General formulation of general-relativistic higher-order gauge-invariant perturbation theory

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Gauge-invariant treatments of general-relativistic higher-order perturbations on generic background spacetime is proposed. After reviewing the general framework of the second-order gauge-invariant perturbation theory, we show the fact that the linear-order metric perturbation is decomposed into gauge-invariant and gauge-variant parts, which was the important premis of this general framework. This means that the development the higher-order gauge-invariant perturbation theory on generic background spacetime is possible. A remaining issue to be resolve is also disscussed.

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Perturbation theories are powerful techniques in many area of physics and lead phyisically fruiteful results. In particualr, in general relativity, the construction of exact soltuions is not so easy and known exact solutions are often too idealized, though there are many known exact solutions to the Einstein equation[1]. Of course, some exact solutions to the Einstein equation well describe our universe, or gravitational field of stars and black holes. However, in natural phenomena, there always exist "fluctuations". To describe this, the *linear* perturbation theories around some background spacetime are developed[2], and are used to describe fluctuations of our universe, gravitational field of stars, and gravitational waves from strongly gravitating sources.

Besides the development of the general-relativistic linear-order perturbation theory, higher-order general-relativistic perturbations also have very wide applications, for example, cosmological perturbations[3–7], black hole perturbations[8], and perturbation of a neutron star[9]. In spite of these applications, there is a delicate issue in the treatment of general-relativistic perturbations, which is called *gauge issue*. General relativity is based on general covariance. Due to this general covariance, the *gauge degree of freedom*, which is an unphysical degree of freedom of perturbations, arises in general-relativistic perturbations. To obtain physical results, we have to fix this gauge degree of freedom or to treat some invariant quantities in perturbations. This situation becomes more complicated in higher-order perturbations. Thus, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to this motivation, the general framework of higher-order general-relativistic gauge-invariant perturbation theory has been discussed[10, 11] and applied to cosmological perturbations[6, 7]. However, this framework is based on a conjecture (Conjecture 1 below) which roughly states that *we have already known the procedure to find gauge-invariant variables for a linear-order metric perturbations*. The main purpose of this letter is to give the outline of a proof of this conjecture. Due to this proof, a formulation of the higher-order general-relativistic gauge-invariant perturbation theory is almost completed on generic background spacetime.

Now, we review the framework of the gauge-invariant perturbation theory[10, 11]. In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $(\mathcal{M}, \bar{g}_{ab})$, which is our nature itself, and we want to describe $(\mathcal{M}, \bar{g}_{ab})$ by perturbations. The other is the background spacetime $(\mathcal{M}_0, g_{ab})$, which is prepared as a reference by hand. We note that these two spacetimes are distinct.

Further, in any perturbation theory, we always write equations for the perturbation of the variable Q like

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p). \tag{1}$$

Equation (1) gives a relation between variables on different manifolds. Actually, $Q(\text{“}p\text{”})$ in Eq. (1) is a variable on $\mathcal{M}$, while $Q_0(p)$ and $\delta Q(p)$ are variables on $\mathcal{M}_0$. We regard Eq. (1) as a field equation and this is an implicit assumption of the exitence of a point identification map $\mathcal{M}_0 \to \mathcal{M} : p \in \mathcal{M}_0 \mapsto \text{“}p\text{”} \in \mathcal{M}$. This idenification map is a *gauge choice* in perturbation theories[12].

To develop this understanding of the "gauge", we introduce an infinitesimal parameter λ and $(n+1)+1$-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$ $(n+1 = \dim \mathcal{M})$ so that $\mathcal{M}_0 = \mathcal{N}|_{\lambda=0}$ and $\mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda}$. On $\mathcal{N}$, the gauge choice is regarded as a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. Further, we introduce a gauge choice $\mathcal{X}_\lambda$ as an exponential map with a generator ${}^{\mathcal{X}}\eta^a$ which is chosen so that its integral curve in $\mathcal{N}$ is transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$[4, 5]. Points lying on the same integral curve are regarded as the "same" by the gauge choice $\mathcal{X}_\lambda$.

