Thermodynamics of Slowly Rotating Charged Black Holes in anti-de Sitter Einstein-Gauss-Bonnet Gravity

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By using a new approach, we demonstrate analytic expressions for slowly rotating Gauss-Bonnet charged black holes with negative cosmological constant. Up to the linear order of the rotating parameter a, the mass, Hawking temperature and entropy of the charged black holes get no corrections from rotation.

I. INTRODUCTION

The thermodynamics of black holes in anti-de Sitter spaces have attracted a great deal of attention. one reason for this is the role of AdS/CFT. It is well-known that the AdS Schwarzschild black hole is thermodynamically unstable when the horizon radius is small, while it is stable for large radius; there is a phase transition, named Hawking-Page transition [1], between the large stable black hole and a thermal AdS space. This phase transition is explained by Witten as the confinement/deconfinement transition of Yang-Mills theory in the AdS-CFT correspondence [2]. Since the thermodynamics of black holes has a deep connect with quantum theory of gravity, they may be modified by using other theories of gravity with higher derivative curvature terms. In the AdS/CFT correspondence, the higher derivative terms can be regarded as the corrections of large N expansion in the dual conformal field theory. In general, the higher powers of curvature can give rise to a fourth or even higher order differential equation for the metric, and it will introduce ghosts and violate unitarity. So, the higher derivative terms may be a source of inconsistencies. However,

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Zwiebach and Zumino [3] found that the ghosts can be avoided if the higher derivative terms only consist of the dimensional continuations of the Euler densities, leading to second order field equations for the metric. This higher derivative theory is so-called Lovelock gravity [4], and the equations of motion contain the most symmetric conserved tensor with no more than second derivative of the metric. In this paper, we restrict ourselves to explore the first three terms of the Lovelock gravity, corresponding to the cosmological constant, Einstein and Gauss-Bonnet terms respectively. In recent years, there have been considerable works for understanding the role of the higher curvature terms from various points of view, especially for higher dimensional black hole physics. the analytic expression of static and spherically symmetric Gauss-Bonnet black hole solutions have been investigated in [5]. The thermodynamics of the uncharged static spherically Gauss-Bonnet black hole solutions have been considered in [6] and of charged solutions in [7, 8].

It is of interest to generalize these static and spherically symmetric black hole solutions by including the effects of rotation. In the AdS/CFT correspondence, the rotating black holes in AdS space are dual to certain CFTs in a rotating space [9], while charged ones are dual to CFTs with chemical potential [10]. In general relativity, the most general higher dimensional rotating black holes in AdS space have been investigated in [11]. While, since the equations of motion of Lovelock gravity are highly nonlinear, it is rather difficult to obtain the explicit rotating black hole solutions. Recently, some numerical results about the existence of five-dimensional rotating Gauss-Bonnet black holes with angular momenta of the same magnitude have been presented in [12]. Besides, it is worth to mention that some rotating black brane solutions have been investigated in Gauss-Bonnet gravity [13]. Nevertheless, these solutions are essentially obtained by a Lorentz boost from corresponding static ones. They are equivalent to static ones locally, although not equivalent globally. In order to find rotating black hole solutions in the presence of dilaton coupling electromagnetic field in Einstein(-Maxwell) theory, Horne and Horowitz [14] first developed a simple method that a small angular momentum as a perturbation was introduced into a non-rotating system, and obtained slowly rotating dilaton black hole solutions. Such so-called slowly rotating black holes have been extensively discussed in general relativity [15]. With the help of it, Kim and Cai [16] presented analytic solutions of slowly rotating uncharged Gauss-Bonnet black holes with one non-vanishing angular momentum and arrived at the asymptotic forms of $g(r)$ and $c(r)$ in charged case, here the rotating parameter a appears as a small quantity. In this paper, we will also analyze slowly rotating charged Gauss-Bonnet black holes. We find that the off diagonal component of the stress-tensor of electromagnetic field is not independent of $c(r)$ in charged case. Then, the equations for $g(r)$ and $c(r)$ become two non-homogeneous differential equations. But, it's possible to get analytic expressions for $g(r)$ and $c(r)$.

The outline of this paper is as follows. In section II, we review Gauss-Bonnet gravity, and obtain the equations of gravitation and electromagnetic fields. By putting a new metric into these equations, the slowly rotating charged black hole solution $f(r)$ and exact expressions for functions $g(r)$ and $c(r)$ are obtained. Later, some related physical properties of black holes are studied there. We finish this paper with some concluding remarks.

