Generalized catalan numbers, sequences and polynomials

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Abstract

In this paper we present an algebraic interpretation for generalized Catalan numbers. We describe them as dimensions of certain subspaces of multilinear polynomials. This description is of utmost importance in the investigation of annihilators in exterior algebras.

1. Introduction

Let V be a vector space over a field F and $X \subseteq V$. An element $\mu = \varsigma_1 + \varsigma_2 + \ldots + \varsigma_n$ of the exterior algebra $E(V)$ of V is said to be neat with respect to X if $\varsigma_i = x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_{n_i}}$ with $x_{i_j} \in X$, $j = 1, 2, \ldots, n_i$ and $\varsigma_1 \wedge \varsigma_2 \wedge \ldots \wedge \varsigma_r \neq 0$. The annihilator of μ in $E(V)$ is described by products of the form

$$(\varsigma_{i_1} - \varsigma_{j_1}) \ldots (\varsigma_{i_r} - \varsigma_{j_r}) \varsigma_{k_1} \ldots \varsigma_{k_t}$$

when $Char(F) = 0$ (see [1]). Dimensions of subspaces of $E(V)$ spanned by certain elements of this type can be used to extend results of [1] to remove the restriction $Char(F) = 0$.

Motivated by this, we continue the study of certain type of ideals of the polynomial ring, studied in [2]. To be more precise, let $F[z] = F[z_1, \ldots, z_n]$ be the ring of polynomials in n indeterminates over F. The symmetric group S_n of degree n acts on this ring canonically as

$$f^\sigma(z) = f(z_{\sigma(1)}, \ldots, z_{\sigma(n)}).$$

Letting

$$p(z) = \begin{cases} (z_1 - z_2) \ldots (z_{2r-1} - z_{2r}) & \text{if } n = 2r \\ (z_1 - z_2) \ldots (z_{2r-1} - z_{2r}) z_{2r+1} & \text{if } n = 2r+1, \end{cases}$$

we can form the cyclic module

$$F[S_n]p(z)$$

over the group ring $F[S_n]$. In [2] it was proved that

$$F[S_n]p(z) = F[H]p(z),$$

where H is the subgroup of S_n fixing each z_{2k}, $k = 1, \ldots, r$. Identifying H with S_{n-r} and setting

$$
\begin{aligned}
z_{2i-1} &= x_i \ \text{ for } i = 1, \ldots, n-r \ , \\
z_{2j} &= y_j \ \text{ for } \ j = 1, \ldots, r, \\
F[z] &= F[x; y] \ \text{ and } \ p(z) = p(x; y) \\
p^\sigma(z) &= p^\sigma(x; y) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n-r)}; y_1, \ldots, y_r) \text{ for } \sigma \in S_{n-r}
\end{aligned}
$$

the results given in Theorem 6 and its corollaries in [2] can be summarized in the following form:

The polynomials $p^\sigma(x; y)$, where σ runs over 231-avoiding permutations in S_{n-r}, form an F-basis for the cyclic module $F[S_n]p(z) = F[S_{n-r}]p(x; y)$ and hence its dimension is the Catalan number C_d where $d = n - r$ is the degree of $p(z)$.

Now we extend the space $F[S_n]p(z) = F[S_{n-r}]p(x; y)$ so as to contain all multilinear polynomials

$$
(z_{i_1} - z_{j_1}) \ldots (z_{i_l} - z_{j_l}) z_{k_1} \ldots z_{k_t} \qquad \text{where } \ \{i_1,...,i_l; j_1,..., j_l; k_1,...,k_t\} = \{1, ..., n\}
$$

and construct a basis for this space by using truncated stack-sortable permutations on more than d letters. To begin with, we give some remarks on (truncated) stack-sortable permutations and generalized Catalan numbers.

2. Truncated stack-sortable permutations and sequences

The sequence $(C_n)_{n \in \mathbb{N}}$ of Catalan numbers where the n-th Catalan number C_n is defined as $\dfrac{1}{n+1}\dbinom{2n}{n}$ occurs in many different situations. On page 219 of his book "Enumerative Combinatorics, Volume 2" R. P. Stanley seeks to show that C_n counts the number of elements in 66 different combinatorial configurations. Among these, the most interesting for the discussion in this paper, are the following:

(a) C_n is the number of *lattice paths* from $(0, 0)$ to (n, n) with steps one unit to the right and one unit upward never rising above the line $y = x$. In the literature these paths are also referred to as *Dyck paths* (see for example, [3] or [4]).