The first- and the second-order perturbations of the variable Q on $\mathcal{M}_\lambda$ are defined by the pulled-back $\mathcal{X}^*_\lambda Q$

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on $\mathcal{M}_0$, which is induced by $\mathcal{X}_\lambda$, and expanded as

$$\mathcal{X}^*_\lambda Q = Q_0 + \lambda \, \pounds_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0} + \frac{1}{2}\lambda^2 \, \pounds^2_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0} + O(\lambda^3), \quad (2)$$

$Q_0 = Q|_{\mathcal{M}_0}$ is the background value of Q and all terms in Eq. (2) are evaluated on $\mathcal{M}_0$. Since Eq. (2) is just the perturbative expansion of $\mathcal{X}^*_\lambda Q_\lambda$, the first- and the second-order perturbations of Q are given by ${}^{(1)}_{\mathcal{X}}Q := \pounds_{{}^{\mathcal{X}}\eta} Q\big|_{\mathcal{M}_0}$ and ${}^{(2)}_{\mathcal{X}}Q := \pounds^2_{{}^{\mathcal{X}}\eta} Q\Big|_{\mathcal{M}_0}$, respectively.

When we have two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ with the generators ${}^{\mathcal{X}}\eta^a$ and ${}^{\mathcal{Y}}\eta^a$, respectively, and when these generators have the different tangential componetns to each $\mathcal{M}_\lambda$, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as *different gauge choices*. The *gauge-transformation* is regarded as the change of the gauge choice $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, which is given by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$. The diffeomorphism Φ_λ does change the point identification. Φ_λ induces a pull-back from the representation $\mathcal{X}^*_\lambda Q_\lambda$ to the representation $\mathcal{Y}^*_\lambda Q_\lambda$ as $\mathcal{Y}^*_\lambda Q_\lambda = \Phi^*_\lambda \mathcal{X}^*_\lambda Q_\lambda$. From general arguments of the Taylor expansion[5], the pull-back Φ^*_λ is expanded as

$$\begin{aligned}\mathcal{Y}^*_\lambda Q_\lambda &= \mathcal{X}^*_\lambda Q_\lambda + \lambda \pounds_{\xi_{(1)}} \mathcal{X}^*_\lambda Q_\lambda \\ &\quad + \frac{1}{2}\lambda \left(\pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}}\right) \mathcal{X}^*_\lambda Q_\lambda + O(\lambda^3), \quad (3)\end{aligned}$$

where $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are the genertors of Φ_λ. From Eqs. (2) and (3), each order gauge-transformation is given as

$$\begin{aligned}{}^{(1)}_{\mathcal{Y}}Q - {}^{(1)}_{\mathcal{X}}Q &= \pounds_{\xi_{(1)}} Q_0, \quad (4)\\ {}^{(2)}_{\mathcal{Y}}Q - {}^{(2)}_{\mathcal{X}}Q &= 2\pounds_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}}Q + \left\{\pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}}\right\} Q_0. \quad (5)\end{aligned}$$

We also employ the *order by order gauge invariance* as a concept of gauge invariance[7]. We call the kth-order perturbation ${}^{(p)}_{\mathcal{X}}Q$ is gauge invariant iff ${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q$ for any gauge choice $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$.

