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A short survey on mathematical work of Cemal Koç

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I first met Cemal Koç during Spring semester of 1977. He was then a young associate professor at the Middle East Technical University, Ankara, and I was a Ph.D. student attending his graduate course on noncommutative rings. I had previously attended lectures in Germany given by leading experts on several topics in algebra, yet the course I took that year from Cemal Koç was the one that has left a strong impression on me as a live example of how one should lecture and teach. Later, on many other occasions, I had the chance to observe his enthusiasm for and dedication to teaching. Being a first grade researcher as well, he possessed a combination of talents that one does not find very often. Unfortunately, I won't be able to give you a taste of what his teaching was like but instead I shall try to summarize his mathematical research.

Cemal Koç was born in Kırcaali, Bulgaria, in 1943. After a short while, he and his family had to immigrate to Turkey. His family struggled through many economic problems and they had to move a lot, which affected his education as well. It started in Samsun, then continued in Tokat for a while, and ended in 1963 with his graduation from the department of mathematics and astronomy of Ankara University. After graduation he began work as a math teacher in a high school. But immediately he realized he would not be happy and efficient there. In the same year he started to work as an assistant professor in the department of mathematics at Ege University. This was the beginning of his academic career in mathematics.

In his Ph.D., Cemal Koç worked on a problem in differential geometry under the supervision of Muzaffer Kula and completed it in 1968. His thesis, which is also his first academic work, is entitled "n-boyutlu uzayda bir eğrinin ardışık oskülator küre merkezlerinin limiti ve infinitezimal öğelerinin belirlenmesi." His later research, however, has no relation to this area and is devoted to some problems in algebra. This unusual progress in his mathematical career was essentially due to his acquaintance with Masotoshi Ikeda, who was then a young associate professor at Ege University. The influence of Ikeda's personality and the way he did mathematics had a decisive role in the development of Cemal's life as a mathematician. Although Cemal Koç had a chance of a promising career at Ege University, he did not hesitate to follow İkeda to the Middle East Technical University in Ankara, when İkeda was invited in 1969 by Cahit Arf to join METU (see [9]). A year later he was appointed as an assistant professor.

His research can be classified essentially under two headings, namely, commutativity of rings and Clifford algebras. Both of these topics arose during discussions in the departmental seminars conducted jointly by Arf and İkeda.

1. His research on the commutativity of rings

The following classical theorems motivated much research on the topic of rings satisfying certain polynomial identities.

Theorem (Jacobson) *If R is a ring, and if for any $x \in R$ there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$, then R is commutative.*

Theorem (Herstein) *If R is a ring and if for any $x, y \in R$ there exists an integer $n(x,y) > 1$ such that $(xy - yx)^{n(x,y)} = xy - yx$ then R is commutative.*

İkeda had also been working on related problems in this area. One of his results in 1951 is a generalization of a theorem by Kaplansky related to commutativity of rings. It is likely that Koç was motivated by the ongoing discussions on these problems during algebra seminars at METU at the time and started his research in 1973 by giving a new and elementary proof of Herstein's theorem; see [3]. The following year in a joint work with İkeda, he proved the following theorems.

Theorem 1 *Let R be a ring. Suppose that for any $a, b \in R$, there exists an integer $m(a,b) > 1$ such that $[a, b, .., b]_k^{m(a,b)} = [a, b, .., b]_k$, for some $k \in \mathbb{N}$. Then,*

i. The commutator ideal of R is contained in the prime radical of R;

ii. The prime radical is the set of all nilpotent elements of R;

iii. For any commutator a, we have that $a^n = a$ for some integer $n > 1$ implies $a = 0$;

iv. R satisfies the k-th Engel condition;

v. The prime radical of R is locally nilpotent.

Theorem 2 *Let R be a ring and n and k two non-negative integers. If there exists a monomial $M(X,Y)$ in the noncommutative indeterminates X and Y different from $Y^m X$, $m \geq 1$, such that $b^n [a, c, .., c]_k = M([a, c, .., c]_k, b)$ for any $a, b, c \in R$, then the commutator ideal of R is contained in the prime radical $P(R)$ of R. Furthermore, $P(R)$ is the set of nilpotent elements of R and is locally nilpotent.*

These results are serious generalizations of theorems by Herstein [14], Smiley [22] , Putcha-Yaqub [20] and Putcha-Wilson-Yaqub [21].

Cemal Koç also coadvised Abdullah Harmancı, a Ph. D. student of İkeda, working on similar problems, during İkeda's visit to the United States.

