Do the Ricci and energy-momentum tensors have "duality" in the context of their Lie symmetries?

arXiv:1011.5163v1 [gr-qc] 23 Nov 2010

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Abstract: The Ricci and energy-momentum tensors have the same algebraic symmetries. In the Einstein equations they look "dual" to each other, in that interchanging them and inverting the gravitational coupling leaves the equations invariant. It may then be expected that their differential symmetry Lie algebras would also be identical. Using cylindrically symmetric static spacetimes it is shown that they are not identical and neither algebra is a subset of the other.

1. Introduction

Lie symmetries of various geometrical and physical quantities in general relativity have been studied for some time [1, 2]. Isometries, or Killing vectors (KVs), the vector fields along which the metric tensor, $\mathbf{g}$, remains invariant under Lie transport, have been used to construct new solutions of the Einstein field equations (EFEs)

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}; (a, b = 0, 1, 2, 3) \tag{1.1}$$

where $\mathbf{R}$ is the Ricci tensor, $\mathbf{T}$ the energy-momentum tensor, R the Ricci scalar and κ the gravitational coupling $8\pi G/c^4$. While the KVs give the symmetries inherent in the space itself, invariance under the Lie transport of the energy-momentum tensor gives the symmetries of the matter content of the space (called matter collineations or MCs [3, 4]), and hence is more relevant physically. Since it appears in the EFEs with the Ricci tensor, the symmetries of the Ricci tensor (called Ricci collineations or RCs [1, 2]) are also physically relevant. These vector fields also provide invariant bases for the classification of the solutions of the EFEs. A vector field ξ is an MC if the Lie derivative of the energy-momentum tensor vanishes along ξ

$$\pounds_\xi \mathbf{T} = 0. \tag{1.2}$$

In component form, the MC equation (1.2) takes the form

$$\xi^c T_{ab,c} + T_{ac}\xi^c_{,b} + T_{bc}\xi^c_{,a} = 0. \tag{1.3}$$

Here comma denotes a partial derivative with respect to the coordinates. For four dimensional space these are ten coupled partial differential equations which are to be solved for the four components of the vector $\xi = (\xi^0, \xi^1, \xi^2, \xi^3)$. If the energy-momentum tensor in the last equation is replaced by the Ricci tensor, the vector field is an RC, and if it is replaced by the Riemann curvature tensor, the vector field is called a curvature collineation (CC) [1, 2]. It is well known that every KV is a CC and every CC, in turn, is an RC but the converse is not true in general. Mutual relationships between different spacetime symmetries are represented graphically in the inclusion diagram in Ref. [5]. An interesting question arises here about the place of MCs in this diagram, and that is the subject of this paper.

There has been recent interest in the study of RCs of plane symmetric [6], spherically symmetric [7], cylindrically symmetric [8] and various other classes of spacetimes [9]. As mentioned earlier, if the components of the energy-momentum tensor, T_{ab}, in Eq.(1.3) are replaced by those of the Ricci tensor, we get RCs. Due to the similarity of the mathematical form of the Ricci and the energy-momentum tensors, and similarity of their collineation equations, attempts were made to obtain the results for MCs from those of RCs [10, 11] by replacing the Ricci tensor by the energy-momentum tensor. These attempts assume that because of the identity of the algebraic symmetries of the two tensors their differential symmetries would also be identical i.e. their corresponding algebras would be same. In this paper we show that this is not true. Another aim of this paper is to investigate the next and more important question of "duality" between these two tensors, as regards their collineations. To achieve this, we construct all possible (inclusion) relationships between RCs and MCs. We find that cylindrically symmetric static spacetimes provide a very useful framework for this investigation, as all the components of the Ricci tensor are independent (which is not the case in spherical symmetry, for example). This fact gives rise to a whole lot of possibilities for the relationship between RCs and MCs. We investigate all these possibilities here and demonstrate that there is no inclusion relation between the two algebras.

The plan of the paper is as follows. In the next section we give the MC equations and discuss their solution. In Section 3 all the possibilities of relationships between MCs and RCs are identified and specific examples provided for each of these cases. The concluding remarks are given in Section 4. Tables 1-5 which summarize the solutions of the MC equations are provided in the Appendix.