Based on the above set up, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations[10]. First, we expand the metric on the physical spacetime $\mathcal{M}_\lambda$, which is pulled back to the background spacetime $\mathcal{M}_0$ through a gauge choice $\mathcal{X}_\lambda$ as

$$\mathcal{X}^*_\lambda \bar{g}_{ab} = g_{ab} + \lambda {}_{\mathcal{X}}h_{ab} + \frac{\lambda^2}{2} {}_{\mathcal{X}}l_{ab} + O^3(\lambda). \quad (6)$$

Although the expression (6) depends entirely on the gauge choice $\mathcal{X}_\lambda$, henceforth, we do not explicitly express the index of the gauge choice $\mathcal{X}_\lambda$ in the expression if there is no possibility of confusion. The important premise of our proposal was the following conjecture[10] for h_{ab} :

Conjecture 1. *For a second-rank tensor h_{ab}, whose gauge transformation is given by (4), there exist a tensor $\mathcal{H}_{ab}$ and a vector X^a such that h_{ab} is decomposed as*

$$h_{ab} =: \mathcal{H}_{ab} + \pounds_X g_{ab}, \quad (7)$$

where $\mathcal{H}_{ab}$ and X^a are transformed as

$${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi^a_{(1)} \quad (8)$$

under the gauge transformation (4), respectively.

We call $\mathcal{H}_{ab}$ and X^a are the *gauge-invariant part* and the *gauge-variant part* of h_{ab}, respectively.

Although Conjecture 1 is nontrivial on generic background spacetime, once we accept this conjecture, we can always find gauge-invariant variables for higher-order perturbations[10]. Using Conjecture 1, the second-order metric perturbation l_{ab} is decomposed as

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Y - \pounds^2_X\right) g_{ab}, \quad (9)$$

where ${}_{\mathcal{Y}}\mathcal{L}_{ab} - {}_{\mathcal{X}}\mathcal{L}_{ab} = 0$ and ${}_{\mathcal{Y}}Y^a - {}_{\mathcal{X}}Y^a = \xi^a_{(2)} + [\xi_{(1)}, X]^a$. Furthermore, using the first- and second-order gauge-variant parts, X^a and Y^a, of the metric perturbations, gauge-invariant variables for an arbitrary tensor field Q other than the metric can be defined by

$$\begin{aligned}{}^{(1)}\mathcal{Q} &:= {}^{(1)}Q - \pounds_X Q_0, \quad (10)\\ {}^{(2)}\mathcal{Q} &:= {}^{(2)}Q - 2\pounds_X {}^{(1)}Q - \left\{\pounds_Y - \pounds^2_X\right\} Q_0. \quad (11)\end{aligned}$$

These definitions (10) and (11) also imply that any perturbation of first and second order is always decomposed into gauge-invariant and gauge-variant parts as

$$\begin{aligned}{}^{(1)}Q &= {}^{(1)}\mathcal{Q} + \pounds_X Q_0, \quad (12)\\ {}^{(2)}Q &= {}^{(2)}\mathcal{Q} + 2\pounds_X {}^{(1)}Q + \left\{\pounds_Y - \pounds^2_X\right\} Q_0, \quad (13)\end{aligned}$$

respectively.

Actually, the perturbations of the Einstein tensor are given in the same form Eqs. (12) and (13) :

$$\begin{aligned}\bar{G}_a{}^b &= G_a{}^b + \lambda {}^{(1)}G_a{}^b + \frac{1}{2}\lambda^2 {}^{(2)}G_a{}^b + O(\lambda^3), \quad (14)\\ {}^{(1)}G_a{}^b &= {}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] + \pounds_X G_a{}^b, \quad (15)\\ {}^{(2)}G_a{}^b &= {}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] \\ &\quad + 2\pounds_X {}^{(1)}\bar{G}_a{}^b + \left\{\pounds_Y - \pounds^2_X\right\} G_a{}^b, \quad (16)\end{aligned}$$

where ${}^{(1)}\mathcal{G}_a{}^b[*]$ is the gauge-invariant linear terms and ${}^{(2)}\mathcal{G}_a{}^b[*,*]$ are collections of quadratic terms of gauge-invaraint linear metric perturbations. On the other hand, the energy momentum tensor on $\mathcal{M}_\lambda$ is also expanded as

$$\bar{T}_a{}^b = T_a{}^b + \lambda {}^{(1)}T_a{}^b + \frac{1}{2}\lambda^2 {}^{(2)}T_a{}^b + O(\lambda^3), \quad (17)$$

