The regular cosmic string in Born-Infeld gravity

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It is shown that Born-Infeld gravity –a high energy deformation of Einstein gravity– removes the singularities of a cosmic string. The respective vacuum solution results to be free of conical singularity and closed timelike curves. The space ends at a minimal circle where the curvature invariants vanish; but this circle cannot be reached in a finite proper time.

We shall investigate high energy deformations of General Relativity (GR) based on the following three guiding principles:

a) The theory must reduce to General Relativity in the low field limit.

b) The spacetime dynamics must be described by second order equations.

c) It must provide a proper treatment in order to avoid singularities.

Many deformations can accomplish the condition (a), but just a few of them satisfy (b).[20] Lovelock's extension of GR does it [4], but it only departs from GR for dimensions bigger than four. In turn, a few candidates were proposed following the guidelines (a) and (c) (see e.g. the non-exhaustive list [5]-[10]). Remarkably, there is no conceptual framework accommodating the three requirements. To face the item (c) we will consider one of the prototypes of a GR singular structure: the circular symmetric vacuum solution in (2+1) dimensions [11, 12], or its (3+1)-dimensional analogue, the cosmic string [13]

$$ds^2 = d(t + 4J\,\theta)^2 - d\rho^2 - (1 - 4\mu)^2\,\rho^2\,d\theta^2 - dz^2. \tag{1}$$

In (2+1) dimensions (z is absent), this metric solves the Einstein equations for $T^{00} = \mu\,\delta(x, y)$ and $T^{0i} = (J/2)\,\epsilon^{ij}\,\partial_j\delta(x, y)$ (Cartesian coordinates x, y are defined as usual, and we use hereafter $G = 1$). So the solution looks as a particle of mass μ and spin J at the origin [12]. However, no gravitational field surrounds the particle since the metric is manifestly flat, as is typical of vacuum solutions to Einstein equations without cosmological constant in (2+1)-dimensions. Actually the particle shows itself by means of topological properties: i) the space displays a conical singularity, because the mass μ amounts a deficit angle $8\pi\mu$; ii) the Minkowskian time $t' \equiv t + 4J\,\theta$ suffers a jump $8\pi\,J$ when the circle is completed. Added to this singular structure, the *cosmon* –as named in Ref. [14]– also exhibits violation of causality because closed timelike curves (CTC) occur. In fact, the interval (1) on the closed curves of constant (t, ρ, z) is

$$ds^2 = \left(\frac{16J^2}{M^2} - \rho^2\right)\,M^2\,d\theta^2\,, \qquad M \equiv 1 - 4\mu, \tag{2}$$

which means that curves with radio $\rho < 4J/M$ are CTC. This unpleasant feature was avoided in Ref. [14] by resorting to proper boundary conditions. However, we will show that the entire singular structure of cosmons is prevented by a high energy deformation of GR developed in Refs. [15]-[17].

As the starting point for describing the gravitational field, we will propose a *determinantal* Born-Infeld (BI) Lagrangian density [17]

$$\mathcal{L}_{\mathbf{BIG}} = -\lambda/(16\,\pi)\,\left(\sqrt{|g_{\mu\nu} - 2\lambda^{-1}F_{\mu\nu}|} - \sqrt{|g_{\mu\nu}|}\right), \tag{3}$$

where $|\ |$ stands for the absolute value of the determinant. The structure (3) resembles the one used by Born and Infeld in their non linear electrodynamics [18]. In our case, the field tensor $F_{\mu\nu}$ will depend on the dynamical variable

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describing the gravitational field. The constant λ in Eq. (3) has the units of $F_{\mu\nu}$ and controls the low energy limit alluded in the requirement (a). In fact, since

$$\sqrt{|I - 2\lambda^{-1}F|} = 1 - \lambda^{-1}Tr(F) + \mathcal{O}(\lambda^{-2}), \tag{4}$$

then the Lagrangian (3) goes to

$$\mathcal{L}_{\mathbf{G}} \;=\; 1/(16\,\pi)\,\sqrt{|g_{\mu\nu}|}\;Tr(F), \tag{5}$$