2. Matter collineations of cylindrically symmetric static spacetimes

The line element for the general cylindrically symmetric static spacetimes in (t, ρ, θ, z) coordinates can be written as [1]

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - a^2 e^{\lambda(\rho)}d\theta^2 - e^{\mu(\rho)}dz^2, \tag{2.1}$$

where the minimal symmetry is given by the three Killing vectors, ∂_t, ∂_θ, ∂_z. For this metric the only non-zero components of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \tfrac{e^\nu}{4}\left(2\nu'' + \nu'^2 + \nu'\lambda' + \nu'\mu'\right),\\
R_{11} &= -\left(\tfrac{\nu''}{2} + \tfrac{\lambda''}{2} + \tfrac{\mu''}{2} + \tfrac{\nu'^2}{4} + \tfrac{\lambda'^2}{4} + \tfrac{\mu'^2}{4}\right),\\
R_{22} &= -\tfrac{a^2 e^\lambda}{4}\left(2\lambda'' + \nu'\lambda' + \lambda'^2 + \lambda'\mu'\right),\\
R_{33} &= -\tfrac{e^\mu}{4}\left(2\mu'' + \nu'\mu' + \lambda'\mu' + \mu'^2\right).
\end{aligned} \tag{2.2}$$

Here ′′′ denotes differentiation with respect to ρ. The Ricci scalar is given by

$$R = \nu'' + \lambda'' + \mu'' + \frac{1}{2}\left(\nu'^2 + \lambda'^2 + \mu'^2 + \nu'\lambda' + \nu'\mu' + \lambda'\mu'\right). \tag{2.3}$$

Using the EFEs (Eq. 1.1), the general form of the energy-momentum tensor, T_b^a, becomes

$$\begin{aligned}
T_0^0 &= -\tfrac{1}{4}\left(2\lambda'' + 2\mu'' + \lambda'^2 + \mu'^2 + \lambda'\mu'\right), \\
T_1^1 &= -\tfrac{1}{4}\left(\nu'\lambda' + \nu'\mu' + \lambda'\mu'\right), \\
T_2^2 &= -\tfrac{1}{4}\left(2\nu'' + 2\mu'' + \nu'^2 + \mu'^2 + \nu'\mu'\right), \\
T_3^3 &= -\tfrac{1}{4}\left(2\nu'' + 2\lambda'' + \nu'^2 + \lambda'^2 + \nu'\lambda'\right).
\end{aligned} \tag{2.4}$$

Now, the solution of Eqs. (1.3) for the energy-momentum tensor is similar to the one given in Ref. [8], and it can be written simply by replacing the components of the Ricci tensor there by those of the energy-momentum tensor. Therefore, we will not give the MC vectors for different cases and their corresponding Lie algebras and Lie groups here again and the reader is referred to Ref. [8] for all these details. However, we will reproduce the tables of the main results here as we will need to refer to them frequently in the next section. It may be pointed out here again that during the course of solution of the MC (or RC) equations one gets different cases which are characterized by the constraints on the components of the energy-momentum (or Ricci) tensor. We will be using the same notation and case numbering here as used in Ref. [8] for easy comparison. In fact, if we solve Eqs. (1.3) for a general second rank, symmetric and diagonal tensor A_{ab}, we not only get the KVs [12] and RCs [8] for cylindrically symmetric static spacetimes but also find the MCs explicitly. This means that these tables can be used to obtain complete information on these three symmetries. There is one point however that while the Ricci and the energy-momentum tensors can be degenerate (i.e. the determinant is zero) as well as non-degenerate (i.e. the determinant is non-zero), the metric tensor cannot be degenerate. We see that when the Ricci tensor is non-degenerate, the Lie algebra of the RCs is always finite-dimensional. However, when it is degenerate, it admits a finite-dimensional Lie algebra only when $R_{11} = 0$, $R_{ii} \neq 0$, $i = 0, 2, 3$. This holds for MCs also. Tables 1-5 are for finite-dimensional Lie algebras only. The numbers in the last column indicate the dimension of the Lie algebra admitted by ξ and equation numbers there refer to those in Ref. [8]. Further, as we are dealing with diagonal tensors, for simplicity we will write R_i and T_i for R_{ij} and $T_{ij} (i = j)$, respectively.