and its first- and the second-order perturbations ${}^{(1)}T_a{}^b$ and ${}^{(2)}T_a{}^b$ are decomposed as Eqs. (12) and (13) :

$$\begin{aligned}{}^{(1)}T_a{}^b &= {}^{(1)}\mathcal{T}_a{}^b + \pounds_X T_a{}^b, \quad (18)\\ {}^{(2)}T_a{}^b &= {}^{(2)}\mathcal{T}_a{}^b + 2\pounds_X {}^{(1)}T_a{}^b + \left\{\pounds_Y - \pounds^2_X\right\} T_a{}^b. \quad (19)\end{aligned}$$

These were confirmed in the case of a perfect fluid, an imperfect fluid, and a scalar field[7].

Imposing order by order Einstein equations

$$G_a{}^b = 8\pi T_a{}^b, \quad {}^{(p)}G_a{}^b = 8\pi{}^{(p)}T_a{}^b, \quad (p=1,2), \quad (20)$$

the first- and the second-order perturbation of the Einstein equations are automatically given in gauge-invariant form :

$$\begin{aligned} &{}^{(1)}\mathcal{G}_a{}^b[\mathcal{H}] = 8\pi G^{(1)}\mathcal{T}_a{}^b, \\ &{}^{(1)}\mathcal{G}_a{}^b[\mathcal{L}] + {}^{(2)}\mathcal{G}_a{}^b[\mathcal{H},\mathcal{H}] = 8\pi G \;{}^{(2)}\mathcal{T}_a{}^b. \end{aligned} \quad (21)$$

Further, the perturbative equations of motion for matter fields, which are derived from the divergence of the energy momentum tensor, are also automatically given in gauge-invariant form[7].

Thus, based only on Conjecture 1, we have developed the general framework of second-order general relativistic perturbation theory. We also note that this general framework of the second-order gauge-invariant perturbation theory are independent of the explicit form of the background metric g_{ab}, except for Conjecture 1.

Now, we give the outline of a proof of Conjecture 1. To do this, we only consider the background spacetimes which admit ADM decomposition[15]. Therefore, the background spacetime $\mathcal{M}_0$ considered here is $n+1$-dimensional spacetime which is desribed by the direct product $\mathbb{R}\times\Sigma$. Here, $\mathbb{R}$ is a time direction and Σ is the spacelike hypersurface ($\dim\Sigma = n$). The background metric g_{ab} is given as

$$g_{ab} = -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i+\beta^i dt)_a(dx^j+\beta^j dt)_b. \quad (22)$$

In this letter, we only consider the case where $\alpha = 1$ and $\beta^i = 0$, for simlicity.

To consider the decomposition (7) of h_{ab}, first, we consider the components of the metric h_{ab} as

$$\begin{aligned} h_{ab} &= h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)_{(a}(dx^i)_{b)} \\ &\quad + h_{ij}(dx^i)_a(dx^j)_b. \end{aligned} \quad (23)$$

Under the gauge-transformation (4), these components $\{h_{tt}, h_{ti}, h_{ij}\}$ are transformed as

$$\begin{aligned} {}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} &= 2\partial_t\xi_t, && (24) \\ {}_{\mathcal{Y}}h_{ti} - {}_{\mathcal{X}}h_{ti} &= \partial_t\xi_i + D_i\xi_t + 2K^j{}_i\xi_j, && (25) \\ {}_{\mathcal{Y}}h_{ij} - {}_{\mathcal{X}}h_{ij} &= 2D_{(i}\xi_{j)} + 2K_{ij}\xi_t. && (26) \end{aligned}$$

where K_{ij} is the extrinsic curvature of Σ and D_i is the covariant derivative associate with the metric q_{ij} ($D_i q_{jk} = 0$). In our case, $K_{ij} = -\frac{1}{2}\partial_t q_{ij}$.