(b) The Catalan number C_n is the number of finite sequences $s = (s_1, s_2, \ldots, s_{2n})$ of $2n$ terms in which half of them are 0 and the others are 1 and

$$
\sum_{i=1}^t s_i \geq \frac{t}{2}
$$

for any $t \leq 2n$. We will denote the set of such sequences by Sq_n.

(c) A permutation $[a_1, a_2, \ldots, a_n]$ on a set of positive integers for which there exists no $i < j < k$ with $a_k < a_i < a_j$ is called a *stack-sortable permutation* or a 231-*avoiding permutation*. The Catalan number C_n is the number of such permutations on a set of n integers. In the sequel the set of stack-sortable permutations on a specified set of n integers will be denoted by St_n.

(d) In [2] it was proved that $C_d = \dim_F(F[S_n]p(z))$, where d is the integral part of $\dfrac{n+1}{2}$.

Now, we note some natural generalizations by considering the numbers $C(n,m)$ for $m \leq n$ defined by

$$C(n,m) = \binom{n+m}{m} - \binom{n+m}{m-1} = \frac{n-m+1}{n+1}\binom{n+m}{m},$$

with the usual convention $\binom{k}{l} = 0$ when $l > k$ or $l < 0$. Since $C(n,n) = C_n$, they obviously generalize Catalan numbers, and they are referred as *generalized Catalan numbers*.

(a′) $C(n,m)$ is the number of *lattice paths* from $(0,0)$ to (n,m) with steps one unit to the right and one unit upward never rising above the line $y = x$. This can be seen by using D. André's ingenious "reflection principal" (see for example [3], pgs 230, 265.).

(b′) In [2] to each stack-sortable permutation $\tau \in S_n$ it was assigned a sequence $\varphi_\tau = (s_1, s_2, \ldots, s_n; s_1', s_2', \ldots, s_n')$ of 1's and 0's by the rule

$$s_{\tau(n)} = 1, \text{ and } s_n' = 0$$

$$s_{\tau(k)} = \begin{cases} 1 & \text{if} \quad \tau(k) < \tau(k+1) \\ 0 & \text{if} \quad \tau(k) > \tau(k+1) \end{cases}, \text{ and } s_k' = 1 - s_{\tau(k)} \text{ for } k < n.$$

Further, it was proven for $\sigma, \tau \in St_n$ that we have evaluations $p^\sigma(\varphi_\sigma) = \pm 1$, and $p^\sigma(\varphi_\tau) = 0$ when $\sigma > \tau$; that is to say when

$$\exists\, k \in \{1, \ldots, n\} \text{ such that } \sigma(i) = \tau(i) \text{ for all } i > k \text{ and } \sigma(k) > \tau(k).$$

This proves in particular that the composite map

$$\tau \to \varphi_\tau = (s_1, s_2, \ldots, s_n; s_1', s_2', \ldots, s_n') \to s_\tau = (s_1, s_1', s_2, s_2', \ldots, s_n, s_n')$$

is one to one. Now, we will prove that $s_\tau = (s_1, s_1', s_2, s_2', \ldots, s_n, s_n')$ is a sequence in Sq_n described in (b) above. This will establish a canonical bijection between 231-avoiding permutations and Dyck path through stack-sortable polynomials. Although there are various other constructions of bijections of this type, our canonical bijection will be efficiently used in the rest of this paper (see for example, [4]).

Lemma 1 *We have* $\{s_\tau \mid \tau \in St_n\} = Sq_n$.

Proof. Since $s_{\tau(i)} = 1 - s_i'$, the equality of the number 0's and 1's in s_τ is obvious. So we need to show that $\sum_{i=1}^{t} s_\tau[i] \geq \frac{t}{2}$ for $\tau \in St_n$. We proceed by induction on n. It is obvious for $n \leq 2$. Assume that $n \geq 3$ and the inequality is true for any stack-sortable permutation of degree $l < n$. Let $k = \tau^{-1}(n)$. By stack-sortability of τ, $\tau(i) < k$ for any $i < k$ and for $j \geq k$ we have $k \leq \tau(j) \leq n$.