II. SLOWLY ROTATING BLACK HOLES IN ADS SPACE

A. Action and Black Hole Solutions

The action of Gauss-Bonnet gravity in the presence of electromagnetic field can be written as

$$\mathcal{I} = \frac{1}{16\pi G}\int d^D x\sqrt{-g}(-2\Lambda + \mathcal{L}_1 + \alpha\mathcal{L}_2 - 4\pi G F_{\mu\nu}F^{\mu\nu}), \tag{1}$$

where α is the Gauss-Bonnet coefficient with dimension $(length)^2$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is electromagnetic field tensor with a vector potential A_μ. The Einstein term $\mathcal{L}_1$ equals to R, and the Gauss-Bonnet term $\mathcal{L}_2$ is $R_{\mu\nu\sigma\kappa}R^{\mu\nu\sigma\kappa} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. It is easy to find that the solution is asymptotically flat for $\Lambda = 0$, AdS for negative value of Λ and dS for positive value of Λ. We discuss the case of asymptotically AdS solutions in this paper.

Varying the action with respect to the metric tensor $g_{\mu\nu}$ and electromagnetic tensor field $F_{\mu\nu}$, the equations for gravitation and electromagnetic fields are

$$\Lambda g_{\mu\nu} + G^{(1)}_{\mu\nu} + \alpha G^{(2)}_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0. \tag{3}$$

Here $T_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ is the energy-momentum tensor of electromagnetic field, $G^{(1)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is Einstein tensor, and $G^{(2)}_{\mu\nu}$ is Gauss-Bonnet tensor given as

$$G^{(2)}_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_\nu{}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^\sigma{}_\nu + RR_{\mu\nu}) - \frac{1}{2}\mathcal{L}_2 g_{\mu\nu}. \tag{4}$$

Usually, the action Eq. (1) is supplemented with surface terms (a Gibbons-Hawking surface term) whose variation will cancel the extra normal derivative term in deriving the equation of motion Eq. (2). However, these surface terms is not necessary in our discussion and will be neglected. For Gauss-Bonnet gravity, the nontrivial third term requires the dimension(D) of spacetime satisfying $D \geq 5$.

We assume that the metric of slowly rotating spacetime is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \sum_{i=j=3}^{D} r^2 h_{ij}dx^i dx^j - 2ar^2 g(r)h_{44}dtd\phi, \tag{5}$$

where $h_{ij}dx^i dx^j$ represents the metric of a $(D-2)$ dimensional hyper-surface with constant curvature scalar $(D-2)(D-3)k$ and volume Σ_k, here k is a constant. The latter is the unit metric on S^{n-1}, R^{n-1}, or H^{n-1}, respectively, for $k = 1$, 0 or -1.

For the convenience future, we introduce

$$\Lambda = -\frac{(D-1)(D-2)}{2l^2}, \quad \tilde{\alpha} = \alpha(D-3)(D-4). \tag{6}$$

For $g(r) = 0$, we can arrive at the function $f(r)$ in charged case [8, 17]

$$f(r) = k + \frac{r^2}{2\tilde{\alpha}}\Big[1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2} + \frac{4\tilde{\alpha}m}{r^{D-1}} - \frac{4\tilde{\alpha}q^2}{r^{2D-4}}}\Big], \tag{7}$$

where m and q are related to the total mass $M = \frac{(D-2)\Sigma_k}{16\pi G}m$ and total charge $Q^2 = \frac{2\pi(D-2)(D-3)}{G}q^2$ of spacetimes.

In [16], the slowly rotating Gauss-Bonnet black holes have been studied in charged case. Kim and Cai reported expressions for $g(r)$ and $c(r)$ in asymptotic form. However, it is possible to present the analytic expressions for $g(r)$ and $c(r)$ by adopted a new approach. In case of $g(r) \neq 0$, the metric function $f(r)$ still keep the form Eq. (16). Here, we introduce $f(r) = k - r^2\varphi_*$ with

$$\varphi_* = -\frac{1}{2\tilde{\alpha}}\Big[1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2} + \frac{4\tilde{\alpha}m}{r^{D-1}} - \frac{4\tilde{\alpha}q^2}{r^{2D-4}}}\Big]. \tag{8}$$