Inspecting gauge-transformation rules (25) and (26), we introduce a new symmetric tensor $\hat{H}_{ab}$ whose components are given by

$$\hat{H}_{tt} := h_{tt}, \quad \hat{H}_{ti} := h_{ti}, \quad \hat{H}_{ij} := h_{ij} - 2K_{ij}X_t. \quad (27)$$

Here, we assume the existence of the variable X_t whose gauge-transformation rule is given by ${}_{\mathcal{Y}}X_t - {}_{\mathcal{X}}X_t = \xi_t$. This assumption is confirmed later soon. Since the components $\hat{H}_{ti}$ and $\hat{H}_{ij}$ are a vector and a symmetric tensor on Σ, respectively, $\hat{H}_{ti}$ and $\hat{H}_{ij}$ are decomposed as[13]

$$\begin{aligned} \hat{H}_{ti} &= D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, && (28) \\ \hat{H}_{ij} &= \frac{1}{n} q_{ij} h_{(L)} \\ &\quad + 2\left(D_{(i}h_{(TV)j)} - \frac{1}{n}q_{ij}D^l h_{(TV)l}\right) \\ &\quad + h_{(TT)ij}, \quad D^i h_{(TT)ij} = 0, && (29) \\ h_{(TV)i} &= D_i h_{(TVL)} + h_{(TVV)i}, \quad D^i h_{(TVV)i} = 0. && (30) \end{aligned}$$

The one-to-one correspondence between $\{\hat{H}_{ti},\ \hat{H}_{ij}\}$ and $\{h_{(VL)}, h_{(V)i}, h_{(L)}, h_{(TVL)}, h_{(TVV)i}, h_{(TT)ij}\}$ guaranteed by the existence of the Green functions of operators $\Delta := D^i D_i$ and $\mathcal{D}^{ij} := q^{ij}\Delta + \left(1-\frac{2}{n}\right)D^i D^j + {}^{(n)}R^{ij}$, where ${}^{(n)}R^{ij}$ is the Ricci curvature on Σ. Here, we assume thier existence. Gauge-transformation rules for $\{h_{tt},\ h_{(VL)},\ h_{(V)i},\ h_{(L)},\ h_{(TVL)},\ h_{(TVV)i},\ h_{(TT)ij}\}$ are summarized as

$$\begin{aligned} {}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} &= 2\partial_t\xi_t, \quad {}_{\mathcal{Y}}h_{(TT)ij} - {}_{\mathcal{X}}h_{(TT)ij} = 0, && (31) \\ {}_{\mathcal{Y}}h_{(VL)} - {}_{\mathcal{X}}h_{(VL)} &= \partial_t\xi_{(L)} + \xi_t + \Delta^{-1}\left[2D_i\left(K^{ij}D_j\xi_{(L)}\right) + D^k K\xi_{(V)k}\right], && (32) \\ {}_{\mathcal{Y}}h_{(V)i} - {}_{\mathcal{X}}h_{(V)i} &= \partial_t\xi_{(V)i} + 2K^j{}_i D_j\xi_{(L)} + 2K^j{}_i\xi_{(V)j} - D_i\Delta^{-1}\left[2D_i\left(K^{ij}D_j\xi_{(L)}\right) + D^k K\xi_{(V)k}\right], && (33) \\ {}_{\mathcal{Y}}h_{(L)} - {}_{\mathcal{X}}h_{(L)} &= 2D^i\xi_i, \quad {}_{\mathcal{Y}}h_{(TVL)} - {}_{\mathcal{X}}h_{(TVL)} = \xi_{(L)}, \quad {}_{\mathcal{Y}}h_{(TVV)l} - {}_{\mathcal{X}}h_{(TVV)l} = \xi_{(V)l}, \quad . && (34) \end{aligned}$$