3. Matter and Ricci Collineations

In what follows we write "finite (or infinite) MCs", in place of "MCs having finite (or infinite) dimensional Lie algebra", for the sake of brevity. Similarly, we write "non-degenerate (or degenerate) MCs" when we mean "MCs for the non-degenerate (or degenerate) energy-momentum tensor". The same holds for RCs also. We find that depending upon whether the MCs and RCs are degenerate or non-degenerate, finite or infinite, all possible relationships between them can be written in the form of the following table, where the last column gives the example number for the corresponding case.

Possible relationships between MCs and RCs				
Non-Degenerate MCs	(Finite MCs)	Non-Degenerate RCs	(Finite RCs)	3.1
		Degenerate RCs	Finite RCs	3.2
			Infinite RCs	3.3
Degenerate MCs	Finite MCs	Non-Degenerate RCs	(Finite RCs)	3.4
		Degenerate RCs	Finite RCs	3.5*
			Infinite RCs	3.6*
	Infinite MCs	Non-Degenerate RCs	(Finite RCs)	3.7
		Degenerate RCs	Finite RCs	3.8
			Infinite RCs	3.9
*Examples for these cases have not been provided				

The metrics for all these possibilities have been constructed with the exception of two cases. The examples of these spacetimes given below also demonstrate the procedure of finding MCs, RCs and KVs from Tables 1-5 in the Appendix. We shall call MCs (or RCs) *proper* if they are not KVs.

3.1 Non-degenerate (finite) MCs; non-degenerate (finite) RCs

In this case both the energy-momentum and the Ricci tensor are non-degenerate having finite MCs and RCs. Consider the metic

$$ds^2 = \cosh^2 k\rho dt^2 - d\rho^2 - a^2 \left(\cosh k\rho\right)^{-1} d\theta^2 - \left(\cosh k\rho\right)^{-1} dz^2, \qquad (3.1)$$

For this metric the components of energy-momentum tensor are

$$\begin{aligned} T_0 &= \cosh^2 k\rho \frac{k^2}{4}\left(4 - 7\tanh^2 k\rho\right), \\ T_1 &= -\frac{3k^2}{4}\tanh^2 k\rho, \\ T_2 &= a^2 (\cosh k\rho)^{-1} \frac{k^2}{4}\left(2 + \tanh^2 k\rho\right) \ , \\ T_3 &= (\cosh k\rho)^{-1} \frac{k^2}{4}\left(2 + \tanh^2 k\rho\right) \ . \end{aligned}$$

This is an anisotropic fluid with energy density positive for $0 \leq \rho < \frac{1}{k}\tanh^{-1}\frac{2}{\sqrt{7}}$ and negative for $\rho \geq \frac{1}{k}\tanh^{-1}\frac{2}{\sqrt{7}}$. However, with a cosmological constant greater than $\frac{3}{4}k^2$, the energy density becomes positive definite. For this metric the components for R_{ab} are

$$\begin{aligned} R_0 &= k^2, R_1 = -\frac{3k^2}{2}\tanh^2 k\rho, \\ R_2 &= \frac{k^2}{2}(\sec hk\rho)^3, R_3 = \frac{k^2}{2}(\sec hk\rho)^3. \end{aligned}$$

It admits 4 MCs (Case AIIa(2)), 7 RCs (Case BIVb3(ii)γ_2) and 4 KVs and, therefore, is a case of proper RCs.

3.2 Non-degenerate (finite) MCs; degenerate and finite RCs

Here we provide an example of a metric with non-degenerate energy-momentum tensor and degenerate Ricci tensor with both MCs and RCs finite. Consider

$$ds^2 = (\rho/\rho_0)^{2a} dt^2 - d\rho^2 - (\rho/\rho_0)^{2b} \alpha^2 d\theta^2 - (\rho/\rho_0)^{2c} dz^2, \tag{3.2}$$

where, $a = (1 \pm \sqrt{3})/2$, $b = c = 1/2$, and one gets $R_1 = 0$ and R_i are non-zero constants for $i = 0, 2, 3$. For $a = (1 + \sqrt{3})/2$, we have T_{ab} in component form

$$\begin{aligned} T_0 &= \frac{\rho^{\sqrt{3}-1}}{4\rho^{\sqrt{3}+1}}, T_1 = (\frac{3}{4} + \frac{\sqrt{3}}{2})\rho^{-2}, \\ T_2 &= -\frac{\alpha^2(2+\sqrt{3})}{4\rho\rho_0}, T_3 = -\frac{2+\sqrt{3}}{4\rho\rho_0}\ . \end{aligned}$$