when $F_{\mu\nu}$ takes values much smaller than λ. So, Born-Infeld gravity will reduce to GR if $Tr(F) = -R$. On the other hand, the requirement (b) implies that the quantities involved in the Lagrangian (3) should depend just on the dynamical variables and their first derivatives. This condition could seem to be out of tune with GR, since the curvature scalar depends on the second derivatives of its dynamical field (the metric). However, there is a beautiful way of circumventing this blind alley, by moving out towards an equivalent formulation of GR which depends just on first derivatives of its dynamical variables. Indeed, Einstein theory can be rephrased in a spacetime possessing absolute parallelism [19] where the *vielbein* $\{e^a(x)\}$ –a set of 1-forms– plays the role of the gravitational potentials, while the metric $\mathrm{g}(x) = \eta_{ab}\, e^a(x)\otimes e^b(x)$ is a subsidiary field. The absolute parallelism is ruled by the Weitzenböck connection $\Gamma^{\lambda}_{\mu\nu} \equiv e_a{}^{\lambda}\, e^a{}_{\mu,\,\nu}$, where $\{e_a{}^{\lambda}(x)\}$ is the vector basis dual to the *vielbein*. This connection is metric compatible and curvatureless: Weitzenböck spacetime is flat, but it possesses torsion $T^{\rho}{}_{\mu\nu} = e^{\rho}{}_a\,(e^a{}_{\nu,\,\mu} - e^a{}_{\mu,\,\nu})$ –i.e., $T^a = de^a$–, which is the agent where the gravitational degrees of freedom are encoded. The equivalence between both pictures reveals itself when one discovers that the Levi-Civita curvature scalar R can be rewritten as

$$4\,R[e^a] \;=\; 2\,W_1 \;-\; W_2 \;-\; 4\,W_3 \;+\; 8\,\mathrm{e}^{-1}\,(\mathrm{e}\,T^{\mu}{}_{\mu}{}^{\rho})_{,\,\rho}\;, \tag{6}$$

where $\mathrm{e} = \sqrt{|\mathrm{g}|}$ is the determinant of the matrix $e^a{}_{\mu}$, and the invariants W_i are

$$W_1 = T^{\mu\nu}{}_{\rho}\,T^{\rho}{}_{\mu\nu}\,,\quad W_2 = T_{\rho}{}^{\mu\nu}\,T^{\rho}{}_{\mu\nu}\,,\quad W_3 = T^{\rho\nu}{}_{\rho}\,T^{\mu}{}_{\mu\nu}\,. \tag{7}$$

Remarkably, the advantage of the teleparallel picture of gravity is that the total divergence term in Eq. (6) can be ruled out to work just with a first order Lagrangian $\mathcal{L}(e^a, \partial e^a)$, which will preserve the dynamical content of Einstein theory anyway. This distinctive feature makes Weitzenböck spacetime a privileged geometric structure to formulate modified theories of gravitation, since it guarantees that any modified Lagrangian in this language will assure second order field equations. This mechanism is essential to satisfy the requirement (b), otherwise fourth order differential equations would be obtained. Summarizing, we are building a theory of gravity where the fundamental piece is the set of 2-forms $T^a = de^a$ (notice the similarity with Yang-Mills theories), so guaranteeing the requirement (b). To accomplish the requirement (a) we should demand that

$$4\,Tr(F) \;=\; -2\,W_1 \;+\; W_2 \;+\; 4\,W_3 \;\equiv\; 4\,S_{\rho}{}^{\mu\nu}\,T^{\rho}{}_{\mu\nu}, \tag{8}$$

where

$$S_{\rho}{}^{\mu\nu} = -\frac{1}{4}\left(T^{\mu\nu}{}_{\rho} - T^{\nu\mu}{}_{\rho} - T_{\rho}{}^{\mu\nu}\right) + \frac{1}{2}\left(\delta^{\mu}_{\rho}\;T^{\theta\nu}{}_{\theta} - \delta^{\nu}_{\rho}\;T^{\theta\mu}{}_{\theta}\right). \tag{9}$$

The more general tensor $F_{\mu\nu}$ fulfilling the condition (8) reads $F_{\mu\nu} = \alpha\,S_{\mu\lambda\rho}\,T_{\nu}{}^{\lambda\rho} + \beta\,S_{\lambda\mu\rho}\,T^{\lambda}{}_{\nu}{}^{\rho}$, $\alpha + \beta = 1$. For simplicity we will set $\beta = 0$; thus, in four dimensions the theory reduces to

$$\mathcal{L}_{\mathbf{BIG}}[e^a] = -\lambda/(16\,\pi)\left(\sqrt{|g_{\mu\nu} - 2\lambda^{-1}S_{\mu\lambda\rho}\,T_{\nu}{}^{\lambda\rho}|} - \sqrt{|g_{\mu\nu}|}\right). \tag{10}$$

