}{\partial r}\right|_{r=r_0} > -3\frac{\ell^2}{mr_0^4} = \frac{3}{r_0}F(r)|_{r=r_0}, \tag{25}$$

or

$$\left.\frac{\partial F}{\partial r}\right|_{r=r_0} < -\frac{3}{r_0}F(r_0). \tag{26}$$

This inequality must be satisfied in order to have stable circular orbits at $r = r_0$.

From equation (10) and (16), the force between two particles in a d-dimensional space is given by

$$F = -\frac{4\pi G^{(3)}}{\text{vol}(S^{d-1})}\frac{mM}{r^{d-1}} = -\frac{2\Gamma(\frac{d}{2})G^{(3)}}{\pi^{\frac{d}{2}-1}}\frac{mM}{r^{d-1}}. \tag{27}$$

Hence,

$$\frac{dF}{dr} = \frac{2\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}-1}}G^{(3)}Mm(d-1)r^{-d}. \tag{28}$$

By substituting equations (27) and (28) evaluated at $r = r_0$ into equation (26), we obtain

$$\frac{d-1}{r_0^d} < \frac{3}{r_0^d}, \tag{29}$$

or $d < 4$. This means that the maximum value of the spatial dimension d is 3, and it is the upper bound we were looking for.

4 CONCLUDING REMARKS

We have shown that the spatial dimension must satisfy $d < 4$ to have stable circular orbits. This means that for the planetary motion in a space-time of five dimensions or higher, the circular orbits become instable. This result could be derived because the gravitational force in d dimensions possesses the same mathematical form as those for the central potentials studied by Bertrand. Although the problem we have treated in this paper is a texbook one [10], as we have shown above, its solution is non-trivial. Moreover, as was emphasized in the introduction, the concept of stability is studied in many current research areas of physics. Being this paper a comprehensive and fully detailed work, it can be a helpful pledge for the students to make a glance for advanced topics.

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