For this metric R_{ab} has the following components

$$R_0 = \frac{(1+\sqrt{3})^2 \rho^{\sqrt{3}-1}}{4\rho_0^{\sqrt{3}+1}}, R_1 = 0,$$

$$R_2 = -\frac{\alpha^2(1+\sqrt{3})}{4\rho_0\rho}, R_3 = -\frac{(1+\sqrt{3})}{4\rho_0\rho}.$$

It admits 5 MCs (AIIb1(i)β), 5 RCs (Case IIBd4(i)) and 4 KVs (Case AIIa(2)), and therefore, is a case of proper MCs and RCs.

3.3 Non-degenerate (finite) MCs; degenerate and infinite RCs

Here we discuss the example of non-degenerate energy-momentum and degenerate Ricci tensors with finite MCs but infinite dimensional RC algebra.

$$ds^2 = e^{A\rho}\left(dt^2 - dz^2\right) - d\rho^2 - a^2 d\theta^2, \tag{3.3}$$

A is a non-zero constant. For this metric the components of T_{ab} are

$$T_0 = -\frac{e^{A\rho}A^2}{4}, T_1 = \frac{A^2}{4},$$
$$T_2 = \frac{3a^2A^2}{4}, T_3 = \frac{e^{A\rho}A^2}{4}.$$

R_{ab} has the following components

$$R_0 = \frac{e^{A\rho}A^2}{2}, R_1 = -\frac{A^2}{2},$$
$$R_2 = 0, R_3 = -\frac{e^{A\rho}A^2}{2}.$$

It has 7 MCs (Case AIa1(i)), RCs have infinite dimensional Lie algebra (Case (III)) and 7 KVs (Case AIa1(i)). It is anti-Einstein and anisotropic with negative energy.

3.4 Degenerate and finite MCs; non-degenerate (finite) RCs

One of the examples of metrics with degenerate energy-momentum tensor with finite MCs and non-degenerate Ricci tensor with finite RCs is provided here.

$$ds^2 = (\rho/\rho_0)^{-1/2} dt^2 - d\rho^2 - (\rho/\rho_0)\,\alpha^2 d\theta^2 - (\rho/\rho_0)\,dz^2 \tag{3.4}$$

Taking $a = -1/4$, $b = c = 1/2$ in metric (A3) gives the above metric. For this metric the components of T_{ab} are

$$T_0 = \frac{\rho_0^{1/2}}{4\rho^{5/2}}, T_1 = 0,$$
$$T_2 = -\frac{\alpha^2}{16\rho\rho_0}, T_3 = -\frac{1}{16\rho\rho_0}.$$

and components of R_{ab} are

$$R_0 = \frac{\rho_0^{1/2}}{16\rho^{5/2}}, R_1 = \frac{3}{16\rho^2},$$
$$R_2 = \frac{\alpha^2}{8\rho\rho_0}, R_3 = \frac{1}{8\rho\rho_0}.$$

It has 5 MCs (Case II Bd4(i)), 5 RCs (Case AIIb1(i)β) and 4KVs (Case AIIa(2)).

3.5 Degenerate and finite MCs; degenerate and finite RCs

This is the case where an example has eluded our attempts. It would really be interesting to see if an example of a spacetime exists for which both the RCs and MCs are finite and degenerate. The necessary conditions for this to happen are that R_{11} and T_{11} are zero and other components non-zero. Alternatively, a proof of non-existence of such a space would also be very interesting.