Since the black hole rotates along the direction ϕ, it will generate a magnetic field. Considering this effect, we get the gauge potential

$$A_\mu dx^\mu = A_t dt + A_\phi d\phi. \tag{9}$$

Here we assume $A_\phi = -aQc(r)h_{44}$. As a result, the electro-magnetic field associated with the solution are

$$F_{tr} = -A_t', \quad F_{r\phi} = -aQc'(r)h_{44}, \quad F_{\theta\phi} = -aQc(r)h_{44}'. \tag{10}$$

where Q, an integration constant, is the electric charge of the black hole and a prime denotes the derivative with respect to r. Form t-component of electromagnetic field equation $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$, one can find $F_{tr} = \frac{Q}{4\pi r^{D-2}}$, which is the same as the static form. Unlike the static case, there exist the ϕ-component of the electromagnetic field equation, and then the equation for function $c(r)$ reads [16]

$$(r^{D-4}f(r)c'(r))' - 2k(D-3)r^{D-6}c(r) = \frac{g'(r)}{4\pi}. \tag{11}$$

Meanwhile, there exist off diagonal $t\phi$ component of equations of motion. It was considered that this equation decoupled from function $c(r)$ in [16]. While, one can easily verify that $t\phi$ component is concerned with functions $g(r)$ and $c(r)$. A tedious computation leads to a following equation

$$r^D(1+2\tilde{\alpha}\varphi_*)g(r)' = 4GQ^2c(r) + C_3. \tag{12}$$

We substitute metric Eq. (5) into the action Eq. (1). It is interesting to notice that the action Eq. (1) is in the absence of parameter a and φ_* is determined by solving for the real roots of the following second-order polynomial equation [7]

$$\frac{1}{l^2} + \varphi_* + \tilde{\alpha}\varphi_*^2 = \frac{m}{r^{D-1}} - \frac{q^2}{r^{2D-4}}. \tag{13}$$

We can easily verify by drawing a parallel between the Eqs.(12) and (13)

$$[m(1-D) + \frac{2q^2(D-2)}{r^{D-3}}]\frac{g(r)'}{\varphi_*'} = 4GQ^2c(r) + C_3. \tag{14}$$

Let the constant $C_3 = m(D-1)$ and $g(r) = -\varphi_*$, one can find two explicit solutions for functions $g(r)$ and $c(r)$

$$\begin{aligned} g(r) &= -\varphi_* \\ &= \frac{1}{2\tilde{\alpha}}\Big[1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2} + \frac{4\tilde{\alpha}m}{r^{D-1}} - \frac{4\tilde{\alpha}q^2}{r^{2D-4}}}\Big] \qquad (15) \\ c(r) &= -\frac{1}{4\pi(D-3)r^{D-3}}. \qquad (16) \end{aligned}$$

Apparently the expressions for $g(r)$ and $c(r)$ still satisfy the ϕ-component of the electromagnetic field equation Eq. (11).

B. Physical Properties

In this subsection, we analyze the physical properties of slowly rotating charged black holes. Shown in Eq. (16), the slowly rotating charged solution $f(r)$ is independent of a. Most interesting physical properties depend only on a^2, but one can still extract some useful information from it.

For slowly rotating solution, the horizon r_+ is still determined by the equation $f(r_+) = 0$. Then, r_+ is related to the mass (m) and charge (q) of the black hole by the relation

$$r_+^{2D}(r_+^4/l^2 + kr_+^2 + k^2\tilde{\alpha}) = mr_+^{D+5} - q^2 r_+^8. \tag{17}$$

So, one can write the gravitational mass of black holes

$$M = \frac{(D-2)\Sigma_k r_+^{D-1}}{16\pi G}[\frac{1}{l^2} + kr_+^{-2} + k^2\tilde{\alpha}r_+^{-4} + \frac{q^2}{r_+^{2D-4}}]. \tag{18}$$