Also, let ν be the restriction of τ to $\{1, \ldots, k-1\}$. Then $\nu \in St_{k-1}$. Let $\mu \in S_{n-k+1}$ be defined by $\mu(j) = \tau(k - 1 + j) - (k - 1)$. Then $\mu \in St_{n-k+1}$ and the sequence $(s_1, s_1', \ldots, s_n, s_n')$ corresponding to τ is obtained by concatenating the sequences corresponding to ν and μ. So if $k > 1$, our induction hypothesis gives the result. So we can assume that $\tau(1) = n$.

Let
$$\mu = [\tau(2), \tau(3), \ldots, \tau(n)].$$

Clearly $\mu \in St_{n-1}$. Let $s_\tau = (a_1, a_1', \ldots, a_n, a_n')$ and $s_\mu = (b_1, b_1', \ldots, b_{n-1}, b_{n-1}')$. We see that

$$(b_1, b_1', \ldots, b_{n-1}, b_{n-1}') = (a_1, a_2', a_2, a_3', a_3, \ldots, a_{n-1}', a_{n-1}, a_n'),$$

i.e. $b_i = a_i$ and $b_i' = a_{i+1}'$ for $i = 1, 2, \ldots, n-1$. By induction we assume that for any t the sequence consisting of the first t terms of the sequence s_μ satisfies $\sum_{i=1}^{t} s_\mu[i] \geq \frac{t}{2}$. As we have

$$\sum_{i=1}^{t} s_\tau[i] - \sum_{i=1}^{t-2} s_\mu[i] = a_1' + a_u,$$

where u is the greatest integer in $\frac{t+1}{2}$, and $a_1' = 1$ we see that $\sum_{i=1}^{t} s_\tau[i] \geq \frac{t}{2}$. We have shown that $\{s_\tau \mid \tau \in St_n\} \subseteq Sq_n$ and equality holds since $|St_n| = |Sq_n| = C_n$. $\square$

Take two positive integers n and p with $p \leq n$ and consider sequences s of $2n$ terms $s[i] \in \{0,1\}$, such that,

$$\sum_{i=1}^{t} s[i] \geq \frac{t}{2} \quad \text{for all } t \leq 2n-p \text{ and } s[2n-p+1] = \cdots = s[2n] = 0.$$

$$\sum_{i=1}^{2n} s[i] = n$$

Obviously, labelling steps to the right as 1 and those upward as 0 such sequences are seen to be in one-to-one correspondence with paths described in (a') above, with $m = n - p$. Hence their number is $C(n,m) = C(n, n-p)$. They will be called *p-truncated sequences* in Sq_n and their set will be denoted by $Sq_n^{(p)}$.

(c') A stack-sortable permutation $\sigma \in St_n$ will be said to be $p-truncated$ if its sequence s_σ is p-truncated. The set of all p-truncated stack-sortable permutations is denoted by $St_n^{(p)}$. Obviously, $St_n^{(1)} = St_n$ and $St_n^{(p)} \subset St_n^{(q)}$ when $q < p$.

Now we characterize truncated stack sortable permutations intrinsically.

Theorem 2 *For any $0 \leq p \leq m$, we have*

$$St_m^{(p)} = \{\tau \in St_m \mid \tau(r+1) < \tau(r+2) < \cdots < \tau(m) \leq d\},$$

where d is the integral part of $\frac{2m-p+1}{2}$ *and $r = 2m - p - d$.*

Proof. Let τ be a stack-sortable permutation which satisfies

$$\tau(r+1) < \tau(r+2) < \cdots < \tau(m) \le d.$$

Then, $s_{\tau(r+1)} = \cdots = s_{\tau(m)} = 1$, therefore

$$s'_{r+1} = \cdots = s'_m = 0$$

and further

$$\tau^{-1}(k) \le r \ \text{ for } k = d+1, d+2, \cdots, m.$$

On the other hand, it follows from this and the stack-sortability of τ that

$$\tau^{-1}(m) < \tau^{-1}(m-1) < \cdots < \tau^{-1}(d+1) \le r$$

and hence

$$\tau\tau^{-1}(k) > \tau(\tau^{-1}(k)+1) \ \text{ for } k = d+1, d+2, \cdots, m,$$

which in turn yields $s_{\tau\tau^{-1}(d+1)} = \cdots = s_{\tau\tau^{-1}(m)} = 0$, namely

$$s_{d+1} = \cdots = s_m = 0,$$

in addition to $s'_{r+1} = \cdots = s'_m = 0$, which is obtained above. Thus $s_\tau \in Sq_m^{(p)}$ and hence $\tau \in St_m^{(p)}$.