3.6 Degenerate and finite MCs; degenerate and infinite RCs

Here also we have not been able to find an example for which both the MCs and RCs are degenerate but the former is finite and the latter is infinite. But this is where the question of "duality" between the energy-momentum and the Ricci tensors become important because we have its "mirror" example in Case 3.8, where although both are degenerate the MCs are infinite while the RCs are finite. Non-existence of an example here would imply that there is no "duality" between the two tensors as far as their collineations is concerned.

3.7 Degenerate and infinite MCs; non-degenerate (finite) RCs

Here we provide a metric with infinite dimensional MC algebra and finite RCs.

$$ds^2 = (\rho/\rho_0)^{2a} dt^2 - d\rho^2 - (\rho/\rho_0)^{4/3} \alpha^2 d\theta^2 - (\rho/\rho_0)^{4/3} dz^2 \tag{3.5}$$

Choosing $a \neq 4/3,\ 0\ ,\ 2/3\ ,\ -1/3\ ,\ 1$ and $b = c = 2/3$ in metric (A3) gives above metric. For this metric the components of T_{ab} are

$$T_0 = 0, T_1 = \frac{4(a+1/3)}{3\rho^2},$$
$$T_2 = -\frac{\alpha^2(3a - 9a^2 + 2)}{9\rho^{2/3}\rho_0^{4/3}}, T_3 = -\frac{(3a - 9a^2 + 2)}{9\rho^{2/3}\rho_0^{4/3}}.$$

and R_{ab} has the following components

$$R_0 = \rho_0^{-2a} a(a+1/3)\rho^{2a-2}, R_1 = -\frac{(9a^2-9a-4)}{9\rho^2},$$
$$R_2 = -\frac{2\alpha^2(3a+1)}{9\rho_0^{4/3}\rho^{2/3}}, R_3 = -\frac{2(3a+1)}{9\rho_0^{4/3}\rho^{2/3}}.$$

It admits infinite dimensional MCs (Case (I)), 5 RCs (CaseA IIb1(i)β) and 4 KVs (Case AIIa(2)).

3.8 Degenerate and infinite MCs; degenerate and finite RCs

Here both the energy-momentum and the Ricci tensors are degenerate with finite RCs and infinite MCs.

$$ds^2 = (\rho/\rho_0)^{8/3} dt^2 - d\rho^2 - (\rho/\rho_0)^{4/3} \alpha^2 d\theta^2 - (\rho/\rho_0)^{4/3} dz^2 . \tag{3.6}$$

Taking $a = 4/3$, $b = c = 2/3$ in metric (A3) gives the above metric. The components of T_{ab} are

$$T_0 = 0, T_1 = \frac{20}{9}\rho^{-2},$$
$$T_2 = \frac{10\alpha^2}{9\rho^{2/3}\rho_0^{4/3}}, T_3 = \frac{10}{9\rho^{2/3}\rho_0^{4/3}}.$$

for this metric R_{ab} has the following components

$$R_0 = \frac{20\rho^{2/3}}{9\rho_0^{8/3}}, R_1 = 0,$$
$$R_2 = -\frac{10\alpha^2}{9\rho^{2/3}\rho_0^{4/3}}, R_3 = -\frac{10}{9\rho^{2/3}\rho_0^{4/3}}.$$

It admits MCs having infinite dimensional Lie algebra (CaseI), 5 RCs (Case IIBd4(i)) and 4 KVs (AIIa(2)).

3.9 Degenerate and infinite MCs; degenerate and infinite RCs

The case of infinite dimensional algebras for both the MCs and RCs when the two tensors are degenerate is discussed here.

$$ds^2 = cosh^2(A+B\rho)dt^2 - d\rho^2 - a^2 d\theta^2 - dz^2, \tag{3.7}$$

A and α are constants. It is a Bertotti-Robinson-like metric. Components of T_{ab} are as follows

$$T_0 = 0, T_1 = 0,$$
$$T_2 = a^2 B^2, T_3 = B^2.$$

and R_{ab} has the following components

$$R_0 = B^2 \cosh^2(A + B\rho), R_1 = B^2,$$
$$R_2 = 0, R_3 = 0.$$

It has infinite dimensional Lie algebras both for MCs (case (IX)) and RCs (Case (X)) 6 KVs (Case AIIb2(ii)α_2).