We first find the variable X_t in Eq. (27). From the above gauge-transformation rules, we see that the combination

$$X_t := h_{(VL)} - \partial_t h_{(TVL)} - \Delta^{-1}\left[2D_i\left(K^{ij}D_j h_{(TVL)}\right) + D^k K h_{(TVV)k}\right] \quad (35)$$

satisfy ${}_{\mathcal{Y}}X_t - {}_{\mathcal{X}}X_t = \xi_t$. We also find the variable X_i

$$X_i := h_{(TV)i} = D_i h_{(TVL)} + h_{(TVV)i} \quad (36)$$

satisfy the gauge-transformation rule ${}_{\mathcal{Y}}X_i - {}_{\mathcal{X}}X_i = \xi_i$.

Inspecting gauge-transformation rules (31)–(34) and using the variables X_t and X_i defined by Eqs. (35)–(36), we find gauge-invariant variables as follows:

$$-2\Phi := h_{tt} - 2\partial_t \hat{X}_t, \quad -2n\Psi := h_{(L)} - 2D^i \hat{X}_i, \tag{37}$$

$$\begin{aligned} \nu_i := & h_{(V)i} - \partial_t h_{(TVV)i} \\ & -2K^j{}_i \left(D_j h_{(TVL)} + h_{(TVV)j}\right) \\ & +D_i \Delta^{-1} \left[2D_i \left(K^{ij} D_j h_{(TVL)}\right) \right. \\ & \left. \qquad + D^k K h_{(TVV)k}\right], \end{aligned} \tag{38}$$

$$\chi_{ij} := h_{(TT)ij}. \tag{39}$$

Actually, it is straightforward to confirm the gauge-invariance of these varaibles.

In terms of the variables Φ, Ψ, ν_i, χ_{ij}, X_t, and X_i, original components of h_{ab} is given by

$$h_{tt} = -2\Phi + 2\partial_t X_t, \tag{40}$$

$$h_{ti} = \nu_i + D_i X_t + \partial_t X_i + 2K^j{}_i X_j, \tag{41}$$

$$h_{ij} = -2\Psi q_{ij} + \chi_{ij} + D_i X_j + D_j X_i + 2K_{ij} X_t. \tag{42}$$

Comparing Eq. (7), a natural choice of $\mathcal{H}_{ab}$ and X_a are

$$\begin{aligned} \mathcal{H}_{ab} = & -2\Phi (dt)_a (dt)_b + 2\nu_i (dt)_{(a} (dx^i)_{b)} \\ & + \left(-2\Psi q_{ij} + \chi_{ij}\right) (dx^i)_a (dx^i)_b, \end{aligned} \tag{43}$$

$$X_a = X_t (dt)_a + X_i (dx^i). \tag{44}$$

These show that the linear-order metric perturbation h_{ab} is decomposed into the form Eq. (7).

In summary, we showed the outline of a proof of Conjecture 1 which is the important premise of our general framework of gauge-invariant perturbation theory. Although we only consider the background spacetime with $\alpha = 1$ and $\beta^i = 0$, the above proof is extended to general case[14].

In our proof, we assumed the existence of the Green functions for the derivative operators Δ and $\mathcal{D}^{ij}$. This implies that we have ingored the modes which belong to the kernel of these derivative operators. To includes these modes into our consideration, different treatments of perturbations will be necessary. We call this problem as *zero-mode problem*. Even in the cosmological perturbations, zero-mode problem exists. We leave the resolution of this zero-mode problem as a future work.

Although this zero-mode problem should be resolved, we confirmed the important premise of our general framework of second-order gauge-invaraint perturbation theory on generic background spacetime. Due to this, we have the possibility of applications of our framework for the second-order gauge-invariant perturbation theory to perturbations on generic background spacetime. Actually, in the cosmological perturbation case, we have developed the second-order cosmological perturbations along this general framework[6, 7]. The similar development will be also possible for the any order perturbation in two-parameter case[10]. Therefore, we may say that the wide appicaltions of our gauge-invariant perturbation theory are opened. We also leave these developments as future works.

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