Let us now analyze the requirement (c). We will show that BI gravity (10) prevents the singular structure of the GR cosmon metric (1). First of all, we remark that the cosmon was endowed with spin J by considering the coordinate $t = t' - 4J\theta$ as the continuous time in the place of the Minkowskian time t'; in (2+1) dimensions, this procedure led to a non-zero T^{0i} at the origin. The mass μ came from a similar procedure. In BI gravity, instead, the equations determine not the metric but the vielbein; they contain more integration constants than GR does. In particular, J and $M \equiv 1 - 4\mu$ will appear as labels for the family of cylindrically symmetric vielbeins solving the dynamical equations. We found that the solution of the motion equations is given by the vielbein [17]

$$e^0 \;=\; d(t + 4J\theta)\,,\quad e^1 \;=\; Y(\rho)\,d\rho\,,\quad e^2 \;=\; \rho\,M\,d\theta\,,\quad e^3 \;=\; dz, \tag{11}$$

where the function $Y(\rho)$ solves the cubic equation

$$Y^2(\rho) - Y^3(\rho) = -\frac{16\,J^2}{\lambda M^2}\Big(\rho^2 - \frac{16\,J^2}{M^2}\Big)^{-2}. \tag{12}$$

The vielbein (11) implies the metric

$$ds^2 = d(t + 4J\,\theta)^2 - Y^2(\rho)d\rho^2 - M^2\,\rho^2\,d\theta^2 - dz^2, \tag{13}$$

then the function $Y(\rho)$ constitutes the sole difference between GR and BI gravity. GR is recovered when $\lambda \to \infty$, as expected from the low energy limit (5). In fact, Eq. (12) says that $Y \to 1$ and the metric (13) acquires the flat form of Eq. (1). For finite values of λ and far away from the string, the low field regime is also restituted. Nevertheless, the BI cosmic string is a geometry with a J-depending non-constant scalar curvature $R = -2Y'/(\rho Y^3)$ depicted in Figure 1 for several values of λ.

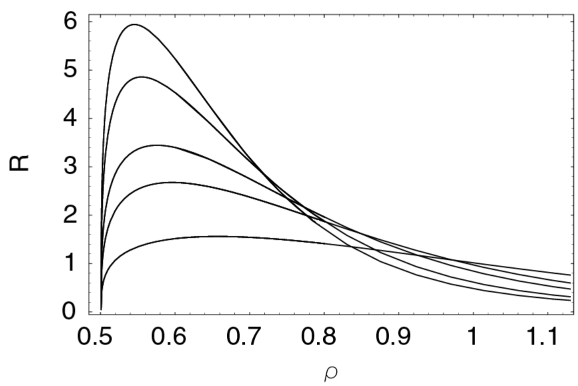

FIG. 1: Scalar curvature $R(\rho)$ for $J/M = 1/8$ in units of length (so, R vanishes at $\rho_o = 1/2$). Following the maximum of the curves from bottom to top, it is $\lambda = 1, 3, 5, 10, 15$ in units of inverse square length.

According to Eq. (12), Y goes to infinity as ρ approaches the value $\rho_o = 4J/M$. At this minimal circle, R and other curvature invariants such as $R^{\mu\nu}R_{\mu\nu}$ and $R^{\lambda}{}_{\mu\nu\rho}\,R_{\lambda}{}^{\mu\nu\rho}$ become zero. Thus, the spacetime is flat not only far from the string (GR region) but at ρ_o as well. It is easy to verify that the proper time to reach ρ_o diverges [17], which means that BI gravity avoids the conical singularity. Moreover, the CTC's are prevented too. Even though the interval on the closed curves of constant (t, ρ, z) does not differ from the one of Eq. (2) –since ρ is constant, then $Y(\rho)$ does not play any role–, however CTC's are forbidden because the lower bound of the radial coordinate ρ implies that the interval (2) is negative definite. In this way, the same mechanism responsible for the taming of the conical singularity in BI gravity also provides a natural chronological protection.

It is worth mentioning that the coordinate change $d\xi = Y(\rho)\,d\rho$ in the vielbein (11) allows to regard the curved geometry (13) as a space of a variable deficit angle ranging from $8\pi\mu$ at spatial infinity, to 2π at ρ_0. This feature might have important observational implications on the lensing effect. As another remarkable physical consequence, BI gravity seems to forbid the possibility of packing energy in arbitrarily small regions. Differing from GR, any junction of this vacuum solution with an inner solution has to be made at a radius bigger than $\rho_o = 4J/M$. This property is another manifestation of the Born-Infeld regularization program. As a final remark, we emphasize that the BI determinantal gravity (10) departs from a mere "$f(T)$" theory even at the lowest order in λ^{-1} (T stands for the invariant $S_{\rho}{}^{\mu\nu}\,T^{\rho}{}_{\mu\nu}$). This feature is essential for smoothing the cosmic string singularities, since the solution has $T = 0$ [16].

Acknowledgments

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[20] For instance, the so called "$f(R)$" theories [1], lead to fourth order equations. They can be reformulated as scalar-tensor theories to give second order equations; however this procedure results in violations of the equivalence principle [2, 3]. Moreover, vacuum singular GR solutions remain as solutions of smooth $f(R)$ theories, so threatening the requirement (c).