4. Conclusion

We have studied the relationship between the Lie symmetries or collineations of the two second rank tensors, the energy-momentum and the Ricci tensors, which are mathematically very similar. In particular, we investigate whether or not this similarity and their duality in the EFEs is preserved by their collineations also. For this purpose we have used the framework of cylindrically symmetric static manifolds. The KVs and RCs of these spaces have been classified earlier [12, 8]. While KVs have a finite dimensional Lie algebra always, RCs and MCs can admit infinite dimensional Lie algebra as well. Similarly, RCs and MCs can be degenerate or non-degenerate. In this way we see that, in all, there are a total of nine types of relationships between RCs and MCs which are formulated in a table in Section 3. To show that they are not just symmetries, to which no solutions of EFEs exist, we have explicitly constructed examples for all of these cases, except for the two cases 3.5 and 3.6. For these two cases we have not been able to provide any example, nor have we managed to prove that they do not exist. Unless and until the examples for these two cases are provided the question of "duality" of the Lie symmetries of the energy-momentum and the Ricci tensors will remain open.

It is worth while explaining the problem in finding the examples. Despite the apparent duality of the tensors in the EFEs, there is an enormous difference in the differential equations defining the tensors. At the very least, this complicates the equations to the point that while we can construct the solutions for the cases for the Ricci tensor, we are unable to do so for the energy-momentum tensor. It appears to be a distinct possibility that there is no duality between the two tensors because of the difference in the differential equations yielding the cases. It may be that the answer to our question will come by investigating the structure of the two differential equations.

Appendix

The tables in the appendix summarize the solutions of Eq.(1.3). These are, in fact, obtained by changing the components of the Ricci tensor in Ref. [8] by those of the energy-momentum tensor. Thus the equation numbers in the last columns of these tables refer to the equations in Ref. [8].

Acknowledgments

KS acknowledges a research grant from the Higher Education Commission of Pakistan. MZ gratefully acknowledges the Internal Research Grant: IG/SCI/DOMS/10/01 by Sultan Qaboos University, Sultanate of Oman.

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Table 1: The Non-Degenerate Case A(I) $\quad T_0' \neq 0$

(I) $\left(\frac{T_2}{T_3}\right)' \neq 0$	(a) $T_2' = 0\,,$ $T_3' \neq 0$	(1) $\alpha = 0,\ \beta = 0$	(i) $\left(\frac{T_0}{T_3}\right)' = 0$			7 MCs (Eqs. 16)
			(ii) $\left(\frac{T_0}{T_3}\right)' \neq 0$			4 MCs (Eqs. 17)
		(2) $\alpha \neq 0,\ \beta = 0$	(i) $\alpha > 0$			3 MCs
			(ii) $\alpha < 0$	(α) $\left(\frac{T_0}{T_3}\right)' \neq 0$		3 MCs
				(β) $\left(\frac{T_0}{T_3}\right)' = 0$		4 MCs (Eqs. 18)
		(3) $\alpha = 0,\ \beta \neq 0$				Similar to (2)
		(4) $\alpha \neq 0,\ \ \beta \neq 0$	(i) $\alpha > 0,\ \beta > 0$	(α) $\beta \int \frac{\sqrt{T_1}}{T_0} d\rho - \frac{T_3'}{2T_0\sqrt{T_1}} \neq 0$	(α_1) $\left(\frac{T_0}{T_3}\right)' = 0$	4 MCs (Eqs. 19)
					(α_2) $\left(\frac{T_0}{T_3}\right)' \neq 0$	3 MCs
				(β) $\beta \int \frac{\sqrt{T_1}}{T_0} d\rho - \frac{T_3'}{2T_0\sqrt{T_1}} = 0$		3 MCs
			(ii) $\alpha > 0,\ \beta < 0$			Similar to (i)
			(iii) $\alpha > 0,\ \beta > 0$			Similar to (i)
			(iv) $\alpha < 0,\ \beta < 0$			Similar to (i)
	(b) $T_2' \neq 0\,,$ $T_3' = 0$					Similar to (a)

Definitions

$$\alpha = \frac{T_0}{\sqrt{T_1}}\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)' \quad k_1 = -\frac{T_0'}{2T_0\sqrt{T_1}}$$

$$\beta = \frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)' \quad k_2 = -\frac{T_3'}{2T_3\sqrt{T_1}}$$