Conversely, if s_τ is p-truncated for $\tau \in St_m$, then

$$s_{d+1} = \cdots = s_m = 0 = s'_{r+1} = \cdots = s'_m$$

and hence,

$$s_{\tau(r+1)} = \cdots = s_{\tau(m)} = 1.$$

Therefore,

$$\tau(r+1) < \tau(r+2) < \cdots < \tau(m).$$

If we had $\tau(m) = d + t$ for some $t \ge 1$ we would get $s_{\tau(m)} = s_{d+t} = 0$ which contradicts $s_{\tau(m)} = 1$. Thus

$$\tau(r+1) < \tau(r+2) < \cdots < \tau(m) \le d$$

that is $\tau \in \{\tau \in St_m \mid \tau(r+1) < \tau(r+2) < \cdots < \tau(m) \le d\}$. This completes the proof. $\square$

Now, we are in a position to give our basis construction for certain subspaces of multilinear polynomials. For this purpose we introduce the algebra A which is the quotient ring of the polynomial ring $F[x;y]$ by the ideal J generated by $\{x_1^2, \ldots, x_{n-r}^2; y_1^2, \ldots, y_r^2\}$. Letting $\xi_i = x_i + J$, and $\eta_j = y_j + J$ we see that $A = F[\xi;\eta] = F[\xi_1, \ldots, \xi_{n-r}; \eta_1, \ldots, \eta_r]$ with $\xi_i^2 = 0$ and $\eta_j^2 = 0$ for $i = 1, 2, \ldots n-r$ and $j = 1, 2, \ldots r$. It is a graded ring with respect to the ordinary grading induced from degrees of polynomials, say

$$A = F \oplus A_1 \oplus \cdots \oplus A_n,$$

and its ideal $\mathcal{I}$ generated by $F[S_{n-r}]p(\xi;\eta)$ is a graded ideal:

$$\mathcal{I} = \mathcal{I}_d \oplus \mathcal{I}_{d+1} \oplus \cdots \oplus \mathcal{I}_n = \bigoplus_{m=d}^{n} \mathcal{I}_m \quad \text{where } d = n - r = \deg(p(x;y)).$$

Note that $d = r = \dfrac{n}{2}$ or $d = r + 1 = \dfrac{n+1}{2}$ according to n is even or odd and any homogenous component $\mathcal{I}_m$ consists of all linear combinations of products of the form

$$(\xi_{i_1} - \eta_{j_1}) \ldots (\xi_{i_t} - \eta_{j_t}) \xi_{k_1} \ldots \xi_{k_p} \eta_{l_1} \ldots \eta_{l_q},$$

where $\{i_1, ..., i_t \; ; \; j_1, ..., j_t \; ; \; k_1, ..., k_p; l_1, ..., l_q\} = \{1, ..., n\}$, $t + p + q = m$ and $q = p$ or $p - 1$ according as n is even or odd. We also note that $\mathcal{I}_m = \mathcal{I}_d A_l$ when $m = d + l$. Furthermore we notice that the natural algebra homomorphism $\pi : F[x;y] \to F[\xi;\eta]$ for which

$$\pi(x_k) = \xi_k \text{ and } \pi(y_k) = \eta_k$$

induces a linear isomorphism between $\mathcal{I}_m$ and I_m, the space of multilinear polynomials whose images are contained in $\mathcal{I}_m$. Regarding this isomorphism elements of $F[\xi;\eta]$ will be referred to as multilinear polynomials.