Cemal Koç has two more papers [5] and [6] on this subject. These are joint work with Arif Kaya and appeared in 1976 and 1981, the main theorems of which are

Theorem 3 *Let R be a ring in which for any $x, y \in R$, there exists a positive integer $n = n(x,y)$ such that the relation $(xy)^k = x^k y^k$ is satisfied for any $k \in \{n, n+1, n+2\}$. Assume further that all zero divisors of R are contained in a proper left (right) ideal of R. Then R is commutative.*

Theorem 4 *Let R be a prime ring and T a semicentralizing automorphism of R. Then, T is a commuting automorphism. In particular, a prime ring possessing a nontrivial semicentralizing automorphism is a commutative integral domain.*

Here, an autumorphism T of a ring R is called semicentralizing if for any $x \in R$, we have that $xT(x) - T(x)x$ or $xT(x) + T(x)x$ is in the center of R.

These theorems generalize the results of J. H. Mayne (1976), [18] and L. O. Chung, J. Luh (1978), [16] .

2. His research on Clifford algebras

In 1970 Cemal Koç published the paper "On a generalization of Clifford algebras" [1], where he answered a problem suggested by Cahit Arf which can be stated as follows.

Let V be a vector space over a field K and Q a quadratic form on V. The Clifford Algebra $C = C(V, Q)$ of the quadratic form Q on V has the property that any isometry of V with respect to Q can be extended to an automorphism of C leaving V invariant as a subspace and conversely every automorphism of C mapping V onto itself induces an isometry on V with respect to Q. Does this property characterize the Clifford algebra of Q on V?

Cemal Koç answered this question in the negative by constructing algebras having this property which are not Clifford algebras. He denotes them by $C_{r,\varphi}(Q, V)$, where r is a positive integer and $\varphi(\xi) = \sum_{i=0}^{m} a_i \xi^i$ is a monic irreducible polynomial over K with $a_{m-1} \neq 0$. He calls them (r, φ)-Clifford algebras. In the case that $\varphi(\xi) = \xi - 1$, the notation is simplified to $C_{r,\varphi}(Q, V) = C_r(Q, V)$ and it is proven that if K is a field with q elements and V is of dimension greater than 2 then $C_r(Q, V)$ is infinite dimensional for every $r > q(q+1)/2$; in particular, it is not a Clifford algebra.

It is possibly due to this success that he was granted a NATO Foreign Research Scholarship by Tubitak with which he spent a year at the University of London where he was able to improve his proficiency in algebra.

The following year, together with İkeda, they proved that (with the above notation) if $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the roots of φ in the algebraic closure of K with multiplicities $e_1, e_2, \ldots, e_t$, respectively, then $C_{r,\varphi}(Q, V)$ is infinite dimensional if $\max\{re_i \mid i = 1, 2, ..., t\} \geq 4$ and $\dim(V) > 1$, [2].

His research was interrupted for a year by his military service obligation. When he returned to METU in 1973, he continued his research and focused on the commutativity of rings.

Cemal Koç became an associate professor in 1977, but his academic career was suspended between 1983 and 1989 as the new political authority had forced him to leave the University. Upon his return in 1990 he continued his research on generalizations of Clifford algebras, which were defined and studied in the mean time by other mathematicians and some of which he had already worked out before. His first paper from that period is "Clifford algebras of d-forms", [8] in 1995. In that paper the concept of ω-Clifford Algebras is introduced as follows.

Let K be a field containing a primitive d-th root of unity ω and let f be a diagonal form of degree d on the K-space V with basis $\{e_1, \ldots, e_n\}$ such that $f(e_i) = a_i \neq 0$ for all i. The algebra $T(V)/I(V, f)$, where $I(V, f)$ is the ideal of the tensor algebra $T(V)$ generated by

$$\underbrace{e_i \otimes \cdots \otimes e_i}_{d} - a_i, \qquad i = 1, 2, ..., n$$

and

$$\underbrace{e_i \otimes \cdots \otimes e_i}_{d-1} \otimes e_j \otimes e_i \otimes \underbrace{e_j \otimes \cdots \otimes e_j}_{d-1} - \ \omega a_i a_j \qquad \text{for} \quad j > i,$$

is called the ω-Clifford Algebra of V associated to the d-form f. For $d = 2$, it becomes the classical Clifford algebra associated to a non degenerate quadratic form on a vector space V with an orthogonal basis $\{e_1, \ldots, e_n\}$.

The ω-Clifford algebra can be considered as an algebra generated by n elements $e_1, \ldots, e_n$ subject to the relations

$$\begin{aligned} e_i^d &= a_i, \qquad i = 1, ..., n, \\ e_j e_i &= \omega e_i e_j, \quad \text{for } j > i, \end{aligned}$$

and as such denoted by $(a_1, ..., a_n)_\omega^d$. These algebras satisfy the following universal property: Given a linear map φ from V to an algebra A such that $(\varphi(e_i))^d = a_i 1_A$, $i = 1, ..., n$ and $\varphi(e_j)\varphi(e_i) = \omega\varphi(e_i)\varphi(e_j)$ for $j > i$, there exists a unique algebra homomorphism $\phi : (a_1, ..., a_n)_\omega^d \to A$ which extends φ.