Table 2: The Non-Degenerate Case A(I) $T_0' \neq 0$ (continued)

(I) $\left(\frac{T_2}{T_3}\right)' \neq 0$	(c) $T_2' \neq 0\,,$ $T_3' \neq 0$	(1) $\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)' \neq 0\,,$ $\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)' \neq 0$	(i) $\left(\frac{T_2}{T_0}\right)' = 0,$ $\left(\frac{T_3}{T_0}\right)' \neq 0$		4 MCs (Eqs. 16)
			(ii) $\left(\frac{T_2}{T_0}\right)' \neq 0,$ $\left(\frac{T_3}{T_0}\right)' = 0$		Similar to (i)
			(iii) $\left(\frac{T_2}{T_0}\right)' \neq 0,$ $\left(\frac{T_3}{T_0}\right)' \neq 0$		3 MCs
		(2) $\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)' = 0\,,$ $\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)' = 0$	(i) $\left(\frac{T_2}{T_0}\right)' = 0,$ $\left(\frac{T_3}{T_0}\right)' \neq 0$		5 MCs (Eqs. 22)
			(ii) $\left(\frac{T_2}{T_0}\right)' \neq 0,$ $\left(\frac{T_3}{T_0}\right)' = 0$		Similar to (i)
			(iii) $\left(\frac{T_2}{T_0}\right)' \neq 0,$ $\left(\frac{T_3}{T_0}\right)' \neq 0$	(α) $\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)' = 0$	4 MCs (Eqs. 23)
				(β) $\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)' \neq 0$	3 MCs

Table 3: The Non-Degenerate Case A(II) $T_0' \neq 0$

(II) $\left(\frac{T_2}{T_3}\right)' = 0$	(a) $\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)' \neq 0$	(1) $\left(\sqrt{\frac{T_0}{T_2}}\right)' = 0$				6 MCs (Eqs. 24)
		(2) $\left(\sqrt{\frac{T_0}{T_2}}\right)' \neq 0$				4 MCs (Eqs. 25)
	(b) $\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)' = 0$	(1) $\alpha \neq 0$	(i) $\left(\frac{T_2}{T_0}\right)' \neq 0$	(α) $\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)' \neq 0$		4 MCs (Eqs. 26)
				(β) $\left(\frac{T_0'}{2T_0\sqrt{T_1}}\right)' = 0$		5 MCs (Eqs. 27)
			(ii) $\left(\frac{T_2}{T_0}\right)' = 0$			10 MCs (Eqs. 28)
		(2) $\alpha = 0$,	(i) $\left(\frac{(\sqrt{T_0})'}{\sqrt{T_1}}\right)' = 0$			10 MCs (Eqs. 29)
			(ii) $\left(\frac{(\sqrt{T_0})'}{\sqrt{T_1}}\right)' \neq 0$	(α) $\left[\frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0\sqrt{T_1}}\right)'\right]' = 0$	(α_1) $\eta = 0$	6 MCs (Eqs. 30)
					(α_2) $\eta \neq 0$	6 MCs (Eqs. 31)
				(β) $\left[\frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0\sqrt{T_1}}\right)'\right]' \neq 0$		4 MCs (Eqs. 32)

Definitions	$\alpha = \frac{T_2'}{T_2\sqrt{T_1}}$	$\eta = \frac{T_0}{2\sqrt{T_1}}\left(\frac{T_0'}{T_0\sqrt{T_1}}\right)'$