3. An F-basis for the ideal $\mathcal{I}$

As we pointed out in (d) above it was proved in [2] that $\mathcal{I}_d$ is of dimension C_d, and it has the canonical basis $\{p^\sigma(\xi;\eta) \mid \sigma \in St_d\}$. Therefore, $\mathcal{I}_{d+l} = \mathcal{I}_d A_l$ is spanned by the products $p^\sigma(\xi,\eta)M$ where M is a monomial in A_l and $\sigma \in St_d$. Since

$$(\xi_{\sigma(1)} - \eta_1) \ldots (\xi_{\sigma(i)} - \eta_i) \ldots (\xi_{\sigma(r)} - \eta_r)\eta_i$$
$$= (\xi_{\sigma(1)} - \eta_1) \ldots (\xi_{\sigma(i-1)} - \eta_{i-1})(\xi_{\sigma(i+1)} - \eta_{i+1}) \ldots (\xi_{\sigma(r)} - \eta_r)\xi_{\sigma(i)}\eta_i$$
$$= -(\xi_{\sigma(1)} - \eta_1) \ldots (\xi_{\sigma(i)} - \eta_i) \ldots (\xi_{\sigma(r)} - \eta_r)\xi_{\sigma(i)},$$

$\mathcal{I}_{r+l}$ is spanned by the products

$$p^\sigma(\xi,\eta)M = (-1)^l \prod_{j \in L} \xi_{\sigma(j)}\eta_j \prod_{i \in R-L} (\xi_{\sigma(i)} - \eta_i),$$

where $M = \prod_{j \in L} \xi_{\sigma(j)}$, $L \subseteq R = \{1, 2, \ldots, d\}$ with $|L| = l$ and $\sigma \in S_d$.

In order to investigate the homogeneous component $\mathcal{I}_m$ we introduce the algebra $\overline{A} = F[\overline{\xi};\overline{\eta}] = F[\overline{\xi}_1, \ldots, \overline{\xi}_m; \overline{\eta}_1, \ldots, \overline{\eta}_m]$ subject to the relations $\overline{\xi}_i^2 = 0$ and $\overline{\eta}_i^2 = 0$ for $i = 1, 2, \ldots m$ as above. For any positive integer d, we consider the natural projection θ of the algebra $\overline{A} = F[\overline{\xi};\overline{\eta}] = F[\overline{\xi}_1, \ldots, \overline{\xi}_m; \overline{\eta}_1, \ldots, \overline{\eta}_m]$ onto $A = F[\xi;\eta] = F[\xi_1, \ldots, \xi_d; \eta_1, \ldots, \eta_{n-d}]$ for which

$$\theta(\overline{\xi}_i) = \begin{cases} \xi_i & \text{for } i = 1, \ldots, d \\ 0 & \text{for } i = d+1, \ldots, m \end{cases} \quad ; \quad \theta(\overline{\eta}_i) = \begin{cases} \eta_i & \text{for } i = 1, \ldots, n-d \\ 0 & \text{for } i = n-d+1, \ldots, m \end{cases}$$

Now, we can prove the following theorem.

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Theorem 3 *For the integers m and n as before, the set*

$$\{\theta(p^\tau(\overline{\xi};\overline{\eta})) \mid \tau \in St_m^{(2m-n)}\}$$

is a basis for $\mathcal{I}_m$.

Proof. It is obvious that $\theta(p^\tau(\overline{\xi};\overline{\eta})) \in \mathcal{I}_m$ and hence $\theta(F[S_m]p(\overline{\xi};\overline{\eta})) = \mathcal{I}_m$. Thus for any $\theta(f(\overline{\xi},\overline{\eta})) \in \mathcal{I}_m$ we have

$$
\begin{aligned}
\theta(f(\overline{\xi},\overline{\eta})) &= \theta\left(\sum_{\tau \in St_m} a_\tau p^\tau(\overline{\xi};\overline{\eta})\right) \\
&= \sum_{\tau \in St_m} a_\tau \theta(p^\tau(\overline{\xi};\overline{\eta})).
\end{aligned}
$$

For each stack-sortable τ, either $\tau(r+l) > d$ for some $l \geq 1$ or $\tau(r+l) \in \{1, 2, \cdots, d\}$ for all $l \geq 1$. In the former case $\theta(p^\tau(\overline{\xi};\overline{\eta})) = 0$ and in the latter case by permuting $\tau(r+1), \cdots, \tau(m)$ to bring them into the natural order and leaving others as they stand we obtain a new permutation ρ satisfying

$$\rho(r+1) < \cdots < \rho(m) \leq d$$

which is still stack-sortable contained in $St_m^{(2m-n)}$. This yields $\theta(p^\tau(\overline{\xi};\overline{\eta})) = \theta(p^\rho(\overline{\xi};\overline{\eta}))$. Thus any $\theta(f(\overline{\xi},\overline{\eta})) \in \mathcal{I}_m$ can be written in the form