In this paper Koç proves the following theorems.

Theorem 5 *If $a_1, ..., a_n$ are non zero elements of a field containing a primitive d-th root of unity then*

i. We have canonical graded isomorphisms

$$(a_1, ..., a_n)_\omega^d \cong (a_1, ..., a_s)_\omega^d \widehat{\otimes} (a_{s+1}, ..., a_n)_\omega^d \cong (a_1)_\omega^d \widehat{\otimes} ... \widehat{\otimes} (a_n)_\omega^d,$$

for any $1 < s < n$, where $\widehat{\otimes}$ stands for the graded tensor product defined by ω.

ii. $\dim(a_1, ..., a_n)_\omega^d = d^n$.

iii. The dimension of the vector space of homogeneous elements of the $\mathbb{Z}/d\mathbb{Z}$ graded algebra $(a_1, ..., a_n)_\omega^d$ of fixed degree is d^{n-1}.

Theorem 6 *Let K be a field containing a primitive d-th root of unity ω and let $C = (a_1, \ldots, a_n)_\omega^d$. Then, there exists an element a in K uniquely and explicitly defined by $a_1, ..., a_n$ such that if $L = K[x]/(x^d - a)$, then*

i. If n is odd, the center of C is L and it is a central simple L-algebra when $x^d - a$ is irreducible over K. The homogeneous part C_0 of degree 0 is a central K-algebra.

ii. If n is even C is a central simple K-algebra. The center of the homogeneous part C_0 of degree 0 is L and C_0 is a central simple L-algebra when $x^d - a$ is irreducible over K.

In these years Cemal Koç has supervised several Master's and Ph. D. theses, most of which where related to Clifford algebras:

1996, Azumaya Algebras Arising From The Clifford Algebras (F. N. Akbulut)

1997, Clifford-Littlewood-Eckmann Groups (B. A. Nalbantoğlu)

1998, Structure Theory of Central Simple $\mathbb{Z}_p$-Graded Algebras (Y. Kurtulmaz - Ph. D.).

The first two of these were surveying work on the subject indicated by their titles. From the third came an original result, which was unfortunately never published.

In 1999 Cemal Koç retired from METU and joined Doğuş University in İstanbul, which at that time was in the process of establishment. He made every effort to ensure that the basic mathematics courses offered by the department were of a certain standard both in content and quality. He was literally the founder of the mathematics department. In many of his speeches he stated that he was very pleased to be a part of Doğuş University and to be able to contribute to its development.

Meanwhile, he also initiated an annual mathematics competition for high school students in İstanbul sponsored by Doğuş University. One could say that one of his most important achievements was to conduct a series of algebra seminars with regular participation from various universities.

Although he was seriously occupied with administrative responsibilities he had always some time devoted to research. One of the results of that period directly related to generalizations of Clifford algebras is "*C-lattices and decompositions of Generalized Clifford Algebras*", [12]. In this paper, he gives simplified proofs of the structure theorems for the algebras of the form $(\omega^{m_1}, \omega^{m_2}, \ldots, \omega^{m_n})^d_\omega$ which were previously obtained by Lam and Smith (1989–1991), [17] and proven by the machinery of Clifford-Littlewood-Eckmann Groups and their representations.

Exterior algebra of a vector space V is the classical Clifford algebra on V corresponding to the zero quadratic form. In this sense, Koç's results (joint with S. Esin) in "*Annihilators of principal ideals in the exterior algebra*," [10] are also related to Clifford algebras. Let V be a finite dimensional vector space with a basis $X = \{x_1, x_2, ..., x_n\}$. Let $E = E(V)$ be the exterior algebra on V and M_i, for $i = 1, \ldots, s$, non-empty disjoint subsets of X. Let μ_i be the product of the elements in M_i with the ordering of increasing indices and $\mu = \mu_1 + \cdots + \mu_s$. Generalizing a result of İ.Dibağ, [15], they prove in [10] the following:

Theorem 7 *With the above notation if all $|M_i|$ are odd the principal ideal (μ) of E is of dimension*

$$\frac{\dim(E)}{2}(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2}) \cdots (1 - 2^{1-n_s})).$$

Furthermore, the annihilator $Ann(\mu)$ is generated by μ and the products $P_J = \prod_{t=1}^{s} x_{tj_t}$ where $x_{ij} \in M_i$, $i = 1, ..., s$ and is of dimension

$$\frac{\dim(E)}{2}(1 + (1 - 2^{1-n_1})(1 - 2^{1-n_2}) \cdots (1 - 2^{1-n_s})).$$

The investigation of even case is much more subtle and there are some further results in the same paper concerning this case too but they are omitted here for the sake of simplicity of presentation.