Then, the Hawking temperature of the black hole can be obtained by required the absence of conical singular at the horizon in the Euclidean of the black hole solution. It is the same as the static case

$$T = \frac{f'(r_+)}{4\pi} = \frac{\Gamma(r_+)}{4\pi r_+(r_+^2 + 2k\tilde{\alpha})}, \tag{19}$$

where $\Gamma(r_+) = (D-1)r_+^4/l^2 + (D-3)kr_+^2 + (D-5)k^2\tilde{\alpha} - (D-3)q^2/r_+^{2D-8}$. Thus, the angular momentum of the black hole

$$J = \frac{2aM}{D-2} = \frac{a\Sigma_k r_+^{D-1}}{8\pi G}[\frac{1}{l^2} + kr_+^{-2} + k^2\tilde{\alpha}r_+^{-4} + \frac{q^2}{r_+^{2D-4}}]. \tag{20}$$

Usually, the entropy of black hole satisfies the so-called area law of entropy which states that the black hole entropy equals to one-quarter of the horizon area [18], [19]. It applies to all kinds of black holes and black strings of Einstein gravity [20]. However, in higher derivative gravity, the area law of the entropy is not satisfied in general [21]. Since black hole can be regard as a thermodynamic system, it obeys the first law of thermodynamics $dM = TdS + \omega_H dJ$. Through the angular velocity ω_H, one can get the entropy of black hole.

For the slowly rotating solution, the stationarity and rotational symmetry metric Eq. (5) admits two commuting Killing vector fields

$$\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{\phi} = \frac{\partial}{\partial \phi}. \tag{21}$$

The various scalar products of these Killing vectors can be expressed through the metric components as follows

$$\begin{aligned}\xi_{(t)} \cdot \xi_{(t)} &= g_{tt} = -f(r), \\ \xi_{(t)} \cdot \xi_{(\phi)} &= g_{t\phi} = -ar^2 g(r) h_{44}, \\ \xi_{(\phi)} \cdot \xi_{(\phi)} &= g_{\phi\phi} = r^2 h_{44}.\end{aligned}$$

To examine further properties of the slowly rotating black holes, as well as physical processes near such a black hole, we introduce a family of locally non-rotating observers. The coordinate angular velocity for these observers that move on orbits with constant r and θ and with a four-velocity u^{μ} such that $u \cdot \xi_(\phi) = 0$ is given by [22],

$$\begin{aligned}\Omega &= -\frac{g_{t\phi}}{g_{\phi\phi}} = ag(r) \\ &= \frac{a}{2\tilde{\alpha}}\Big[1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2} + \frac{4\tilde{\alpha}m}{r^{D-1}} - \frac{4\tilde{\alpha}q^2}{r^{2D-4}}}\Big].\end{aligned} \tag{22}$$

In contrast to the case of an ordinary kerr black hole in asymptotically flat spacetime, the angular velocity does not vanish at spatial infinity

$$\Omega_{\infty} = \frac{a}{2\tilde{\alpha}}\Big(1 - \sqrt{1 - \frac{4\tilde{\alpha}}{l^2}}\Big) = \frac{a}{l_{eff}^2}. \tag{23}$$

When approaching the black hole horizon, the angular velocity turns to be $\Omega_H = ag(r_+) = -a\varphi(r_+) = -\frac{ak}{r_+^2}$. This Ω_H can be thought as the angular velocity of the black hole. The relative angular velocity with respect to a frame static at infinity is defined by

$$\omega_H = \Omega_H - \Omega_{\infty} = -a(\frac{k}{r_+^2} + \frac{1}{l_{eff}^2}). \tag{24}$$

Therefore, we get the entropy of slowly rotating black hole up to the linear order of the rotating parameter a

$$S = \frac{\Sigma_k}{4G} r_+^{D-2}[1 + \frac{2(D-2)k\tilde{\alpha}}{(D-4)r_+^2}]. \tag{25}$$

III. CONCLUDING REMARKS

By introducing a small angular momentum, we discussed slowly rotating Gauss-Bonnet charged black holes. For charged case, the vector potential has an extra nonradial component $A_\phi = -aQc(r)h_{44}$ due to the rotation of the black holes. In addition, since the off-diagonal component of the stress-tensor of electro-magnetic field was related to $c(r)$, the equations for $g(r)$ and $c(r)$ become two non-homogeneous differential equations. Then, the analytic solutions for $c(r)$ and $g(r)$ have been separately expressed as $c(r) = -\frac{1}{4\pi(D-3)r^{D-3}}$ and $g(r) = -\varphi_*$ by using a new approach, while the function $f(r)$ still keep the form of the static solution. Up to the linear order of the rotating parameter a, the expressions of the mass, temperature, and entropy for the charged black holes got no correction from rotation.

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