Table 4: The Non-Degenerate Case B $T_0' = 0$

(I) $T_2' = 0\,,\ T_3' = 0$						10 MCs (Eqs. 33)
(II) $T_2' = 0\,,\ T_3' \neq 0$	(a) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' \neq 0$					4 MCs (Eqs. 34)
	(b) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' = 0$	(1) $k_1 > 0$				6 MCs (Eqs. 35)
		(2) $k_1 < 0$				Similar to (1)
		(3) $k_1 = 0$				6 MCs (Eqs. 36)
(III) $T_2' \neq 0\,,\ T_3' = 0$						Similar to (II)
(IV) $T_2' \neq 0\,,\ T_3' \neq 0$	(a) $\left[\frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'\right]' \neq 0$	(1) $\left(\frac{T_2}{T_3}\right)' \neq 0$				3 MCs
		(2) $\left(\frac{T_2}{T_3}\right)' = 0$				4 MCs (Eqs. 37)
	(b) $\left[\frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'\right]' = 0$	(1) $k_3 > 0$	(i) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' \neq 0$			4 MCs (Eqs. 38)
			(ii) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' = 0$			4 MCs (Eqs. 38)
		(2) $k_3 < 0$				Similar to (1)
		(3) $k_3 = 0$	(i) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' \neq 0$			4 MCs (Eqs. 38)
			(ii) $\left[\frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'\right]' = 0$	(α) $k_1 > 0$		4 MCs (Eqs. 38)
				(β) $k_1 < 0$		Similar to (2)
				(γ) $k_1 = 0$	(γ_1) $\left(\frac{T_3}{T_2}\right)' \neq 0$	4 MCs (Eqs. 39)
					(γ_2) $\left(\frac{T_3}{T_2}\right)' = 0$	7 MCs (Eqs. 40)

Definitions

$$k_1 = \frac{T_3}{\sqrt{T_1}}\left(\frac{T_3'}{2T_3\sqrt{T_1}}\right)'$$

$$k_3 = \frac{T_2}{\sqrt{T_1}}\left(\frac{T_2'}{2T_2\sqrt{T_1}}\right)'$$

Table 5: The Degenerate Case II $T_1 = 0\,,\, T_0 \neq 0\,,\, T_2 \neq 0\,,\, T_3 \neq 0$

(A) $T_0' = 0$	$T_2' \neq 0\,,\, T_3' \neq 0$	(1) $\left(\frac{T_2' T_3}{T_2 T_3'}\right)' = 0$		4 MCs (Eqs. 41)
		(2) $\left(\frac{T_2' T_3}{T_2 T_3'}\right)' \neq 0$		3 MCs
	Otherwise			Infinitely many MCs
(B) $T_0' \neq 0$	(a) $\left(\frac{T_0}{T_2}\right)' = 0, \left(\frac{T_0}{T_3}\right)' = 0$			10 MCs (Eqs. 42)
	(b) $\left(\frac{T_0}{T_2}\right)' = 0, \left(\frac{T_0}{T_3}\right)' \neq 0$	(1) $\left(\frac{T_3' T_0}{T_3 T_0'}\right)' = 0$		5 MCs (Eqs. 43)
		(2) $\left(\frac{T_3' T_0}{T_3 T_0'}\right)' \neq 0$		4 MCs (Eqs. 44)
	(c) $\left(\frac{T_0}{T_2}\right)' \neq 0, \left(\frac{T_0}{T_3}\right)' = 0$			Similar to (b)
	(d) $\left(\frac{T_0}{T_2}\right)' \neq 0, \left(\frac{T_0}{T_3}\right)' \neq 0$	(1) $\left(\frac{T_0' T_2}{T_0 T_2'}\right)' \neq 0, \left(\frac{T_0' T_3}{T_0 T_3'}\right)' \neq 0$	(i) $\left(\frac{T_2}{T_3}\right)' = 0$	4 MCs (Eqs. 45)
			(ii) $\left(\frac{T_2}{T_3}\right)' \neq 0$	3 MCs
		(2) $\left(\frac{T_0' T_2}{T_0 T_2'}\right)' \neq 0\,, \left(\frac{T_0' T_3}{T_0 T_3'}\right)' = 0$		4 MCs (Eqs. 45)
		(3) $\left(\frac{T_0' T_2}{T_0 T_2'}\right)' = 0\,, \left(\frac{T_0' T_3}{T_0 T_3'}\right)' \neq 0$		Similar to (2)
		(4) $\left(\frac{T_0' T_2}{T_0 T_2'}\right)' = 0\,, \left(\frac{T_0' T_3}{T_0 T_3'}\right)' = 0$	(i) $\left(\frac{T_2}{T_3}\right)' = 0$	5 MCs (Eqs. 46)
			(ii) $\left(\frac{T_2}{T_3}\right)' \neq 0$	4 MCs