$$\theta(f(\overline{\xi},\overline{\eta})) = \sum_{\rho \in St_m^{(2m-n)}} b_\rho \theta(p^\rho(\overline{\xi};\overline{\eta})),$$

which shows that $\{\theta(p^\rho(\overline{\xi};\overline{\eta})) \mid \rho \in St_m^{(2m-n)}\}$ spans $\mathcal{I}_m$. As for linear independence, we note first that, for any $\sigma, \tau \in St_m^{(2m-n)}$, by writing

$$\overline{\varphi}_\sigma = (s_1, \ldots, s_d; s_1', \ldots, s_{n-d}'),$$

when

$$\varphi_\sigma = (s_1, \ldots, s_m; s_1', \ldots, s_m') = (s_1, \ldots, s_d, 0, \ldots, 0; s_1', \ldots, s_{n-d}', 0, \ldots, 0)$$

we have

$$p^\tau(\varphi_\sigma) = \theta(p^\tau(\overline{\xi};\overline{\eta}))(\overline{\varphi}_\sigma).$$

So a relation

$$\sum_{\tau \in St_m^{(2m-n)}} a_\tau \theta(p^\tau(\overline{\xi};\overline{\eta})) = \sum_{i=1}^s a_{\tau_i} \theta(p^{\tau_i}(\overline{\xi};\overline{\eta})) = 0$$

with $a_{\tau_s} \neq 0$ and $\tau_s > \tau_i$ for $i = 1, \ldots, s-1$ would yield

$$0 = \sum_{i=1}^s a_{\tau_i} \theta(p^{\tau_i}(\overline{\xi};\overline{\eta}))(\overline{\varphi}_{\tau_s}) = a_{\tau_s},$$

which is a contradiction. $\qquad\qquad\Box$

As a consequence of this theorem we easily deduce the following theorem.

Theorem 4 *The dimension of the homogeneous component $\mathcal{I}_m$ of the ideal $\mathcal{I}$ of $F[\xi_1, \ldots, \xi_{n-r}; \eta_1, \ldots, \eta_r]$ which is generated by $F[S_{n-r}]p(\xi; \eta)$ is*

$$\dim \mathcal{I}_m = \frac{2m - n + 1}{m + 1}\binom{n}{m}$$

and hence

$$\dim \mathcal{I} = \binom{n}{r}.$$

Proof. By the previous theorem, $(2m - n)$-truncated stack-sortable polynomials form a basis for $\mathcal{I}_m$, and thus its dimension is

$$
\begin{aligned}
\dim \mathcal{I}_m &= C(m, m - (2m - n)) \\
&= C(m, n - m) \\
&= \binom{n}{n - m} - \binom{n}{n - m - 1} \\
&= \frac{2m - n + 1}{m + 1}\binom{n}{m}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\dim \mathcal{I} &= \sum_{m=d}^{n} \dim \mathcal{I}_m = \sum_{m=d}^{n}\left(\binom{n}{n - m} - \binom{n}{n - m - 1}\right) \\
&= \binom{n}{n - d} \\
&= \binom{n}{r}
\end{aligned}
$$

as asserted. □

Finally we restate our results for polynomials.

Corollary 5 *The space I_m spanned by multilinear polynomials*

$$(x_{i_1} - y_{j_1})\ldots(x_{i_t} - y_{j_t})x_{k_1}\ldots x_{k_p}y_{l_1}\ldots y_{l_q}$$

in unknowns $x_{i_1}, \ldots, x_{i_d}; y_{i_1}, \ldots, y_{i_r}$ where

$$\{i_1, ..., i_t; j_1, ..., j_t; k_1, ..., k_p; l_1, ..., l_q\} = \{1, ..., n\}$$

and $t + p + q = m$, has a basis consisting of the $p^\sigma(x; y)$ where σ runs over all $(2m - n)$-truncated polynomials and hence it is of dimension $C(m, n - m)$.

Proof.   As mentioned at the end of Section 2, the natural algebra homomorphism $\pi : F[z] = F[x; y] \to F[\xi; \eta]$ for which

$$\pi(x_k) = \xi_k \text{ and } \pi(y_k) = \eta_k$$

induces a linear isomorphism between I_m and $\mathcal{I}_m$ with $\pi(p^\sigma(x; y)) = p^\sigma(\xi; \eta)$ and the result follows. $\square$

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