Motivated by certain counting problems from that paper [10] in the works [11] and [13], some combinatorial problems are investigated and yet another algebraic characterization of Catalan numbers and some of its generalizations are obtained. One of the main theorems is the following.

Theorem 8 *If we define the action of the symmetric group* S_n *on the polynomial algebra* $F[x_1, ..., x_n, y_1, ..., y_n]$ *in* $2n$ *indeterminates by*

$$f_\sigma = (\sigma f)(x_1, ..., x_n, y_1, ..., y_n) = f(x_{\sigma 1}, ..., x_{\sigma n}, y_1, ..., y_n)$$

and if we denote the polynomial $\prod(x_i - y_i)$ *by* $p = p(x_1, ..., x_n, y_1, ..., y_n)$, *then the set* $\{p_\sigma | \ \sigma \in St_n\}$ *is an* F*-basis for the cyclic submodule* $F[S_n]p$ *and hence its dimension is the Catalan number* $C(n)$. (*Here,* St_n *denotes the set of stack-sortable permutations in* S_n.)

There is yet another interesting result of C. Koç, [7] with the title "*Recurring-with-carry sequences*" which is not related to his main research areas. It is a contribution to the theory of pseudo random generators and generalizes and unifies the theory of add-with-carry and subtract-with-borrow sequences which were introduced by Marsaglia and Zaman in 1991, [19]:

Let $\{F_n\}$ be an r-th order recurring sequence satisfying

$$a_0 F_n + a_1 F_{n-1} + \cdots + a_r F_{n-r} + c = 0, \qquad n = r, r+1, ...$$

where the characteristic polynomial $P(x) = a_0 + a_1 x + \cdots a_r x^r$ is a polynomial of degree r with integer coefficients. Suppose $b \geq 2$ is an integer and we have initial values with $0 \leq F_i < b$ for $i = 0.1....r-1$. Then there exists a uniquely determined sequence $\{x_n\}$ of base-b digits given by

$$\sum_{i=0}^{n} x_i b^i \equiv F_i b^i \pmod{b^{n+1}} \quad \text{for all} \quad n \geq 0.$$

The sequence $\{x_n\}$ is called a recurring-with-carry sequence of base b. We shall assume that c is either 0 or $b-1$ and that $P(b) \neq 0$. Furthermore, we let

$$\theta = \frac{1}{P(b)} \left(\frac{-c}{b-1} + \sum_{i=1}^{r} a_i (x_{r-i} + x_{r-i+1} b + \cdots x_{r-1} b^{i-1}) \right).$$

Theorem 9 *Let* $\{x_n\}$ *be a recurring-with-carry sequence of base* b *which has more than* r *non-zero terms and let* $P(x)$ *be its characteristic polynomial. If* $0 \leq \theta \leq 1$, *for any arbitrary integer* $n \geq r$ *the terms* $x_r, x_{r+1}, ..., x_n$ *are the first* $n - r + 1$ *digits in the reverse order of the base* b *expansion of* $\frac{k}{|P(b)|}$ *for some integer* k *satisfying* $0 \leq k < |P(b)|$. *In addition if* $|P(b)|$ *is prime the period of* $\{x_n\}$ *is the same as that of the base-*b *expansion of* $|P(b)|^{-1}$ *and is equal to the order of the congruence class of* b *in the multiplicative group of no-zero elements of the finite field of integers modulo* $|P(b)|$.

We complete this article by giving a list of mathematical work of Cemal Koç apart from journal papers:

The thesis supervised by C. Koç

1. 1974, Bazı Komutatiflik Teoremleri (Some commutativity theorems) (A. Harmancı) Ph.D.

2. 1979, p-adik Fonksiyonlar (p-adic functions) (M. Soytürk)
3. 1981, Pro C-Groups (S. Yassin)
4. 1992, Galois Calculus and Generalized Drinfeld Modules (Y. Kurtulmaz)
5. 1996, Azumaya Algebras Arising From The Clifford Algebras(F. N. Akbulut)
6. 1997, Clifford-Littlewood-Eckmann Groups (B. A. Nalbantoğlu)
7. 1998, Structure Theory of Zp-central Simple Graded Algebras(Y. Kurtulmaz) Ph. D.
8. 1999 Some invariants of Fields (F. Koyuncu).

Books

1. Soyut Cebir, Anadolu Üniv., Eskişehir , 1991. (with M. Bilhan and İ. Güloğlu).
2. Linear Algebra I, METU, Ankara, 1996.
3. Topics in Linear Algebra, METU, Ankara , 1996 (New edition 2002).
4. Basic Linear Algebra, Matematik Vakfı Yayınları, 1996 (New Edition 2009).

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